Math Modelling Analysis Computing

Nečas Center for Mathematical Modeling

Piezo–viscous fluids flows in lubrication

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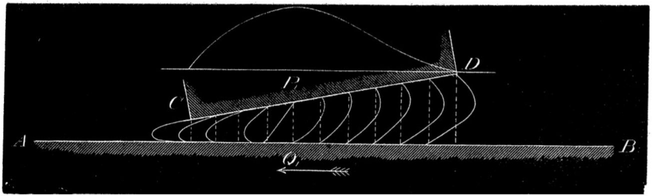
- Elastohydrodynamic lubrication (EHL)
- Reynolds approximation approach
- Piezoviscous fluids: modelling, analysis
- CFD approach in EHL

IV. *On the Theory of Lubrication and its Application to Mr.* Beauchamp Tower's *Experiments, including an Experimental Determination of the Viscosity of Olive Oil.*

By Professor Osborne Reynolds, *LL.D., F.R.S.*

Received December 29, 1885,—Read February 11, 1886.

Fig. 9.



This is the explanation of continuous lubrication.

(Elasto-)hydrodynamic lubrication

- very thin fluid film in between two (elastic) solid surfaces
- extreme pressure, shear rate and temperature variations
- complex problem studied intensively for decades
- increasing requirements in nowadays applications

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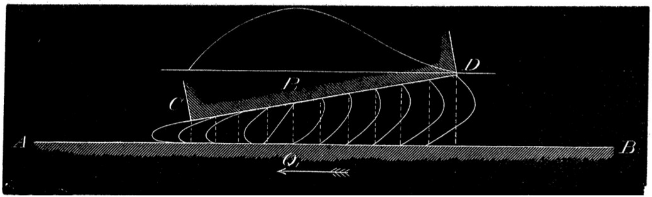

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(Elasto-)hydrodynamic lubrication

- very thin fluid film in between two (elastic) solid surfaces
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- complex problem studied intensively for decades
- increasing requirements in nowadays applications
  - brand new desings being developed
    to reduce friction, extend life and prevent noise
  - thinner lubricant films being used
    (e.g., so that surface topography becomes relevant)

Lugt, Morales–Espejel, *A Review of Elasto-Hydrodynamic Lubrication Theory*, 2011
Evans, Snidle, *The future of engineering tribology*, 2009
Björling, Habchi, Bair, Larsson, Marklund, *Towards the true prediction of EHL friction*, 2013

Basic idea

$$\rho \equiv 1\,, \qquad \operatorname{div} \mathbf{v} = 0$$
$$\mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \mu \Delta \mathbf{v} + \nabla p = \mathbf{b}$$

Reynolds, *On the Theory of Lubrication...*, 1886

Basic idea

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Basic idea

$$
\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \\
\frac{\partial p}{\partial x} &= \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \\
\frac{\partial p}{\partial y} &= \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)
\end{aligned}
$$

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Basic idea

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad (x, y) = (X, \varepsilon Y), \quad (u, v) = (U, \varepsilon V), \quad \text{etc.}$$

$$\frac{\partial p}{\partial x} = \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \qquad \frac{\partial P}{\partial X} = \bar{\mu} \left(\varepsilon^2 \frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} \right),$$

$$\frac{\partial p}{\partial y} = \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \qquad \varepsilon^{-2} \frac{\partial P}{\partial Y} = \bar{\mu} \left(\varepsilon^2 \frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} \right)$$

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\frac{\partial p}{\partial y} &= \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) = 0 \quad \varepsilon^{-2} \frac{\partial P}{\partial Y} = \bar{\mu} \left(\varepsilon^2 \frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} \right) = 0
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... by using the *lubrication assumptions*.

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Basic idea

$$\begin{aligned}\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} &= 0,\\ \frac{\partial p}{\partial x} &= \mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\\ \frac{\partial p}{\partial y} &= \mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)=0\end{aligned}$$

... by using the *lubrication assumptions*. This system can be easily integrated accross the film to obtain the Reynolds' lubrication equation for the pressure

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{h^3}{\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\right)=6\,U\,\frac{\mathrm{d}h}{\mathrm{d}x}\,.$$

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Reynolds approximation approach

Formal asymptotic expansion in ε

$$u^{\varepsilon} = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 \ldots, \qquad \text{and so on}$$

then also allows for

- sequence of corrections
- a priori estimates and rate of convergence (in ε)

Cimatti, Nazarov, Bayada, Chambat, . . . ;
and with focus on Navier–Stokes also:
Elrod, Duvnjak, Marušić–Paloka, Temam, Moise, Ziane,. . .

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Wilkening, *Practical error estimates for Reynolds' lubrication approximation...*, **2009**

$$||u - u^{2k}||_{1,2;(\varepsilon)} + (c_k)||p - p^{2k}||_2 \leq C_k \left(\frac{\varepsilon}{\rho_k}\right)^{2k+2}$$

- complete asymptotic expansion estimates,
  with constants depending on $h(x)$ expressed explicitely
- $\varepsilon \to 0$ **is nice, but the series seems to diverge as** $k!$ **for fixed** $\varepsilon > 0$

Reynolds approximation approach

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{h^3}{\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\right) = 6\,U\,\frac{\mathrm{d}h}{\mathrm{d}x}\,.$$

- directly from physical arguments
- or by using formal asymptotic arguments

- justified on basis of asymptotic a priori arguments
- a posteriori modelling error control would be nice...

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  after including elasticity (EHL), **pressure-thickening**, and shear-thinning

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$$\begin{aligned} \frac{\partial p}{\partial x} &= \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial \mu}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial \mu}{\partial y}\frac{\partial u}{\partial y} \\ &\sim \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \alpha\frac{\partial p}{\partial x}\frac{\partial u}{\partial x} + \alpha\frac{\partial p}{\partial y}\frac{\partial u}{\partial y}\right) \end{aligned}$$

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- validity of the lubrication assumptions is in some important situations questionable, the effect of their violation is not clear
  - **gradient of viscosity becomes dominant in momentum equation**
  - **variations of viscosity with pressure cause secondary pressure gradients**
  - **e.g., no parallel Poiseuille flow for Barus model**
- ...not speaking about thermal effects, for the moment...

Rajagopal, Szeri, *On an inconsistency in the derivation of the equations of EHL*, 2003
Bayada, Cid, García, Vázquez, *A new more consistent Reynolds model...*, 2013

Piezoviscous non-newtonian incompressible fluids

- Governing equations

$$-\operatorname{div}\mathbf{T} = \mathbf{f}, \quad \mathbf{T} = -p\mathbf{I} + 2\mu(p, |\mathbf{D}|)\mathbf{D}, \quad \operatorname{tr}\mathbf{D} = 0$$

- implicit constitutive relation

$$p = -\tfrac{1}{3}\operatorname{tr}\mathbf{T}, \quad \mathbf{T} - \tfrac{1}{3}\operatorname{tr}\mathbf{T} = 2\mu(\tfrac{1}{3}\operatorname{tr}\mathbf{T}, \operatorname{tr}\mathbf{D}^2)\mathbf{D}$$

Rajagopal, *On implicit constitutive theories*, 2003

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Rajagopal, *On implicit constitutive theories*, 2003

- for $|\partial\mathbf{S}/\partial p| > 1$ the elliptic structure is lost (both theory and numerics fails)
- for a subclass (with shear-thinning and with $|\partial\mathbf{S}/\partial p| < 1$), see

Nečas, Málek, Bulíček, Hron, ...,
Lanzendörfer, Stebel, 2009, 2011, 2011

Hirn, Lanzendörfer, Stebel, *Finite element approximation of flow of fluids...*, 2012

Weak formulation

$$(q, \operatorname{div} \boldsymbol{w})_\Omega = 0$$

$$(\boldsymbol{S}(p, \boldsymbol{Dv}), \boldsymbol{Dw})_\Omega - (p, \operatorname{div} \boldsymbol{w})_\Omega = 0, \qquad \boldsymbol{S} = 2\mu(p, |\boldsymbol{Dv}|)\boldsymbol{Dv}$$

Inf–sup inequality and the boundedness of $\partial_p \boldsymbol{S}$

$$0 < \beta \le \inf_{q \in \mathrm{L}^2_{b.c.}(\Omega)} \ \sup_{\boldsymbol{w} \in \mathbf{W}^{1,2}_{b.c.}(\Omega)} \frac{(q, \operatorname{div} \boldsymbol{w})_\Omega}{\|q\|_2 \|\boldsymbol{w}\|_{1,2}}$$

Pressure uniquely determined by velocity?

$$\beta \, \|p^1 - p^2\|_2 \quad \le \quad \|\boldsymbol{S}(p^1, \boldsymbol{Dv}) - \boldsymbol{S}(p^2, \boldsymbol{Dv})\|_2 \le \left\| \int_{p^1}^{p^2} \frac{\partial \boldsymbol{S}(p, \boldsymbol{Dv})}{\partial p} \mathrm{d}p \right\|_2$$

$$\le \quad \gamma_0 \, \|p^1 - p^2\|_2 \qquad \text{provided} \quad \left| \frac{\partial \boldsymbol{S}}{\partial p} \right| \le \gamma_0$$

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    - cavitation (free boundary problem, vaporization in negative pressures)
    - starvation and replenishment (free boundary, fluid–solid interface surface tension)
    - surface roughness, wall slip
    - complex non-newtonian rheologies
    - complex geometries, porous surfaces, etc.

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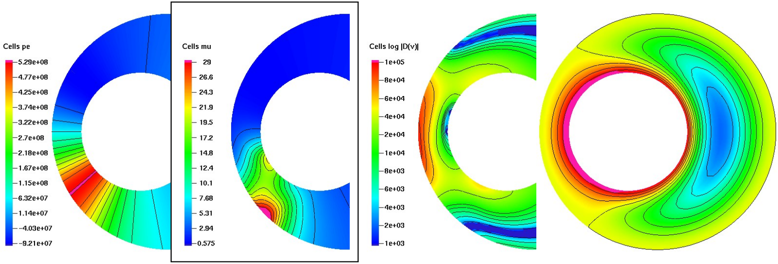
Almqvist, Larsson, *Navier-Stokes approach for thermal EHL line contact solution*, 2002
Almqvist, Almqvist, Larsson, *Comparison between CFD and Reynolds approaches for...*, 2004
Almqvist, Larsson, *Thermal transient rough EHL line contact simulations by aid of CFD*, 2008
Hartinger, Dumont, Ioannides, Gosman, Spikes, *CFD modeling of a thermal...*, 2008

Bruyere, Fillot, Morales–Espejel, Vergne, *CFD and full elasticity model...*, 2012
Knauf, Frei, Richter, Rannacher, *Towards a complete numerical description...*, 2013

- FEM, quadrilaterals, biquadratic velocities, linear discontinuous pressures
- quazi–Newton (FD Jacobian, line search), sparse LU (UMFPACK)



Validation of Reynolds approximation for high pressures

- remains unclear
- CFD approach not succesful (elliptic structure lost for $\mathrm{d}\mu/\mathrm{d}p >> 0$, numerical stability lost even earlier for EHL)
- even the well-posedness of non-reduced equations not clear at all
- Reynolds lubrication assumptions not satisfied, *but...*

...for low to moderade pressures (loads), the CFD approach provides

- access to more complex situations
- a tool for further study (validation) of Reynolds approach
- numerical challenges even in the most simple settings

Conclusions

Thank you for your attention