Homogenization of a non-Newtonian flow through a porous medium

Martin Kalousek

Charles University in Prague

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Homogenization in general

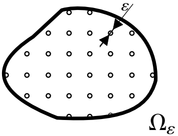
- operator with rapidly oscillating variables

$$-\operatorname{div} A\left(\frac{x}{\varepsilon}, u(x), \nabla u(x)\right) = g(x) \text{ in } \Omega$$

A periodic in first variable

- equations on a domain with a shrinking microstructure

$$-\operatorname{div}(A(x, u(x), \nabla u(x)) = g(x) \text{ in } \Omega_\varepsilon$$



The aim is to establish equation without the dependence on the microstructure whose solution is a good approximation of the solution of an initial problem.

System of interest

ε ratio of the microscopic length and the characteristic length of the porous medium

$$\begin{aligned} -\varepsilon \operatorname{div} T(\varepsilon D\mathbf{u}^\varepsilon) + \nabla p^\varepsilon &= \mathbf{f} \text{ in } \Omega_\varepsilon \\ \operatorname{div} \mathbf{u}^\varepsilon &= 0 \text{ in } \Omega_\varepsilon \\ \mathbf{u}^\varepsilon &= 0 \text{ on } \partial\Omega_\varepsilon \end{aligned} \qquad (\mathrm{GS}_\varepsilon)$$

Here $T : \mathbb{R}^{d\times d}_{sym} \to \mathbb{R}^{d\times d}_{sym}$ is nonlinear.

Generalization of the power law

well-known power law $T(\xi) = |\xi|^{p-2}\xi$ ($p \neq 2$ non-Newtonian fluid)
generalized power law $T(\xi) = \frac{\varphi'(|\xi|)\xi}{|\xi|}$

$\varphi : [0, \infty) \to [0, \infty)$ is N-function if $\exists \varphi'$ such that

1. φ' is (right)continuous, non-decreasing,
2. $\varphi'(0) = 0$,
3. $\varphi'(t) > 0$ for $t > 0$.

Δ_2-condition $\exists c > 0\ \forall t > 0 : \varphi(2t) \leq c\varphi(t)$
examples:
$\varphi(t) = \frac{t^p}{p}$ $p > 0$, $\varphi(t) = \frac{t^2}{\log(t+e)}$, $\varphi(t) = (t+1)\log(t+1) - t$

Sobolev-Orlicz spaces

Let $\Omega \subset \mathbb{R}^d$ be open.
Orlicz space

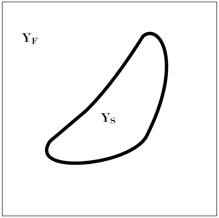
$$L^\varphi(\Omega) = \{u \in L^1_{loc}(\Omega), \int_\Omega \varphi(|u|) < \infty\}$$
$$\|u\|_\varphi = \inf \left\{ \lambda > 0; \int_\Omega \varphi\left(\frac{|u|}{\lambda}\right) \leq 1 \right\}$$

Sobolev-Orlicz space

$$W^{1,\varphi}(\Omega) = \{u \in L^\varphi(\Omega) \colon \nabla u \in L^\varphi(\Omega)\}$$
$$\|u\|_{1,\varphi} = \|u\|_\varphi + \|\nabla u\|_\varphi$$
$$W^{1,\varphi}_{0(,\mathrm{div})}(\Omega) = \overline{C^\infty_{0(,\mathrm{div})}(\Omega)}^{\|\cdot\|_{1,\varphi}}$$

Geometry of a porous medium

$Y = (0,1)^d, \ d = 2, 3$
Y_S solid part of Y, $Y_S \in C^2$
Y_F fluid part of Y



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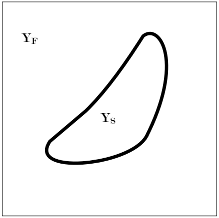

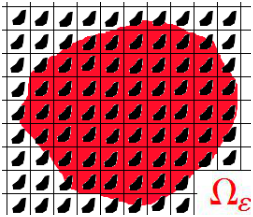

a periodic repetition of
$Y_k^\varepsilon = \varepsilon(Y + k),\ k \in \mathbb{Z}^d$

fluid part of a porous medium $\Omega \in C^{0,1}$
$\Omega_\varepsilon = \Omega \setminus \cup_{k \in \mathbb{Z}^d} Y_S^\varepsilon$

Known results

Tartar '80	derivation of Darcy's law $\mathbf{v} = K(\mathbf{f} - \nabla p)$ via homogenization of Stokes problem
Nguetseng '89	introduction and using of 2-s convergence
Allaire '91	obstacle size affects the form of the homogenized system
Allaire '92	homogenization of unsteady Stokes system
Bourgeat, Mikelić '96	homogenization of stationary p-NS system, the convective term vanishes
Nnang, Tachago '13	2-s convergence in Orlicz setting

The homogenized system

Theorem

Let $\mathbf{u}^0, P, \pi$ *be a unique weak solution of*

$$-\operatorname{div}_y T(|D_y\mathbf{u}^0(x,y)|)+\nabla_y\pi(x,y)=\mathbf{f}(x)-\nabla_x p(x) \ \textit{in}\ \Omega\times Y_F$$

$$\operatorname{div}_y \mathbf{u}^0 = 0 \ \textit{in}\ \Omega\times Y,\ \operatorname{div}_x \int_Y \mathbf{u}^0 = 0 \ \textit{in}\ \Omega$$

$$\mathbf{u}^0 = 0 \ \textit{in}\ \Omega\times Y_S,\ \int_Y \mathbf{u}^0 = 0 \ \textit{on}\ \partial\Omega.$$

Then as $\varepsilon\to 0$

$$\mathbf{u}^\varepsilon \xrightharpoonup{2-s} \mathbf{u}^0,\ \varepsilon D\mathbf{u}^\varepsilon \xrightharpoonup{2-s} D_y\mathbf{u}^0,\ \tilde{p}^\varepsilon(\textit{extended}\ p^\varepsilon) \xrightharpoonup{2-s} p.$$

Two-scale convergence I

Definition

We say that a sequence $\{v^\varepsilon\} \subset L^\varphi(\Omega)$ converges in $L^\varphi(\Omega)$

1. weakly two-scale to some $v^0 \in L^\varphi(\Omega \times Y)(v^\varepsilon \xrightarrow{2-s} v^0)$ if for any $w \in L^{\varphi^*}(\Omega; C_{per}(Y))$ $(w_\varepsilon(x) := w\left(x, \frac{x}{\varepsilon}\right))$

$$\lim_{\varepsilon \to 0} \int_\Omega v^\varepsilon(x) w_\varepsilon(x)\, \mathrm{d}x = \int_\Omega \int_Y v^0(x,y) w(x,y)\, \mathrm{d}y\, \mathrm{d}s.$$

2. strongly two-scale to some $v^0 \in L^\varphi(\Omega \times Y)(v^\varepsilon \xrightarrow{2-s} v^0)$ if for any $\kappa > 0$ and $w \in L^\varphi(\Omega; C_{per}(Y))$ with $\|v^0 - w\|_{L^\varphi}(\Omega \times Y) \le \frac{\kappa}{2}$ there exists $\alpha > 0$ such that for any $\varepsilon \in (0, \alpha)$ $\|v^\varepsilon - w_\varepsilon\|_{L^\varphi(\Omega)} \le \kappa$

Two-scale convergence II

Theorem

From any bounded sequence in $L^{\varphi}(\Omega)$ one can extract a weakly two-scale convergent subsequence in $L^{\varphi}(\Omega)$.

Lemma

Let $\{v_\varepsilon\}, \{\varepsilon \nabla v_\varepsilon\}$ *be bounded in* $L^{\varphi}(\Omega) \Rightarrow \exists v \in L^{\varphi}(\Omega; W^{1,\varphi}_{per}(Y))$, $\{v_{\varepsilon_k}\} : v_\varepsilon \xrightarrow{2-s} v, \ \varepsilon_k \nabla_x v_{\varepsilon_k} \xrightarrow{2-s} \nabla_y v.$

Lemma

Let $v^\varepsilon \xrightarrow{2-s} v^0$ *in* $L^{\varphi}(\Omega)$ *then* $\liminf_{\varepsilon \to 0} \|v^\varepsilon\|_{L^{\varphi}(\Omega)} \geq \|v^0\|_{L^{\varphi}(\Omega \times Y)}.$

Weak solution

Let $\mathbf{f} \in L^{\varphi^*}(\Omega)$. $\mathbf{u}^\varepsilon \in W^{1,\varphi}_{0,\mathrm{div}}(\Omega)$ is a weak solution of $(\mathrm{GS}_\varepsilon)$ if $\forall \mathbf{v} \in W^{1,\varphi}_{0,\mathrm{div}}(\Omega_\varepsilon)$

$$\varepsilon \int_{\Omega_\varepsilon} T(\varepsilon D\mathbf{u}^\varepsilon) D\mathbf{v} = \int_{\Omega_\varepsilon} \mathbf{f}\mathbf{v}$$

Apriori estimates

$$\exists c_1, c_2 > 0 \ \forall \varepsilon : \int_{\Omega_\varepsilon} \varphi(|\varepsilon D\mathbf{u}^\varepsilon|) \leq c_1 \int_{\Omega_\varepsilon} \varphi^*(|\mathbf{f}|)$$

$$\int_{\Omega_\varepsilon} \varphi^*\left(T(\varepsilon D\mathbf{u}^\varepsilon)\right) \leq c_2 \int_{\Omega_\varepsilon} \varphi^*(|\mathbf{f}|).$$

Extensions I

- $\mathbf{u}^\varepsilon = 0$ on $\partial\Omega_\varepsilon \Rightarrow \mathbf{u}^\varepsilon$ extended by zero in $\Omega \setminus \Omega_\varepsilon$ is bounded uniformly with respect to ε in $W^{1,\varphi}_{0,\mathrm{div}}(\Omega)$
- Extension of p^ε is not obvious. We want this extension to be bounded in $L^{\varphi^*}(\Omega)$!

Extensions I

- $\mathbf{u}^\varepsilon = 0$ on $\partial\Omega_\varepsilon \Rightarrow \mathbf{u}^\varepsilon$ extended by zero in $\Omega \setminus \Omega_\varepsilon$ is bounded uniformly with respect to ε in $W^{1,\varphi}_{0,\mathrm{div}}(\Omega)$
- Extension of p^ε is not obvious. We want this extension to be bounded in $L^{\varphi^*}(\Omega)$!
  Extension solved by L. Tartar who introduced and applied the restriction operator.

Restriction operator

Lemma

There exists a restriction operator $R_\varepsilon : W_0^{1,\varphi}(\Omega) \to W_0^{1,\varphi}(\Omega_\varepsilon)$ *with properties:*

R_ε *is linear*

$R_\varepsilon(\mathbf{w}) = \mathbf{w}$ *for* $\mathbf{w} \in W_0^{1,\varphi}(\Omega_\varepsilon)$ *extended by* 0 *on* $\Omega \setminus \Omega_\varepsilon$

$\operatorname{div} \mathbf{w} = 0$ *in* $\Omega \Rightarrow \operatorname{div} R_\varepsilon(\mathbf{w}) = 0$ *in* Ω_ε

$$\|R_\varepsilon(\mathbf{w})\|_{\varphi;\Omega_\varepsilon} \leq c\,(\|\mathbf{w}\|_{\varphi;\Omega} + \varepsilon\|\nabla \mathbf{w}\|_{\varphi;\Omega})$$

$$\|\nabla R_\varepsilon(\mathbf{w})\|_{\varphi;\Omega_\varepsilon} \leq c\left(\frac{1}{\varepsilon}\|\mathbf{w}\|_{\varphi;\Omega} + \|\nabla \mathbf{w}\|_{\varphi;\Omega}\right)$$

Extensions II

Define $G_\varepsilon \in (W_0^{1,\varphi}(\Omega))^*$

$$\langle G_\varepsilon, \mathbf{v} \rangle_{(W_0^{1,\varphi}(\Omega))^*, W_0^{1,\varphi}(\Omega)} = \langle \nabla p^\varepsilon, R_\varepsilon \mathbf{v} \rangle_{(W_0^{1,\varphi}(\Omega_\varepsilon))^*, W_0^{1,\varphi}(\Omega_\varepsilon)} .$$

Extensions II

Define $G_\varepsilon \in (W_0^{1,\varphi}(\Omega))^*$

$$\langle G_\varepsilon, \mathbf{v}\rangle_{(W_0^{1,\varphi}(\Omega))^*, W_0^{1,\varphi}(\Omega)} = \langle \nabla p^\varepsilon, R_\varepsilon \mathbf{v}\rangle_{(W_0^{1,\varphi}(\Omega_\varepsilon))^*, W_0^{1,\varphi}(\Omega_\varepsilon)} .$$

There is $\tilde{p}^\varepsilon \in L^\varphi(\Omega)$ such that $G_\varepsilon = \nabla \tilde{p}^\varepsilon$ and

- $\tilde{p}^\varepsilon = p^\varepsilon$ in Ω_ε
- $\tilde{p}^\varepsilon = \frac{1}{|Y_{F_i}^\varepsilon|} \int_{Y_{F_i}^\varepsilon} p^\varepsilon$ in each $Y_{S_i}^\varepsilon$
- $\|\tilde{p}^\varepsilon\|_{L^{\varphi^*}} \leq c$

Sketch of the proof

After applying extensions

$$\varepsilon\int_\Omega T(\varepsilon D\mathbf{u}^\varepsilon)D\mathbf{v}-\int_\Omega \tilde{p}^\varepsilon\,\mathrm{div}\,\mathbf{v}=\int_\Omega \mathbf{f}\mathbf{v}\ \forall\mathbf{v}\in W_0^{1,\varphi}(\Omega).$$

Sketch of the proof

After applying extensions

$$\varepsilon \int_\Omega T(\varepsilon D\mathbf{u}^\varepsilon) D\mathbf{v} - \int_\Omega \tilde{p}^\varepsilon \operatorname{div} \mathbf{v} = \int_\Omega \mathbf{f}\mathbf{v} \ \forall \mathbf{v} \in W_0^{1,\varphi}(\Omega).$$

Choose $\mathbf{w} \in \mathcal{D}(\Omega; C^\infty_{per}(Y))$, $\operatorname{div}_y \mathbf{w} = 0$ in Ω, put
$\mathbf{v}(x) := \mathbf{w}\left(x, \frac{x}{\varepsilon}\right)$ $\varepsilon \to 0$ and apply

$$\mathbf{u}^\varepsilon \xrightarrow{2-s} \mathbf{u}^0 \text{ in } L^\varphi(\Omega)$$

$$\varepsilon D\mathbf{u}^\varepsilon \xrightarrow{2-s} D_y\mathbf{u}^0 \text{ in } L^\varphi(\Omega)$$

$$T(\varepsilon D\mathbf{u}^\varepsilon) \xrightarrow{2-s} T^0 \text{ in } L^{\varphi^*}(\Omega)$$

$$\tilde{p}^\varepsilon \rightharpoonup p \text{ in } L^{\varphi^*}(\Omega)$$

to obtain

$$\int_\Omega \int_Y T^0 D_y \mathbf{w} - \int_\Omega \int_Y p \operatorname{div} \mathbf{w} = \int_\Omega \int_Y \mathbf{f}\mathbf{w}.$$

Sketch of the proof

After applying extensions

$$\varepsilon \int_\Omega T(\varepsilon D\mathbf{u}^\varepsilon) D\mathbf{v} - \int_\Omega \tilde{p}^\varepsilon \operatorname{div} \mathbf{v} = \int_\Omega \mathbf{f}\mathbf{v} \ \forall \mathbf{v} \in W_0^{1,\varphi}(\Omega).$$

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$\mathbf{v}(x) := \mathbf{w}\left(x, \frac{x}{\varepsilon}\right)$ $\varepsilon \to 0$ and apply

$$\mathbf{u}^\varepsilon \xrightharpoonup{2-s} \mathbf{u}^0 \text{ in } L^\varphi(\Omega)$$

$$\varepsilon D\mathbf{u}^\varepsilon \xrightharpoonup{2-s} D_y \mathbf{u}^0 \text{ in } L^\varphi(\Omega)$$

$$T(\varepsilon D\mathbf{u}^\varepsilon) \xrightharpoonup{2-s} T^0 \text{ in } L^{\varphi^*}(\Omega)$$

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to obtain

$$\int_\Omega \int_Y T^0 D_y \mathbf{w} - \int_\Omega \int_Y p \operatorname{div} \mathbf{w} = \int_\Omega \int_Y \mathbf{f}\mathbf{w}.$$

Minty's argument: $T^0(x,y) = T(D_y \mathbf{u}^0) = \varphi'(|D_y \mathbf{u}^0|) \frac{D_y \mathbf{u}^0}{|D_y \mathbf{u}^0|}$

Nonstationary Stokes

$$
\begin{aligned}
\mathbf{u}^{\varepsilon}_t - \varepsilon \operatorname{div} T(\varepsilon D\mathbf{u}^{\varepsilon}) + \nabla p^{\varepsilon} &= \mathbf{f} \text{ in } (0, T) \times \Omega_{\varepsilon} \\
\operatorname{div} \mathbf{u}^{\varepsilon} &= 0 \text{ in } (0, T) \times \Omega_{\varepsilon} \\
\mathbf{u}^{\varepsilon}(0) &= \mathbf{a}^{\varepsilon} \text{ in } \Omega_{\varepsilon} \\
\mathbf{u}^{\varepsilon} &= 0 \text{ on } \partial\Omega_{\varepsilon}
\end{aligned}
$$

Nonstationary Stokes

$$
\begin{aligned}
\mathbf{u}_t^\varepsilon - \varepsilon \operatorname{div} T(\varepsilon D\mathbf{u}^\varepsilon) + \nabla p^\varepsilon &= \mathbf{f} \text{ in } (0,T)\times\Omega_\varepsilon \\
\operatorname{div}\mathbf{u}^\varepsilon &= 0 \text{ in } (0,T)\times\Omega_\varepsilon \\
\mathbf{u}^\varepsilon(0) &= \mathbf{a}^\varepsilon \text{ in } \Omega_\varepsilon \\
\mathbf{u}^\varepsilon &= 0 \text{ on } \partial\Omega_\varepsilon
\end{aligned}
$$

assumptions

1. a restriction on $N-$function which allows $L^\varphi((0,T)\times Y) = L^\varphi(I; L^\varphi(\Omega))$
2. an embedding $L^\varphi(\Omega) \hookrightarrow L^2(\Omega)$
3. more regular uniformly bounded data $\Rightarrow \|\mathbf{u}_t^\varepsilon\|_{L^2((0,T)\times\Omega)} \le c$
   consequences
   - $\mathbf{u}_t^\varepsilon \xrightarrow{2-s} \mathbf{u}_t^0$ in $L^2((0,T)\times\Omega)$
   - $\|\mathbf{u}_t^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \le c \Rightarrow \|P^\varepsilon\|_{L^{\varphi^*}((0,T)\times\Omega)} \le c$

Homogenized nonstationary NS

$$\mathbf{u}^0_t - \mathrm{div}_y\, T(D_y \mathbf{u}^0) + \nabla_y \pi = \mathbf{f} - \nabla_x p \text{ in } (0,T) \times \Omega \times Y_F$$

$$\mathbf{u}^0(0) = \mathbf{a}^0 \text{ in } \Omega \times Y$$

$$\mathrm{div}_y\, \mathbf{u}^0 = 0 \text{ in } (0,T) \times \Omega \times Y$$

$$\mathrm{div}_x \int_Y \mathbf{u}^0 = 0 \text{ in } (0,T) \times \Omega$$

$$\mathbf{u}^0 = 0 \text{ in } (0,T) \times \Omega \times Y_S$$

Homogenized nonstationary NS

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$$

Thank you for attention!