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Convergence to equilibrium for solutions of an abstract wave equation with general damping function

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1 Another introduction

This work has been inspired by a result presented in Chergui [4]. In particular, Chergui has studied the following semilinear damped wave equation

$$u_{tt} + |u_t|^\alpha u_t - \Delta u = f(u,x). \tag{1}$$

He proved that every bounded solution is relatively compact and that every relatively compact solution converges to an equilibrium point for certain values of α, where the set of admissible α's depends on the Lojasiewicz exponent of the operator $\Delta + f(\cdot,x)$.

The main goal of this paper is to study the above equation with more general damping functions and obtain convergence to equilibrium for relatively compact solutions for a large class of damping functions. We will prove our result in a more general setting assuming an abstract operator $M(u)$ instead of $\Delta u - f(u,x)$.

$$u_{tt}(t) + g(|u_t(t)|)u_t(t) = M(u(t)), \quad t > 0, \tag{2}$$

2 Introduction

Let us denote $V := H^1_0(\Omega)$, $H := L^2(\Omega)$, $V' := H^{-1}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is open and bounded. Let $E \in C^2(V)$, $M := E' \in C^1(V,V')$ and $g : [0,+\infty) \to [0,+\infty)$. Consider the following problem

$$u_{tt} + g(|u_t|)u_t = M(u(t)), \quad t > 0 \tag{3}$$

with initial values

$$u(0) = u_0 \in V, \quad \dot{u}(0) = u_1 \in H.$$

Let us assume there exists a solution $u \in C^1(\mathbb{R}_+, H) \cap C(\mathbb{R}_+, V)$ such that $u_t^2 \cdot g(u_t) \in L^1(\mathbb{R}_+, L^1(\Omega))$ and assume that the trajectory $(u(t), u_t(t))_{t\geq 0}$ is relatively compact in $V \times H$.

Then there exists a sequence $t_n \to +\infty$ such that $(u(t_n), u_t(t_n))$ converge to $(\varphi, \psi) \in V \times H$ and one can show that $\psi = 0$ (see [4]). The question we are interested in is

$$\text{Is } \lim_{t \to +\infty} u(t) = \varphi? \tag{Q}$$

This problem was studied by Chergui in [4] for functions $g(z) = z^\alpha$, $\alpha \in (0, 1)$. The answer was positive (under suitable assumptions on M) if α satisfies the following two conditions:

(1) $0 < \alpha < \frac{\theta}{1-\theta}$, where θ is a 'Lojasiewicz exponent' depending on M and E,

(2) $\alpha < \frac{4}{N-2}$.

Condition (1) says that the damping term $g(|u_t|)u_t$ is not too small near zero (which seems to be reasonable condition). It also estimates the growth at infinity but it can be seen from the proof that we do not need this estimate. In any case, the decay at zero cannot be u_t^2 or faster. Condition (2) says that the growth of g at $+\infty$ is not too fast (a Sobolev imbedding is needed in the proof but it is not clear, whether bigger damping should destabilize the system) and also that the growth of g at zero is not too small (but we will show that this estimate at zero is not necessary). From physical interpretation we would say that the bigger is the damping term, the better will be the convergence or the stabilisation effect.

We give positive answer to the question (Q) for certain more general functions g. Let us first formulate the assumptions and prove the result with these assumptions (Section 2). In Section 3, we give some comments and examples of functions g and finally we reformulate the Theorem with more clear assumptions on the damping function g. In the last section, we mention some corollaries for ordinary second order equations.

3 Main result

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $H := L^2(\Omega)$, $V := H_0^1(\Omega)$ and V' be the dual space to V, $V \hookrightarrow H \hookrightarrow V'$. We will usually denote real numbers by s, r, vectors in $\mathbb{R}^N$ by z, w. Letters u, v will be used for members of V' (and its subspaces V, H) or functions of two variables, e.g. $u \in C(\mathbb{R}_+, H)$. If u is a function of $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$, we often write $u(t)$ instead of $u(t, \cdot)$.

By $|z|$ we denote the norm in $\mathbb{R}^N$ (or absolute value in $\mathbb{R}$). We will denote $\|\cdot\|$ the norm in H and $\|\cdot\|_*$ the norm in V' and similarly the scalar products $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_*$. By $B_V(\varphi, \varepsilon)$ we denote the open ball in V with radius ε and centered at φ. We denote by $K : V' \to V$ the duality mapping given by $\langle u, v \rangle_* = \langle u, Kv \rangle$, $u \in H$, $v \in V'$.

Let $p := \frac{2N}{N+2}$ for $N > 2$ and $p := 1$ for $N \leq 2$. In this way $L^p(\Omega)$ is embedded to V' since V is embedded to $L^q(\Omega)$ with $q \leq \frac{2N}{N-2}$. Since $p < 2$ and $p' = \frac{2N}{N-2} > 2$, we have $V \hookrightarrow L^{p'} \hookrightarrow H \hookrightarrow L^p \hookrightarrow V'$ for $N > 2$, and with $p = 1$, $p' = \infty$ for $N = 2$, too. Further $\langle u, v \rangle_{V',V} = \langle u, v \rangle_{L^p, L^{p'}} = \langle u, v \rangle_H$, $u \in H$, $v \in V$. For a function $\tilde{g} : \mathbb{R} \to \mathbb{R}$ we define $\tilde{g}_1(z) := z\tilde{g}(z)$ and $\tilde{g}_2(z) := z^2\tilde{g}(z)$ and similarly for h we define h_1, h_2. For the damping function $g : \mathbb{R} \to \mathbb{R}$ we define $g_1 : \mathbb{R}^N \to \mathbb{R}^N$ a $g_2 : \mathbb{R}^N \to \mathbb{R}$ by $g_1(z) := g(|z|)z$ and $g_2(z) := \langle g_1(z), z\rangle = g(|z|)|z|^2$.

We introduce our assumptions. Let us start with definition of a KL–function. We say that $\Theta : \mathbb{R}_+ \to \mathbb{R}_+$ is *a KL–function* if it is nondecreasing, sublinear ($\Theta(s+r) \le \Theta(s)+\Theta(r)$ for all $r, s \ge 0$) and satisfies $\Theta(s) > 0$ for all $s > 0$ and $\Theta(s) \le c\sqrt{s}$ for some $c > 0$ and all $s \in [0, \tau]$ (for some $\tau > 0$).

Remark 3.1. *(i) Since the assumptions on Θ below ((e1) and (h2)) involve only arguments near zero, we could define a KL–function on a neighborhood of zero only (any such function can be extended to $\mathbb{R}_+$ such that it has the above properties on the whole $\mathbb{R}_+$.*

(ii) The sublinearity assumption could be weakened to $\Theta(s + r) \le C(\Theta(s) + \Theta(r))$ for some $C > 0$ and all $r, s \ge 0$.

Our assumptions on the operator E are following.

(E) Assume that $E \in C^2(V)$ and $M := E' \in C^1(V, V')$ satisfy:

(e1) there exists a KL–function Θ such that E satisfies the Kurdyka-Łojasiewicz gradient inequality with function Θ on a neighborhood of $N_M := \{\varphi \in V : \ M(\varphi) = 0\}$, i.e., for each $\varphi \in N_M$ there exists η, $C > 0$ such that

$$\|E'(u)\|_* \ge C\Theta(|E(u) - E(\varphi)|) \tag{4}$$

for all $u \in B_V(\varphi, \eta)$,

(e2) for all $u \in V$, $KM'(u)$ extends to a bounded operator on H and $\sup \|KM'(u)\|_{L(H)}$ is finite when u ranges over a compact subset of V.

Let us mention that Chergui ([4]) works with $M(u) = \Delta u - f(x, u)$ which corresponds to $E(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - F(x, u)\mathrm{d}x$, where $F(x, u) := \int_0^u f(x, s)\mathrm{d}s$. It is shown in [4] that (if f satisfies certain assumptions) this operator E satisfies Łojasiewicz gradient inequality

$$\|E'(u)\|_* \ge C(|E(u) - E(\varphi)|)^{1-\theta} \tag{5}$$

with some $\theta \in [0, 1/2)$ in a neighborhood of N_M. Łojasiewicz inequality (5) is a special case of Kurdyka–Łojasiewicz inequality (4) with a KL–function $\Theta(s) = s^{1-\theta}$. It is easy to see that Chergui's operator M satisfies (e2), as well. Conditions (e1), (e2) also appear in [5] (with (5) instead of (4)), where linear damping is considered.

Now, we introduce our assumption on the damping function.

(G) The function $g : [0, +\infty) \to \mathbb{R}_+$ is continuous on $(0, +\infty)$ and there exists $\tau > 0$ such that

(g1) there exist $C_2 > 0$ such that $g(s) \le C_2$ on $[0, \tau)$.

(g2) there exist $C_3 > 0$ such that $C_3 \le g(s)$ on $[\tau, +\infty)$.

(g3) if $N > 2$ then there exist $C_4 > 0$ such that $g(s) \le C_4 s^{4/(N-2)}$ on $[\tau, +\infty)$.

(H) For τ from condition (G) there exists a concave nondecreasing function $h : [0, \tau] \to \mathbb{R}_+$ with $h(0) \ge 0$ such that

(h1) $g \ge h$ on $[0, \tau]$

(h2) function $s \mapsto (\Theta(s)h(\Theta(s)))^{-1}$ belongs to $L^1([0, \tau])$

(h3) function $\psi(s) := h_2(\sqrt{s})$ is convex on $[0,\tau]$.

We can see that no monotonicity is needed, only some estimates from above and from below. Clearly, Chergui's damping function s^α, $\alpha < 4/(N-2)$ satisfies (G)and it also satisfies (H)with $h(s) := s^\alpha$, where our condition (h2) corresponds to Chergui's condition $\alpha \in [0, \theta/(1-\theta))$. This is the condition coupling the damping function g with the operator E.

We say that $u \in W^{1,1}_{loc}(\mathbb{R}_+, V) \cap W^{2,1}_{loc}(\mathbb{R}_+, H)$ is *a strong solution* to (3) if (3) holds in V' for a.e. $t > 0$. The *omega-limit set* of u is

$$\omega_V(u) = \{\varphi \in V : \exists\, t_n \nearrow +\infty,\ \|u(t_n) - \varphi\|_V = 0\}.$$

Theorem 3.2. *Let E and g satisfy* (E), (G)*and* (H). *Let u be a strong solution to* (3) *such that*

$$\bigcup_{t\geq 0}\{(u(t), \dot u(t))\} \text{ is precompact in } V \times H$$

and $\varphi \in \omega_V(u)$. Then $\lim_{t\to+\infty} \|u(t) - \varphi\|_V + \|\dot u(t)\| = 0$.

Remark 3.3. *Assumptions* (G)*and* (H)*say that no monotonicity of g is needed, we need only some estimates near zero and near infinity. Condition* (H)*, which estimates g from below on a neighborhood of zero, is more complicated than the others. In fact, this condition is trivial if $\liminf_{s\to 0+} g(s) > 0$, since then a small constant function h works* (h2) *is satisfied due to $\Theta(s) \leq c\sqrt{s}$).*

If $\liminf_{s\to 0+} g(s) = 0$, then necessarily $h(0) = 0$. Condition (h2) *says that the growth of h at zero must be steep enough. In fact, together with condition* (Θ) *($\Theta(s) \leq c\sqrt{s}$) we have that $h'_+(0) = +\infty$ and if $\lim_{s\to 0+} g(z) = 0$, then also $g'_+(0) = +\infty$. Assumption* (h3) *is satisfied e.g. if h is increasing and h_1 is convex (easy computations). Here the first condition (h increasing) follows from concavity of h and $h'_+(0) = +\infty$ (we can take τ smaller if neccessary). Finally, let us mention that every function*

$$h(s) := s^\alpha(\ln(1/s))^{\alpha_1}(\ln\ln(1/s))^{\alpha_2}\ldots(\ln\ldots\ln(1/s))^{\alpha_n}$$

with $\alpha \in (0,1)$, $n \in \mathbb{N}$, $\alpha_i \in \mathbb{R}$ satisfies condition (h3)*. This last assertion can be shown by computing the first derivative of h and the second derivative of h_1.*

4 An equivalent set of assumptions

In this section, we will introduce another set of assumptions ((G$_0$), ($\tilde{\mathrm{G}}$), (Γ)) and show that these assumptions are equivalent to assumptions (G), (H). These new assumptions are motivated by the proof of Theorem 1.4 in [4]. Reading that proof carefully and trying to minimize the assumptions needed lead us to this set and we will prove the assertion of Theorem 3.2 under these new assumptions. Here we show that the old assumptions imply the new ones. And we also show the opposite implication which says in some sense that these assumptions are the best possible if we want to use the method from [4]. Here are the new assumptions:

(G_0) there exists $c_3 > 0$ such that $g(s) \le c_3 s^{4/(N-2)}$ on a neigborhood of infinity (only if $N > 2$),
($\tilde{G}$) There exists $\tilde{g} : \mathbb{R}_+ \to \mathbb{R}_+$ positive on $(0, +\infty)$, such that
($\tilde{g}1$) there exists $c_1 > 0$ such that $g \ge c_1 \tilde{g}$ on $\mathbb{R}_+$,
($\tilde{g}2$) $\tilde{g}$ is concave on $\mathbb{R}_+$ and $\tilde{g}(0) \ge 0$,
($\tilde{g}3$) function ψ defined by $\psi(s) := \tilde{g}_2(\sqrt{s})$ is convex on $\mathbb{R}_+$,
($\tilde{g}4$) function $s \mapsto (\Theta(s)\tilde{g}(\Theta(s)))^{-1}$ belongs to $L^1([0,1])$.
(Γ) There exists a Young function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ (convex with $\gamma(0) = 0$, $\lim_{s\to+\infty} \gamma(s) = +\infty$) such that
($\gamma1$) there exists $d_1 > 0$ such that $\gamma(|g_1(z)|) \le d_1 g_2(z)$ on $\mathbb{R}^n$,
($\gamma2$) there exists $d_2 > 0$ such that $\gamma(s) \ge d_2 s^2$ on a neighborhood of zero,
($\gamma3$) the function $\tilde{\gamma}$ defined by $\tilde{\gamma}(s) := \gamma(s^{1/p})$ is convex on $\mathbb{R}_+$.
($\gamma4$) for every $K > 0$ there exists $C(K)$ such that for all $s > 0$ it holds that $\gamma(Ks) \le C(K)\gamma(s)$.

We say that function $f : \mathbb{R}_+ \to \mathbb{R}_+$ has *property K* if for every $K > 0$ there exists $C(K)$ such that for all $s > 0$ it holds that $f(Ks) \le C(K)f(s)$. So, ($\gamma4$) says that γ has property K. Typically, nondecreasing functions with polynomial growth do have this property, functions with exponential growth does not.

Lemma 4.1. *Condition* ($\tilde{G}$) *implies that*
($\tilde{g}5$) *$\tilde{g}$ is nondecreasing on $\mathbb{R}_+$,*
($\tilde{g}6$) *$s\tilde{g}'_{\pm}(s) \le \tilde{g}(s)$ on $\mathbb{R}_+$,*
($\tilde{g}7$) *$\tilde{g}$ has property K,*
($\tilde{g}8$) *ψ has property K.*

Proof. ($\tilde{g}5$), ($\tilde{g}6$) follow immediately from ($\tilde{g}2$) and positivity of $\tilde{g}$. ($\tilde{g}7$) holds with $C(K) = 1$ for $K \le 1$ since $\tilde{g}$ is nondecreasing and $C(K) = K$ for $K > 1$ since $\tilde{g}$ is concave and $\tilde{g}(s) \ge 0$. We show that ($\tilde{g}8$) follows from ($\tilde{g}7$). In fact,

$$\psi(Ks) = Ks\tilde{g}(\sqrt{Ks}) \le KsC(\sqrt{K})\tilde{g}(\sqrt{s}) = KC(\sqrt{K})\psi(s).$$

□

Lemma 4.2. *Denote by δ the convex conjugate function to γ. Then* ($\gamma2$) *is equivalent to $\delta(s) \le d_3 s^2$ on a neighborhood of zero for some $d_3 > 0$.*

Proof. By definition $\delta(r) = \sup_{s\ge0}(rs - \gamma(s))$. From the shape of γ it follows that the maximizer s_0 of $rs - \gamma(s)$ is small if r is small. Hence, $\max_{s\ge0}(rs - \gamma(s)) \le \max_{s\ge0}(rs - d_2 s^2) = r^2/(2d_2)^2$. And the converse implication $\gamma(s) = \max_{r\ge0}(sr - \delta(r)) \ge \max_{r\ge0}(sr - d_3 r^2) = s^2/(2d_3)^2$. □

Proposition 4.3. *The following are equivalent*

(i) (G_0), ($\tilde{G}$), (Γ)

(ii) (G), (H).

Proof. (i) ⇒ (ii): Upper bound on $[\tau, +\infty)$, condition (g3), follows from (G)0 on a neighborhood of infinity $[K, +\infty)$ and from continuity of g on the compact interval $[\tau, K]$. Lower bound on $[\tau, +\infty)$, condition (g2), follows from positivity and concavity of $\tilde{g}$ and inequality ($\tilde{\text{g}}$1). Concerning the upper bound on $[0, \tau)$ (condition (g1)) we distinguish two cases. The first case $\lim_{s\to 0+} sg(s) \neq 0$ leads to contradiction. In fact, taking $s_k \to 0$, $s_k > 0$ with $s_k g(s_k) \geq c > 0$ and dividing the inequality in (γ1) by $|g_1(|z|)|$ we obtain

$$\frac{\gamma(s_k g(s_k))}{s_k g(s_k)} \leq d_1 s_k.$$

Here, the right-hand side tends to zero as $k \to \infty$ and the left-hand side does not since $\gamma(r) \geq ar$ for $r \in [c, +\infty)$ for some $a > 0$ (γ is increasing and convex). In the second case $\lim_{s\to 0+} sg(s) = 0$ we have $\gamma(sg(s)) \geq ds^2 g(s)^2$ and $\gamma(g_1(s)) \leq g(s)s^2$, hence $g(s) \leq C$. Condition (H)follows immediately taking $h := c_1\tilde{g}$ on $[0, \tau]$.

(ii) ⇒ (i): (G)0 follows immediately from (g3). To show (Γ) let us define

$$\gamma(s) := \begin{cases} c_1 s^2 & \text{for } s \in [0, \tau) \\ c_2 s^p - c_3 & \text{for } s \in [\tau, +\infty), \end{cases}$$

where c_1, c_2, $c_3 > 0$ are such that γ is continuous in τ and convex, i.e.,

$$\gamma_-(\tau) = c_1\tau^2 = c_2\tau^p - c_3 = \gamma_+(\tau). \tag{6}$$

$$\gamma'_-(\tau) = 2c_1\tau \leq pc_2\tau^{p-1} = \gamma'_+(\tau). \tag{7}$$

Then γ is a Young function and we show that it satisfies (γ2)–(γ4). Since its Young conjugate on $[0, \tau)$ is again a multiple of s^2, (γ2) holds. For $K \leq 1$ (γ4) holds with $C(K) = 1$. For $K > 1$ we distinguish three cases: 1. if $s < \tau/K$, then $\gamma(Ks) = c_1K^2s^2 = K^2\gamma(s)$. 2. if $s \in [\tau/K, \tau]$, then

$$\gamma(Ks) = c_2K^ps^p - c_3 = \frac{c_2K^ps^p - c_3}{c_1s^2} \cdot \gamma(s) \leq \max\left\{\frac{c_2K^ps^p - c_3}{c_1s^2};\ s \in [\tau/K, \tau]\right\} \gamma(s).$$

3. if $s > \tau$, then

$$\gamma(Ks) = c_2K^ps^p - c_3 \leq \frac{c_2K^p\tau^p - c_3}{c_2\tau^p - c_3}\gamma(s),$$

where the last inequality holds since the function

$$s \mapsto \frac{c_2K^ps^p - c_3}{c_2s^p - c_3}$$

is decreasing on $(\tau, +\infty)$. Then we take $C(K)$ as maximum of the three numbers on the right-hand sides and (γ4) is proven.

Condition (γ3) clearly holds on $[0, \tau^p]$, since $\tilde{\gamma}(s) = c_1s^{2/p}$ is convex ($p < 2$). On $(\tau^p, +\infty)$ we have $\tilde{\gamma}(s) = c_2s - c^3$, so it is convex again. Hence, if

$$\tilde{\gamma}'_-(\tau) = \frac{2c_1}{p}\tau^{2/p-1} \leq c_2 = \tilde{\gamma}'_+(\tau) \tag{8}$$

holds, then $\tilde{\gamma}$ is convex on $\mathbb{R}_+$.

Let us take $c_2 > 0$ arbitrarily and then take $c_1 > 0$ small enough such that (7), (8) hold and $c_1\tau^2 < c_2\tau^p$. Finaly, define c_3 such that (6) holds ($c_3 > 0$).

To show (γ1) we first take any z satisfying $|g_1(z)| < \tau$. Then

$$\gamma(|g_1(z)|) = c_1|z|^2 g(|z|)^2 \leq d_1 g(|z|)|z|^2 = d_1 g_2(|z|)$$

if $d_1 \geq c_1 \sup\{g(|z|) : \ |g_1(z)| \leq \tau\}$ (the supremum is finite, since the set is compact and g is bounded on $[0,\tau]$). If z is such that $|g_1(z)| \geq \tau$ then

$$\gamma(|g_1(z)|) \leq c_2|z|^p g(|z|)^p = g(|z|)|z|^2 \cdot c_2 g(|z|)^{p-1}|z|^{p-2}. \tag{9}$$

We need to show that $g(|z|)^{p-1}|z|^{p-2}$ is bounded on $M := \{z : \ |g_1(z)| \geq \tau\}$. It is clear if $p = 1$ since the closure of M does not contain zero. If $p > 1$ and $|z| \geq \tau$, then

$$g(|z|) \leq c_3|z|^{4/(N-2)} = c_3|z|^{\frac{2-p}{p-1}}, \quad \text{therefore} \quad c_2 g(|z|)^{p-1}|z|^{p-2} \leq c_2c_3.$$

For $z \in M$, $z \leq \tau$, both $1/|z|$ and $g(|z|)$ are bounded from above by positive constants, so $g(|z|)^{p-1}|z|^{p-2}$ is bounded. (Γ) is proven.

Now, we will prove ($\tilde{\text{G}}$). If $h(0) > 0$, then g is bounded from below on $\mathbb{R}_+$ by a positive constant and we define $\tilde{g} \equiv 1$. This function satisfies ($\tilde{\text{g}}$1) with c_1 small enough and conditions ($\tilde{\text{g}}$2)–($\tilde{\text{g}}$4) are obvious.

If $h(0) = 0$, then h'_- is positive on a neighborhood of zero (see Remark 3.3). Take $\delta \in (0,\tau)$ such that $h'_-(\delta) > 0$ and $h(\delta) < C_3 := \inf_{s\geq\tau} g(s)$. Let us define

$$\tilde{g}(s) := \begin{cases} h(s)/2 & \text{for } s \in [0,\delta) \\ h(\delta)/2 + c_5/\delta - c_5/s & \text{for } s \in [\delta,+\infty). \end{cases}$$

We show that if c_5 is small enough (in particular $c_5 \leq h(\delta)\delta/2$, $c_5 \leq h'_-(\delta)\delta^2/2$), then $\tilde{g}$ satisfies ($\tilde{\text{G}}$) (observe that for such c_5 we have $\tilde{g}(s) \leq h(\delta)$ for all s).

Clearly, $\tilde{g}$ is positive and continuous on $(0,+\infty)$. We will show ($\tilde{\text{g}}$1) with $c_1 = 1$. For $s \in (0,\delta)$ we have $\tilde{g}(s) \leq h(s) \leq g(s)$. For $s \in [\delta,\tau]$ we have $\tilde{g}(s) \leq h(\delta) \leq h(s) \leq g(s)$. For $s \in (\tau,+\infty)$ we have $\tilde{g}(s) \leq h(\delta) \leq C_3 \leq g(s)$.

Clearly, $\tilde{g}$ is concave on $(0,\delta)$ and $(\delta,+\infty)$. Moreover, we have

$$\tilde{g}'_-(\delta) = h'_-(\delta)/2 \geq c_5/\delta^2 = \tilde{g}'_+(\delta)$$

and $\tilde{g}$ is concave on $\mathbb{R}_+$, i.e. ($\tilde{\text{g}}$2) holds.

We show convexity of function $\psi(s) := \tilde{g}_2(\sqrt{s}) = s\tilde{g}(\sqrt{s})$. On $s \in (0,\delta^2)$ convexity follows from (h3). For $s > \delta^2$ we have

$$\psi''(s) = (sh(\delta)/2 + sc_5/\delta - c_5\sqrt{s}))'' = c_5\frac{1}{4}s^{-3/2} > 0.$$

For $s = \delta^2$ we have

$$\psi'_-(\delta^2) = h(\delta) + \delta^2 h'_-(\delta)\frac{1}{2\delta} > h(\delta), \quad \psi'_+(\delta^2) = h(\delta)/2 + c_5/\delta - c_5\frac{1}{2\delta} < h(\delta).$$

So, ψ is convex on $\mathbb{R}_+$, ($\tilde{\text{g}}$3) holds. Condition ($\tilde{\text{g}}$4) follows immediately from (h2) and the proof is complete. □

5 Proof of Theorem 3.2

Let the assumptions (E), (G), (H) or equivalently (E), (G_0), ($\tilde{G}$), (Γ) hold. We start with the following lemma

Lemma 5.1. *If u is a strong solution to* (3)*, then*

(i) $t \mapsto g_2(u_t(t)) \in L^1(\mathbb{R}_+, L^1(\Omega))$,

(ii) $\|u_t(t)\| \to 0$ *for* $t \to +\infty$,

(iii) $\varphi \in \omega_V(u)$ *then* $M(\varphi) = 0$ *(*$\omega(u) \subset N_M$*).*

Proof. (i) Take the scalar product (resp. duality) of the equation (3) with u_t and integrate over $[s, T]$. We obtain

$$\frac{1}{2}\|u_t(T)\|^2 - \frac{1}{2}\|u_t(s)\|^2 + \int_s^T \int_\Omega g_2(u_t(t))\mathrm{d}t = E(u(T)) - E(u(s)).$$

This implies that $E(u(\cdot)) - \frac{1}{2}\|u_t(\cdot)\|^2$ is nonincreasing. Relative compactness of the range then yields that $t \mapsto g_2(u_t(t)) \in L^1(\mathbb{R}_+, L^1(\Omega))$.

(ii) follows from Theorem 2.8 in [2].

(iii) Let $\varphi \in \omega_V(u)$ and $t_n \to +\infty$, $u(t_n) \to \varphi$ in V. Then

$$u(t_n + s) = u(t_n) + \int_{t_n}^{t_n+s} u_t(r)\mathrm{d}r$$

Since the integral tends to zero in $L^2(\Omega)$, compactness of the trajectory implies that $u(t_n + s) \to \varphi$ for every $s \in [0, 1]$ in V (Otherwise there exists $s \in [0, 1]$ and $\varepsilon > 0$, such that $\|u(t_{n_k}) - \varphi\|_V \geq \varepsilon$, by compactness we can take a subsequence and a $\tilde{\varphi} \neq \varphi$ such that $\|u(t_{n_{k_l}}) - \tilde{\varphi}\|_V \to 0$, so $\|u(t_{n_{k_l}}) - \tilde{\varphi}\|_H \to 0$, so $\tilde{\varphi} = \varphi$, contradiction).

The following equalities hold in V':

$$\begin{aligned} M(\varphi) = \int_0^1 M(\varphi)\mathrm{d}s &= \lim_{n\to\infty} \int_0^1 M(u(t_n + s))\mathrm{d}s \\ &= \lim_{n\to\infty} \int_0^1 u_{tt}(t_n + s) + g_1(u_t(t_n + s))\mathrm{d}s \\ = \lim_{n\to\infty} u_t(t_n + 1) - u_t(t_n) + \int_{t_n}^{t_n+1} g_1(u_t(s))\mathrm{d}s &= \lim_{n\to\infty} \int_{t_n}^{t_n+1} g_1(u_t(s))\mathrm{d}s \end{aligned}$$

(the second equality follows from Lebesgue dominated convergence theorem and the last one from (ii)). We show that last limit is equal to zero. Let τ be from (G). On $\Omega_{s,\tau} := \{x \in \Omega : |u_t(x, s)| < \tau\}$ we have $|g_1(u_t)| \leq C_2|u_t|$ by (g1). On $\Omega'_{s,\tau} := \Omega \setminus \Omega_{s,\tau}$ we have $|g_1(u_t)|^p \leq d_1 g_2(u_t)$ (by (g3) and the choice of p). Hence,

$$\int_{t_n}^{t_n+1} \|g_1(u_t(s))\|_* \mathrm{d}s \leq C \int_{t_n}^{t_n+1} \|g_1(u_t(s))\|_{L^p(\Omega)}\mathrm{d}s$$

$$\leq C\int_{t_n}^{t_n+1}\left(\int_{\Omega_{s,\tau}} C_2|u_t(s,x)|^p\mathrm{d}x\right)^{1/p}\mathrm{d}s + C\int_{t_n}^{t_n+1}\left(\int_{\Omega'_{s,\tau}} d_1 g_2(u_t(s,x))\mathrm{d}x\right)^{1/p}\mathrm{d}s.$$

The first integral converges to zero, since $\|u_t(s)\|_{L^p(\Omega)} \leq c\|u_t(s)\|_{L^2(\Omega)} \to 0$ as $s \to +\infty$ by (ii). The second integral to power p can be estimated by Jensen's inequality by

$$\tilde{C}\int_{t_n}^{t_n+1}\int_\Omega g_2(u_t(s,x))\mathrm{d}x\mathrm{d}s$$

and this tends to zero by (i). □

Lemma 5.2. *There exists $C > 0$ such that for every $v \in L^2(\Omega)$ it holds that*

$$\tilde{g}(\|v\|_*)\|v\|^2 \leq C\psi(\|v\|^2) \quad \text{and} \quad \psi(\|v\|^2) \leq C\int_\Omega g_2(v(x))\mathrm{d}x,$$

where ψ is from ($\tilde{\text{g}}$3).

Proof. The following computation holds

$$\tilde{g}(\|v\|_*)\|v\|^2 \leq \tilde{g}(c\|v\|)\|v\|^2 \leq C\tilde{g}(\|v\|)\|v\|^2 = C\psi(\|v\|^2),$$

since $\tilde{g}$ is nondecreasing (first inequality) and has property K (second inequality). The first estimate is proven. We show the second estimate. By Jensen's inequality (ψ is convex) we have

$$\psi\left(\frac{1}{|\Omega|}\int_\Omega |v|^2\right) \leq \frac{1}{|\Omega|}\int_\Omega \psi(|v|^2).$$

It follows that

$$\int_\Omega g_2(v) \geq \int_\Omega \tilde{g}(|v|)|v|^2 = \int_\Omega \psi(|v|^2) \geq |\Omega|\ \psi\left(\frac{1}{|\Omega|}\int |v|^2\right) = |\Omega|\ \psi(|\Omega|^{-1}\|v\|^2).$$

By property K we have

$$\psi(\|v\|^2) = \psi(|\Omega|\cdot|\Omega|^{-1}\|v\|^2) \leq C(|\Omega|)\psi(|\Omega|^{-1}\|v\|^2)$$

and the assertion follows with $C = C(|\Omega|)|\Omega|^{-1}$. □

Proof of Theorem 3.2. For a strong solution u from the Theorem, let us denote $v(t,x) := u_t(t,x)$. Let us define (for all $s \geq 0$ and $(u,v) \in V \times H$)

$$\Phi(s) := \int_0^s \frac{1}{\Theta(\xi)\tilde{g}(\Theta(\xi))}\mathrm{d}\xi \quad \text{and} \quad \mathcal{E}(u,v) := \Phi(H(u,v)),$$

where

$$H(u,v) = \frac{1}{2}\|v\|^2 - E(u) - \varepsilon\tilde{g}(\|v\|_*)\langle M(u), v\rangle_*$$

and functions $\tilde{g}$, Θ, M, E are defined in the assumptions and $\varepsilon > 0$ (small enough) will be specified later. It is sufficient to show that $\mathcal{E}$ is nonincreasing along solutions and that

$$-\frac{d}{dt}\mathcal{E}(u(t), v(t)) \geq C\|v(t)\|_*$$

holds for almost all $t \in \mathbb{R}_+$ such that $u(t) \in B_V(\varphi, \eta)$, where η is taken from condition (e1). Then the convergence $u(t) \to \varphi$ follows by Corollary 2.9 in [2].

Then for solution $(u(t), v(t))$ of (3) we have

$$\frac{d}{dt}\mathcal{E}(u(t), v(t)) = \frac{1}{\Theta(H(u(t), v(t)))\tilde{g}(\Theta(H(u(t), v(t))))} \cdot \frac{d}{dt}H(u(t), v(t)). \tag{10}$$

Let us fix $t > 0$ and write (u, v) instead of $(u(t), v(t))$. We compute (the first equality holds since $u \in W^{1,1}_{loc}(\mathbb{R}_+, V) \cap W^{2,1}_{loc}(\mathbb{R}_+, H)$ and the second holds since u is a strong solution to (2))

$$\begin{aligned}\frac{d}{dt}H(u(t), v(t)) &= \frac{d}{du}H(u(t), v(t))u_t + \frac{d}{dv}H(u(t), v(t))v_t = \langle v, \dot{v}\rangle - \langle M(u), \dot{u}\rangle \\ &\quad -\varepsilon\big[\tilde{g}'(\|v\|_*)\|v\|_*^{-1}\langle v, \dot{v}\rangle_*\langle M(u), v\rangle_* + \tilde{g}(\|v\|_*)\langle M'(u)\dot{u}, v\rangle_* + \tilde{g}(\|v\|_*)\langle M(u), \dot{v}\rangle_*\big] \\ &= -\langle g_1(v), v\rangle_{V',V} - \varepsilon\big[\tilde{g}'(\|v\|_*)\|v\|_*^{-1}\langle v, M(u)\rangle_*^2 - \tilde{g}'(\|v\|_*)\|v\|_*^{-1}\langle v, g_1(v)\rangle_*\langle M(u), v\rangle_* \\ &\quad +\tilde{g}(\|v\|_*)\langle M'(u)v, v\rangle_* + \tilde{g}(\|v\|_*)\langle M(u), M(u)\rangle_* - \tilde{g}(\|v\|_*)\langle M(u), g_1(v)\rangle_*\big].\end{aligned}$$

The rest of the proof works for weak solutions. Here we used (G_0), which guarrantees that $g_1(v) \in L^p \hookrightarrow V'$ (since $v \in V \hookrightarrow L^{p'}$). For the first term it holds that

$$-\langle g_1(v), v\rangle_{V',V} = -\int_\Omega g_2(v(t,x))\mathrm{d}x.$$

The second term is less or equal to zero. The third term can be estimated (with help of Cauchy–Schwarz inequality, ($\tilde{g}$6) and $g_1(v) \in L^p(\Omega) \hookrightarrow V'$ which follows from (G_0)) by

$$\varepsilon\tilde{g}(\|v\|_*)\|M(u)\|_*\|g_1(v)\|_* \leq \varepsilon c_p\tilde{g}(\|v\|_*)\|M(u)\|_*\|g_1(v)\|_p.$$

The fourth term is estimated (with help of (e2), Cauchy–Schwarz inequality and precompactness of the range of u) by

$$\varepsilon\tilde{g}(\|v\|_*)C\|v\|^2.$$

The fifth term is equal to

$$-\varepsilon\tilde{g}(\|v\|_*)\|M(u)\|_*^2.$$

The last term is estimated by (here we use again Cauchy–Schwarz and $g_1(v) \in L^p(\Omega) \hookrightarrow V'$)

$$\varepsilon\tilde{g}(\|v\|_*)\|M(u)\|_*\|g_1(v)\|_* \leq c_p\varepsilon\tilde{g}(\|v\|_*)\|M(u)\|_*\|g_1(v)\|_p.$$

Alltogether, we have

$$\frac{d}{dt}H(u(t),v(t)) \leq -\int_\Omega g_2(v(t,x))\mathrm{d}x - \varepsilon\tilde{g}(\|v\|_*)\|M(u)\|_*^2 +$$

$$2\varepsilon c_p\tilde{g}(\|v\|_*)\|M(u)\|_*\|g_1(v)\|_p + \varepsilon\tilde{g}(\|v\|_*)C\|v\|^2. \tag{11}$$

By Lemma 5.2, the last term is dominated by the first one. In fact,

$$\varepsilon\tilde{g}(\|v\|_*)C\|v\|^2 \leq \varepsilon\tilde{C}\int_\Omega g_2(v) \leq \frac{1}{4}\int_\Omega g_2(v) \tag{12}$$

if ε is small enough. We show that the third term on the right-hand side of (11) is dominated by the sum of the first and second terms. By Young inequality we have (δ is convex conjugate to γ)

$$\|M(u)\|_*\|g_1(v)\|_p \leq \delta(\|M(u)\|_*/K) + \gamma(K\|g_1(v)\|_p).$$

Since $\|M(u)\|_*$ is bounded, $\|M(u)\|_*/K$ is uniformly small if K is large enough, and by Lemma 4.2 and ($\gamma 2$) we have

$$\delta(\|M(u)\|_*/K) \leq \frac{d_3}{K^2}\|M(u)\|_*^2.$$

Moreover, it holds that

$$\gamma(K\|g_1(v)\|_p) = \tilde{\gamma}\left(\int_\Omega K^p|g_1(v)|^p\right) \leq \frac{1}{|\Omega|}\int_\Omega \tilde{\gamma}(|\Omega|K^p|g_1(v)|^p) = \frac{1}{|\Omega|}\int_\Omega \gamma(|\Omega|^{1/p}K|g_1(v)|)$$

$$\leq \frac{1}{|\Omega|}C(K|\Omega|^{1/p})\int_\Omega \gamma(|g_1(v)|) \leq \tilde{C}(K)\int_\Omega g_2(v).$$

Here the first equality is the definition of $\tilde{\gamma}$, the second inequality is Jensen's inequality ($\tilde{\gamma}$ is convex), the third equality is again definition of $\tilde{\gamma}$, the fourth inequality is property K for γ and the last inequality is ($\gamma 1$). Hence,

$$\|M(u)\|_*\|g_1(v)\|_p \leq \frac{d_3}{K^2}\|M(u)\|_*^2 + \tilde{C}(K)\int_\Omega g_2(v).$$

Taking K so large that $2c_pd_3/K^2 \leq 1/2$ and ε so small that $2\varepsilon c_p\tilde{C}(K)\tilde{g}(\|v\|_*) \leq 1/2$ we obtain

$$2\varepsilon c_p\tilde{g}(\|v\|_*)\|M(u)\|_*\|g_1(v)\|_p \leq \frac{\varepsilon}{2}\tilde{g}(\|v\|_*)\|M(u)\|_*^2 + \frac{1}{2}\int_\Omega g_2(v).$$

Inserting this inequality and (12) into (11) we obtain

$$\frac{d}{dt}H(u(t),v(t)) \leq -\frac{1}{4}\int_\Omega g_2(v(t,x))\mathrm{d}x - \frac{\varepsilon}{2}\tilde{g}(\|v\|_*)\|M(u)\|_*^2,$$

If we estimate the integral by Lemma 5.2 we get

$$\frac{d}{dt}H(u(t),v(t)) \leq -c\tilde{g}(\|v\|_*)\left(\|v\|^2 + \|M(u)\|_*^2\right) \leq -\frac{c}{2}\tilde{g}(\|v\|_*)\left(\|v\| + \|M(u)\|_*\right)^2. \quad (13)$$

So, $\mathcal{E}$ is nonincreasing along solutions.

Now, let us assume $t > 0$ is such that $u = u(t) \in B_V(\varphi, \eta)$. We can write

$$\Theta(H(u,v)) \leq \Theta(\|v\|^2) + \Theta(E(u)) + \Theta(\varepsilon\tilde{g}(\|v\|_*)\|M(u)\|_*\|v\|_*)$$

$$\leq \Theta(\|v\|^2) + c\|M(u)\|_* + \Theta(\varepsilon\tilde{g}(\|v\|_*)\|M(u)\|_*^2) + \Theta(\varepsilon\tilde{g}(\|v\|_*)\|v\|_*^2) \leq C(\|v\| + \|M(u)\|_*)$$

by sublinearity of Θ (first inequality), Kurdyka–Łojasiewicz gradient inequality, Cauchy–Schwarz inequality, monotonicity and sublinearity of Θ (second inequality), $\Theta(s) \leq C\sqrt{s}$ and boundedness of $\tilde{g}(\|v\|_*)$) (last inequality). Using property K of $\tilde{g}$ we have

$$\Theta(H(u,v))\tilde{g}(\Theta(H(u,v))) \leq C(\|v\| + \|M(u)\|_*)\tilde{g}(\|v\| + \|M(u)\|_*). \quad (14)$$

Then by (10), (13) and (14)

$$-\frac{d}{dt}\mathcal{E}(u(t),v(t)) \geq c\frac{\tilde{g}(\|v\|_*)(\|v\| + \|M(u)\|_*)^2}{(\|v\| + \|M(u)\|_*)\tilde{g}(\|v\| + \|M(u)\|_*)} = c\tilde{g}(\|v\|_*)\frac{\|v\| + \|M(u)\|_*}{\tilde{g}(\|v\| + \|M(u)\|_*)}.$$

Since $\|v\| + \varepsilon\|M(u)\|_* \geq c\|v\|_*$ and the function $z \mapsto z/\tilde{g}(z)$ is nondecreasing (this follows from ($\tilde{g}$6)), we have

$$\frac{\|v\| + \|M(u)\|_*}{\tilde{g}(\|v\| + \|M(u)\|_*)} \geq \frac{c\|v\|_*}{\tilde{g}(c\|v\|_*)} \geq \frac{c\|v\|_*}{C(c)\tilde{g}(\|v\|_*)},$$

where the last inequality follows from property K of $\tilde{g}$ (condition ($\tilde{g}$7)). Hence, for t satisfying $u(t) \in B_V(\varphi, \eta)$ we have

$$-\frac{d}{dt}\mathcal{E}(u(t),v(t)) \geq c\tilde{g}(\|v(t)\|_*)\frac{\|v(t)\|_*}{\tilde{g}(\|v(t)\|_*)} = c\|v(t)\|_*.$$

The proof is complete. □

6 Some extensions of the main result

In this section, we show that the main result holds even for more general damping functions and that it can be also applied to ordinary differential equation

$$\ddot{u} + G(u, \dot{u}) = M(u)$$

and it generalizes the result presented in [1].

6.1 More general damping function

Let us consider the following three steps of generalizing the damping funtion g.
1st step. Take $G(u_t)u_t$ with $G : \mathbb{R}^N \to \mathbb{R}_+$ instead of $g(|u_t|)u_t$, in other words, g does not have to be radially symmetric (anisotropic medium).
2nd step. Take $G(u_t)$ with $G : \mathbb{R}^N \to \mathbb{R}^N$ instead of $G(u_t)u_t$ with $G : \mathbb{R}^N \to \mathbb{R}_+$, i.e., the damping function does not act exactly in the direction opposite to velocity. However, it should act almost in that direction, an angle condition $\langle G(z), z\rangle \geq C_5|G(z)||z|$ should hold (with some $c_5 > 0$).
3rd step. Take $G(u, u_t)$ instead of $G(u_t)$, i.e., damping depends on the position (inhomogeneous medium); however, we will assume that all estimates of function G will be independent of the variable u.

Let us reformulate the problem and the assumptions. We consider the following problem

$$u_{tt} + G(u, u_t) = M(u), \quad t > 0. \tag{15}$$

We will replace assumptions (G), (H) by the following
(GG) Function $G : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is continuous and there exists $\tau > 0$ such that
(g1) there exist $C_2 > 0$ such that $|G(w, z)| \leq C_2|z|$ for all $z \in B(0, \tau)$, $w \in \mathbb{R}^N$.
(g2) there exist $C_3 > 0$ such that $C_3|z| \leq |G(w, z)|$ for all $z \in \mathbb{R}^n \setminus B(0, \tau)$, $w \in \mathbb{R}^N$.
(g3) if $N > 2$ then there exist $C_4 > 0$ such that $|G(w, z)| \leq C_4|z|^{4/(N-2)}|z|$ for all $z \in \mathbb{R}^N \setminus B(0, \tau)$, $w \in \mathbb{R}^N$.
(g0) there exist $C_5 > 0$ such that $\langle G(w, z), z\rangle \geq C_5|G(w, z)||z|$ holds on $\mathbb{R}^N \times \mathbb{R}^N$.
(HH) For τ from condition (G) there exists a concave nondecreasing function $h : [0, \tau] \to \mathbb{R}_+$ with $h(0) \geq 0$ such that
(h1) $|G(w, z)| \geq h(|z|)|z|$ for all $z \in B(0, \tau)$, $w \in \mathbb{R}^N$
(h2) function $s \mapsto (\Theta(s)h(\Theta(s)))^{-1}$ belongs to $L^1([0, \tau])$
(h3) function $\psi(s) := h_2(\sqrt{s})$ is convex on $[0, \tau]$.

Note that (h2), (h3) remained unchanged, (g1)–(g3), (h1) were naturally reformulated for function $G(w, z)$ which corresponds to $g(|z|)z$ and the angle condition (g0) was added.

Theorem 6.1. *Let E and G satisfy* (E), (GG) *and* (HH). *Let u be a strong solution to* (15) *such that*

$$\bigcup_{t\geq 0}\{(u(t), \dot{u}(t))\} \textit{ is precompact in } V \times H$$

and $\varphi \in \omega_V(u)$. Then $\lim_{t\to+\infty} \|u(t) - \varphi\|_V + \|\dot{u}(t)\| = 0$.

Proof. Sections 4 and 5 remain valid with $g_1(z)$ replaced by $G(w, z)$ and $g_2(z)$ replaced by $\langle G(w, z), z\rangle$ and three further changes. First, the inequality in ($\tilde{g}1$) has to be replaced by $|G(w, z)| \geq c_1\tilde{g}(|z|)|z|$, z, $w \in \mathbb{R}^N$. Second, in Proposition 4.3, ($\gamma1$) will be proved as follows (with help of the angle condition (g0)). For $|G(w, z)| < \tau$ we have

$$\gamma(|G(w, z)|) = c_1|G(w, z)|^2 \leq \frac{c_1}{C_5}\langle G(w, z), z\rangle \frac{|G(w, z)|}{|z|} \leq d_1\langle G(w, z), z\rangle,$$

where boundedness of $G(w,z)/|z|$ follows as in Proposition 4.3. For $|G(w,z)| \geq \tau$ we have

$$\gamma(|G(w,z)|) = c_2|G(w,z)|^p \leq \frac{c_1}{C_5}\langle G(w,z), z\rangle \frac{|G(w,z)|^{p-1}}{|z|} \leq d_1\langle G(w,z), z\rangle,$$

where the last inequality follows from (g3) by the same argument as in Proposition 4.3.

In the proof of second inequality in Lemma 5.2 we use the angle condition again, so $1/C_5$ appears in the estimate

$$\int_\Omega g_2(v) \geq \int_\Omega \tilde{g}(|v|)|v|^2,$$

reformulated as

$$\int_\Omega \langle G(u,v), v\rangle \geq \frac{1}{C_5}\int_\Omega \tilde{g}(|v|)|v|^2.$$

□

6.2 Ordinary differential equation

In [1] we studied ordinary differential equation

$$\ddot{u} + G(u,\dot{u}) = M(u) \tag{16}$$

with more restrictive assumptions on the damping function G. However, the proof of Theorem 3.2 (resp. Theorem 6.1) works also in case of ordinary differential equation. In fact, all the assertions and proofs of Sections 4 and 5 remain valid if we change the setting in the following way. Let $V = H = V' = \mathbb{R}^N$ and all the norms and scalar products are norms and scalar products in $\mathbb{R}^N$. Moreover, we can take $p = 1$ (the only purpose of p was to make the embedding $V' \hookrightarrow L^p$ continuous, which is is true since the L^p-norm is replaced by the norm in $\mathbb{R}^N$). The growth condition (g3) is not needed (it was needed only to show condition (γ1) in case $p > 1$). Condition (e2) always holds in finite-dimensional settings. Of course, all integrals over Ω and variable x has to be erased in the above sections. So, we have proved the following result.

Theorem 6.2. *Let E and G satisfy* (e1), (g0)–(g2) *of* (GG), *and* (HH) *with $p = 1$. Let $u \in W^{1,\infty} \cap W^{2,1}_{loc}(\mathbb{R}_+, \mathbb{R}^N)$ be a solution to* (16) *and $\varphi \in \omega(u)$. Then $\lim_{t\to+\infty} \|u(t) - \varphi\| + \|\dot{u}(t)\| = 0$.*

This result generalizes Theorem 4 in [1], where we assumed that G is estimated by multiples of a radially symmetric concave function $\tilde{g}$ from below and above, i.e., $c\tilde{g}(|z|)|z|^2 \leq \langle G(w,z), z\rangle \leq C\tilde{g}(|z|)|z|^2$, and we had a condition on ∇G. Moreover, we assumed Θ to be concave, but in fact it was sublinearity, what was needed in the proof of Theorem 4 in [1].

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