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Analysis of strain-limiting models in solid mechanics

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The talk is based on the following results

- M. Bulíček, J. Málek, K. R. Rajagopal and J. R. Walton: **Existence of solutions for the anti-plane stress for a new class of "strain-limiting" elastic bodies**, submitted
- M. Bulíček, J. Málek and E. Süli: **Analysis and approximation of a strain-limiting nonlinear elastic model**, Mathematics and Mechanics of Solids, 2014
- M. Bulíček, J. Málek, K. R. Rajagopal and E. Süli: **On elastic solids with limiting small strain: modelling and analysis**, EMS Surveys in Mathematical Sciences, 2014.
- L. Beck, M. Bulíček, J. Málek and E. Süli: **Analysis and approximation of a strain-limiting nonlinear elastic model II**, in preparation

Linearized nonlinear elasticity

We consider the elastic deformation of the body $\Omega \subset \mathbb{R}^d$ with $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\overline{\Gamma_1 \cup \Gamma_2} = \partial\Omega$ described by

$$\begin{aligned} -\operatorname{div} \mathbf{T} &= \mathbf{f} \qquad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{u}_0 \qquad \text{on } \Gamma_1, \qquad \text{and} \qquad \mathbf{T}\mathbf{n} = \mathbf{g} \quad \text{on } \Gamma_2. \end{aligned} \tag{El}$$

where $\mathbf{u}$ is displacement, $\mathbf{T}$ the Cauchy stress, $\mathbf{f}$ the external body forces, $\mathbf{g}$ the external surface forces and $\boldsymbol{\varepsilon}$ is the linearized strain tensor, i.e.,

$$\boldsymbol{\varepsilon} := \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$$

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$$\boxed{\mathbf{G}(\mathbf{T}, \boldsymbol{\varepsilon}) = \mathbf{0}}$$

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☞ The key assumption in linearized elasticity

$$\boxed{|\varepsilon| \ll 1}. \tag{A}$$

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- ✌ But there is material behavior that suggests

$$\boxed{|\mathbf{T}(x)| \overset{x \to x_0}{\to} \infty \qquad \text{BUT} \qquad |\varepsilon(x)| \ll 1.}$$

Limiting strain model

- Consider implicit models which a priori guarantees $|\varepsilon| \le K$:

$$\boxed{\varepsilon = \lambda_1(|\operatorname{tr}\mathbf{T}|)(\operatorname{tr}\mathbf{T})\mathbf{I} + \lambda_2(|\mathbf{T}|)\mathbf{T} + \lambda_3(|\mathbf{T}^d|)\mathbf{T}^d}, \qquad \text{(L-S)}$$

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$$\int_\Omega \lambda_1(|\operatorname{tr}\mathbf{T}|)|\operatorname{tr}\mathbf{T}|^2 + \lambda_2(|\mathbf{T}|)|\mathbf{T}|^2 + \lambda_3(|\mathbf{T}^d|)|\mathbf{T}^d|^2 = \int_\Omega \mathbf{T}\cdot\varepsilon \le C.$$

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- The reasonable assumptions (∞-Laplacian-like problem):

$$\left.\begin{aligned} \lambda_{1,2,3}(s) &\ge 0, \\ \lambda_3(s) &\ge \frac{\alpha}{s+1}. \end{aligned}\right\} \implies \int_\Omega |\mathbf{T}^d| \le C.$$

- Apriori estimates for $\mathbf{T}^d$ in L^1
- For the convergence at least some monotonicity needed, the minimal assumption:
$$0 \leq \frac{d}{ds}(\lambda_{1,2,3}(s)s). \tag{M}$$
- If we would have an approximative sequence fulfilling
$$\int_{\Omega_0} |(\mathbf{T}^d)^n|^{1+\delta} \leq C(\Omega_0) \qquad \text{for all } \Omega_0 \subset\subset \Omega,$$
$$\implies \mathbf{T}^n \rightharpoonup \mathbf{T} \qquad \text{weakly in } L^1_{loc}.$$

then using (M) we can identify the limit.

- Assume kind of uniform monotonicity, i.e., for some $\alpha, a, K > 0$
$$\frac{\alpha}{(K+s)^a} \leq \frac{d}{dt}(\lambda_3(s)s) \tag{UM}$$

for example
$$\boxed{\lambda_3(s) := \frac{1}{(1+s^a)^{\frac{1}{a}}}}.$$

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Existence via the dual formulation (very similar to plasticity):
Find the (convex) potential $F : \mathbb{R}^{d\times d} \to \mathbb{R}_+$ such that

$$\frac{\partial F(\mathbf{T})}{\partial \mathbf{T}_{ij}} = \frac{\mathbf{T}_{ij}}{(1+|\mathbf{T}|^a)^{\frac{1}{a}}} \; (= \varepsilon_{ij})$$

and define the class of admissible stresses as

$$\mathcal{S} := \{\mathbf{T} \in L^1(\Omega);\ -\operatorname{div}\mathbf{T} = \mathbf{f},\ \mathbf{T}\mathbf{n} = \mathbf{g} \text{ on } \Gamma_1\}.$$

To find a weak solution to the original problem is equivalent to find $\mathbf{T} \in \mathcal{S}$ fulfilling

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Theorem

Let $\Omega \subset \mathbb{R}^d$. There exists a minimizer $\mathbf{T}$ to the potential F, but in the space of measures.

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- Can we do something better "inside" Ω?
- Is there any chance to avoid measures completely and to solve the original problem?

We consider the following special geometry

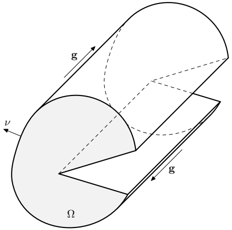

Figure: Anti-plane stress geometry.

and we look for the solution in the following from:

$$\mathbf{u} = \mathbf{u}(x_1, x_2) = (0, 0, u(x_1, x_2)), \quad \mathbf{g}(x) = (0, 0, g(x_1, x_2)),$$

and

$$\mathbf{T}(x) = \begin{pmatrix} 0 & 0 & T_{13}(x_1, x_2) \\ 0 & 0 & T_{23}(x_1, x_2) \\ T_{13}(x_1, x_2) & T_{23}(x_1, x_2) & 0 \end{pmatrix}.$$

Equivalent reformulation

☞ Find $U : \Omega \to \mathbb{R}$ - the Airy stress function such that

$$T_{13} = \frac{1}{\sqrt{2}} U_{x_2} \quad \text{and} \quad T_{23} = -\frac{1}{\sqrt{2}} U_{x_1}.$$

$\Longrightarrow \operatorname{div} \mathbf{T} = \mathbf{0}$ is fulfilled.

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- ☞ U must satisfy ($\varepsilon = \frac{\mathbf{T}}{(1+|\mathbf{T}|^a)^{\frac{1}{a}}}$)

$$\operatorname{div}\left(\frac{\nabla U}{(1+|\nabla U|^a)^{\frac{1}{a}}}\right) = 0 \qquad \text{in } \Omega,$$

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- Dirichlet problem, indeed assume that $\partial\Omega$ is parametrized by $\gamma(s) = (\gamma_1(s), \gamma_2(s))$. Then

$$U(\gamma(s_0)) = a_0 + \sqrt{2}\int_0^{s_0} g(\gamma(s))\sqrt{(\gamma_1'(s))^2 + (\gamma_2'(s))^2}\,ds =: U_0(x).$$

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- ✌ $a \leq 2$ we can localize and prove $\nabla U \in L^\infty_{loc}$
- ✖ $a \in (1, 2)$ the weak solution may not exists eg. for $\Omega = B_2 \setminus B_1$
- ✌ on the flat part of the boundary, one can extend the solution outside

Theorem (anti-plane stress)

Let U_0 be arbitrary. Then there exists unique weak solution U provided that one of the following holds:

- *Ω is uniformly convex, $a > 0$ is arbitrary and U_0 smooth.*
- *$a \in (0, 2)$ and $\partial\Omega = \bigcup_{i=1}^{N} \Gamma_i$ such that either Γ_i is uniformly convex and U_0 is smooth on Γ_i or Γ_i is flat and U_0 is constant there.*

Limiting strain - Reformulation

F1 Find the weak solution

F2 Find $U \in W^{1,1}(\Omega)$ being equal to U_0 on $\partial\Omega$ such that

$$\int_\Omega F(\nabla U) \le \int_\Omega F(\nabla V) \qquad \text{for all } (V - U_0) \in W_0^{1,1}(\Omega).$$

F3 Find $U \in W^{1,1}(\Omega)$ such that

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Theorem (anti-plane stress II)

Let $a \in (0,2]$, U_0 and $\Omega \subset \mathbb{R}^d$ be arbitrary. Then there exists unique weak solution $U \in W^{1,1}(\Omega)$ in the following sense

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Defining $\mathbf{T}_{13} := U_{x_2}$ and $\mathbf{T}_{23} := -U_{x_1}$ we have $\operatorname{div} \mathbf{T} = 0$ but $\mathbf{Tn} = \mathbf{g}$ is not attained but we have “the best approximation".

Consequences for solution in general case

- Bildhauer & Fuchs (2001–): General theory for 1-like Laplacian for $a \in (0, 2]$ - i.e., smoothness locally in Ω, the trace may not be attained; for convex domains everything is nice up to the boundary

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- We cannot solve the problem in general for the Neumann data - counterexamples
- Maybe we can avoid to be $\mathbf{T}$ measure in the interior of Ω

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- Maybe the Dirichlet problem is easier to handle - we do not need the estimates up to the boundary
- But in all cases we need to face the problem with symmetric gradient contrary to the full gradient as in Bildhauer & Fuchs
- Is really the assumption $a \leq 2$ essential? Counterexamples only for non-smooth data

Theorem (Dirichlet data)

Let $\Omega \subset \mathbb{R}^d$, $\lambda_{1,2}$ fulfil (M) *and λ_3 satisfy* (UM) *with $a < \frac{1}{d}$. Then there exists a weak solution $(\mathbf{T}, \mathbf{u})$. Moreover, $\mathbf{u}$ is unique. Further, if either λ_1 or λ_2 are strictly monotone then also $\mathbf{T}$ is unique.*

- ☞ Proper approximation (p-Laplacian)
- ☞ Uniform L^1 estimates
- ☞ Uniform $L^{1+\delta}_{loc}$ estimates by showing that $\mathbf{T} \in \mathcal{N}^{\alpha,1}$ for some $\alpha \in (0,1)$.

Theorem (Periodic data)

Let $\lambda_{1,2}$ fulfil (M) *and λ_3 satisfy* (UM) *with $a < \frac{2}{d}$. Then there exists a weak solution $(\mathbf{T}, \mathbf{u})$. Moreover, $\mathbf{u}$ is unique. Further, if either λ_1 or λ_2 are strictly monotone then also $\mathbf{T}$ is unique.*

- ☞ The same as before but no problem with localization $\Longrightarrow$ better bound for a

Theorem (Periodic data II)

Let $\lambda_{1,2}$ *fulfil* (M) *and* λ_3 *satisfy* (UM) *with* $a > 0$. *Then there exists a* $(\mathbf{T}, \mathbf{u})$ *fulfilling the implicit relation a.e. such that*

$$\mathbf{T} \in L^1, \qquad \varepsilon \in L^\infty, \qquad \frac{\nabla \mathbf{T}}{(1 + |\mathbf{T}|)^{\frac{a+1}{2}}} \in L^2$$

the energy inequality holds, i.e.,

$$\int_\Omega \mathbf{T} \cdot \varepsilon(\mathbf{u}) \le \int_\Omega \mathbf{f} \cdot \mathbf{u},$$

and fulfill the renormalized equation, i.e., for all smooth periodic $\mathbf{v}$ *and all* $g \in \mathcal{D}(\mathbb{R})$ *there holds*

$$\int_\Omega \mathbf{T} \cdot (g(|\mathbf{T}|)\nabla \mathbf{v} + \mathbf{v} \otimes \nabla g(|\mathbf{T}|)) = \int_\Omega g(|\mathbf{T}|)\mathbf{f} \cdot \mathbf{v}.$$

Moreover, if $\mathbf{T} \in L^{a+1}$ *the the solution is weak.*

Theorem (Limiting strain - Dirichlet data)

Consider $\Omega \subset \mathbb{R}^d$ *with continuous boundary and smooth* $\mathbf{f}$*. Then for arbitrary* $a > 0$ *there exists unique* $\mathbf{T} \in L^1(\Omega; \mathbb{R}^{d\times d})$ *and* $\mathbf{u} \in W_0^{1,1}(\Omega; \mathbb{R}^d)$ *such that* $\varepsilon(\mathbf{u}) \in L^\infty(\Omega; \mathbb{R}^{d\times d})$ *solving for all* $\mathbf{v} \in \mathcal{D}(\Omega; \mathbb{R}^d)$

$$\int_\Omega \mathbf{T} \cdot \varepsilon(\mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v}, \qquad \varepsilon(\mathbf{u}) = \frac{\mathbf{T}}{(1 + |\mathbf{T}|^a)^{\frac{1}{a}}}.$$

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Theorem (Plasticity-like models - Neumann data)

Consider $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary and smooth $\mathbf{g}$. Moreover, assume that there exists $\mathbf{T}_0$ fulfilling $\mathbf{T}_0\mathbf{n} = \mathbf{g}$ on $\partial\Omega$ such that $\|\mathbf{T}_0\|_\infty < 1$ and $\operatorname{div} \mathbf{T}_0 = 0$ (necessary compatibility condition). Then for arbitrary $a > 0$ there exists unique $\mathbf{T} \in L^\infty(\Omega; \mathbb{R}^{d\times d})$ and $\mathbf{u} \in L_0^{d'}(\Omega; \mathbb{R}^d)$ such that $\varepsilon(\mathbf{u}) \in L^1(\Omega; \mathbb{R}^{d\times d})$ solving for all $\mathbf{v} \in \mathcal{C}^\infty(\Omega; \mathbb{R}^d)$

$$\int_\Omega \mathbf{T} \cdot \varepsilon(\mathbf{v}) = \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{v}, \qquad \mathbf{T} = \frac{\varepsilon}{(1 + |\varepsilon|^a)^{\frac{1}{a}}}.$$