Compressible Navier-Stokes system with entropy transport

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25. 9. 2014

[1]Supported by grant GA13-00522S of the Grant Agency of the Czech Republic

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$\rho(0,x) = \rho_0(x)$, $\rho\mathbf{u}(0,x) = m_0(x)$ and $s(0,x) = s_0(x)$.
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- $\mathbb{S}(u) = \mu(\nabla \mathbf{u} + \nabla \mathbf{u}') + (\eta - 2/3\mu) \operatorname{div} \mathbf{u} \mathbb{I}$,
- $p(\rho, s) = \rho^\gamma \mathcal{T}(s)$, for $\mathcal{T}$ continuous, non-negative and bijective.

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Weak formulation of the system

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For any $\eta \in \mathcal{D}((0,T) \times \Omega)$ (or $\eta \in \mathcal{D}((0,T) \times \Omega)^3$)

$$\int_{(0,T)\times\Omega} \rho \partial_t \eta + \int_{(0,T)\times\Omega} \rho \mathbf{u} \nabla \eta = 0$$

$$\int_{(0,T)\times\Omega} \rho \mathbf{u} \partial_t \eta + \int_{(0,T)\times\Omega} \rho \mathbf{u} \otimes \mathbf{u} \nabla \eta + \int_{(0,T)\times\Omega} p(\rho, s) \operatorname{div} \eta$$

$$- \mu \int_{(0,T)\times\Omega} \nabla \mathbf{u} \nabla \eta - \int_{(0,T)\times\Omega} (\lambda + \mu) \operatorname{div} \mathbf{u} \operatorname{div} \eta = 0$$

$$\int_{(0,T)\times\Omega} s \partial_t \eta + \int_{(0,T)\times\Omega} s \mathbf{u} \nabla \eta - \int_{(0,T)\times\Omega} s \operatorname{div} \mathbf{u} \eta = 0$$

A couple of results

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- E. Feireisl, 2004, for $p(\rho, \vartheta) = \rho^\gamma + \vartheta\rho$, $\gamma > 3/2$ and equation for thermal energy of parabolic type instead of the transport equation
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- It is possible to obtain a stability result for $p(\rho, s) = (\rho/s)^\gamma$, $\gamma > 3/2$ with $s_0 \subseteq [1/C, C]$ for some $C > 0$.

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A priori estimates

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Weak stability

Let $(\rho_n, \mathbf{u}_n, s_n)$ be a weak solution of NSwET with $p(\rho,s) = (\rho/s)^\gamma$. Let the sequence be uniformly bounded in spaces given by a priori estimates and $\rho_n \in L^2(L^2(\Omega))$ (not uniformly). Then the weak limit $(\rho, \mathbf{u}, s)$ is also a weak solution.

Renormalization

Transport equation

If $s \in L^2((0,T); L^p(\Omega))$ and $\mathbf{u} \in L^2((0,T); W^{1,p/(p-1)}(\Omega))$ satisfy

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$$\partial_t B(s) + \mathbf{u} \cdot \nabla B(s) = 0 \quad \text{in } \mathcal{D}'(\Omega)$$

for any suitable function B.

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If $\rho \in L^2((0,T); L^p(\Omega))$ and $\mathbf{u} \in L^2((0,T); W^{1,p/(p-1)}(\Omega))$ satisfy

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$$\partial_t B(\rho) + \operatorname{div}(B(\rho)\mathbf{u}) = (B'(\rho)\rho - B(\rho)) \operatorname{div} \mathbf{u} \quad \text{in } \mathcal{D}'(\Omega)$$

Problems with renormalization

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Answer

One can show that $\rho \in L^2((0,T); L^2(\Omega))$ and we can renormalize "for free".

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Due to the isentropic theory we have $\tilde{\rho}_n \to \tilde{\rho}$ strongly in $\mathcal{C}([0,T]; L^{\gamma}(\Omega))$. And therefore

$$\rho \leftharpoonup \rho_n = s_n \tilde{\rho}_n \rightharpoonup s\tilde{\rho}$$

and $(\rho, \mathbf{u}, s)$ satisfies the weak form of the momentum equation for $p(\rho, s) = (\rho/s)^{\gamma}$.

Passing to limit in the transport equation

For any $\eta \in \mathcal{D}((0, T) \times \Omega)$ we have

$$\int_{(0,T)\times\Omega} s_n \partial_t \eta + \int_{(0,T)\times\Omega} s_n \mathbf{u}_n \nabla \eta - \int_{(0,T)\times\Omega} s_n \operatorname{div} \mathbf{u}_n \eta = 0.$$

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Letting $n \to \infty$ we obtain

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- Due to the Arzela-Ascoli theorem $s_n \to s \in \mathcal{C}([0,T]; L^p_\omega(\Omega))$ for all $p \in [1,\infty)$ and therefore

$$s_n \mathbf{u}_n \rightharpoonup s\mathbf{u} \quad \text{in } L^2((0,T); L^{2^*}(\Omega)).$$

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Let us take as a test function $\phi \approx \nabla \triangle^{-1} \sigma_n$ in the momentum equation for $(\rho_n, \mathbf{u}_n, s_n)$. Then take $\phi \approx \nabla \triangle^{-1} \sigma$ for the limit of momentum equations.

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- Let $\sigma_n \rightharpoonup^* \sigma$ in $L^\infty((0,T) \times \Omega)$ with $\partial_t \sigma_n + \operatorname{div}(\sigma_n \mathbf{u}_n) = \kappa_n$ for κ_n boudned in $L^2((0,T); L^2(\Omega))$.

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Then after passing to a subsequence, if needed, we obtain

$$\lim_{n\to\infty} \int_0^T \int_{\mathbb{R}^3} \phi\eta \left(\tilde{\rho}_n^\gamma - (2\mu + \lambda) \operatorname{div} \mathbf{u}_n\right) \sigma_n \,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^T \int_{\mathbb{R}^3} \phi\eta \left(\overline{\tilde{\rho}_n^\gamma} - (2\mu + \lambda) \operatorname{div} \mathbf{u}\right) \sigma \,\mathrm{d}x\,\mathrm{d}t$$

for any $\eta \in \mathcal{D}(\Omega)$ and $\phi \in \mathcal{D}((0,T))$.

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