Long time behavior of the quadratic Klein-Gordon equation in the nonrelativistic limit regime

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Abstract

In this paper, we study the asymptotic behavior of the Klein-Gordon equation in the nonrelativistic limit regime. By employing the techniques in geometric optics, we show that the solution of the quadratic Klein-Gordon equation can be approximately described by a linear Schrödinger equation with an error of order $O(\varepsilon)$ over a long time interval of order $O(\varepsilon^{-1})$. With general nonlinearities, we show that the Klein-Gordon equation can be approximated by nonlinear Schrödinger equations over time of order $O(1)$.

Résumé: Comportement au temps long de l'équation de Klein-Gordon quadratique dans le régime de limite non-relativiste.

Dans cet article, nous étudions le comportement asymptotique de l'équation de Klein-Gordon dans le régime de limite non-relativiste. En utilisant les techniques dans l'optique géométrique, nous montrons que la solution de l'équation de Klein-Gordon quadratique peut être approximativement décrite par une équation de Schrödinger linéaire avec une erreur d'ordre $O(\varepsilon)$ dans un intervalle de temps long d'ordre $O(\varepsilon^{-1})$. Avec nonlinéarités générales, nous montrons que l'équation de Klein-Gordon peut être approchée par les équations de Schrödinger nonlinéaires au temps d'ordre $O(1)$.

Contents

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1 Introduction

The Klein-Gordon equation is a relativistic version of the Schrödinger equation and is used to describe the motion of a spinless particle. The non-dimensional Klein-Gordon equation reads as follows

$$\varepsilon^2\partial_{tt}u - \Delta u + \frac{1}{\varepsilon^2}u + f(u) = 0, \quad t \geq 0, \quad x \in \mathbb{R}^d. \tag{1.1}$$

Here $u = u(t,x)$ is a real-valued (or complex-valued) field, and $f(u)$ is a real-valued function (or $f(u) = g(|u|^2)u$ if u is complex-valued). The non-dimensional parameter ε is proportional to the inverse of the speed of light.

For fixed ε, the well-posedness of the Klein-Gordon equation is well studied [5, 6]. In this paper, our concern is the asymptotic behavior of the solution in the nonrelativistic limit ($\varepsilon \to 0$) with real initial data of the form

$$u(0) = u_{0,\varepsilon}, \quad (\partial_t u)(0) = \frac{1}{\varepsilon^2} u_{1,\varepsilon}. \tag{1.2}$$

1.1 Background

The nonrelativistic limit of (1.1)-(1.2) has gained a lot interest both in analysis and in numerical computations, see [17, 22, 18, 12, 13, 15, 20, 1, 2] and references therein. In particular, for complex valued unknown u and nonlinearity $f(u) = \lambda |u|^q u$ with $0 \le q < \frac{4}{d-2}$, Masmoudi and Nakanishi [15] showed that a wide class of solutions u to (1.1)-(1.2) can be described by using a system of coupled nonlinear Schrödinger equations. More precisely, for H^1 initial data of the form

$$u_{0,\varepsilon} = \varphi_0 + \varepsilon \varphi_\varepsilon, \quad u_{1,\varepsilon} = \psi_0 + \varepsilon \psi_\varepsilon, \tag{1.3}$$

it was shown in [15] that

$$u(t,x) = \varepsilon^{it/\varepsilon^2} v_+ + \varepsilon^{-it/\varepsilon^2} \overline{v}_- + R(t,x), \tag{1.4}$$

where $v = (v_+, v_-)$ satisfies

$$2iv_t - \Delta v + \widetilde{f}(v) = 0, \quad v(0) := (\varphi_0 - i\psi_0, \bar{\varphi}_0 + i\bar{\psi}_0), \tag{1.5}$$

with $\widetilde{f}(v) = \big(\widetilde{f}_+(v), \widetilde{f}_-(v)\big)$ defined by

$$\widetilde{f}_\pm(v) := \frac{1}{2\pi} \int_0^{2\pi} f(v_\pm + e^{i\theta} \overline{v}_\mp) d\theta. \tag{1.6}$$

The error term $R(t,x)$ satisfies the following estimate

$$\|R\|_{L^\infty(0,T;L^2)} = o(\varepsilon^{1/2}), \quad \text{for any } T < T^*, \tag{1.7}$$

where T^* is the maximal existence time of the coupled nonlinear Schrödinger equations (1.5).

Furthermore, if $f \in C^2$, for H^3 initial data of the form

$$u_{0,\varepsilon} = \varphi_0 + \varepsilon \varphi_1 + \varepsilon^2 \varphi_{2,\varepsilon}, \quad u_{1,\varepsilon} = \psi_0 + \varepsilon \psi_1 + \varepsilon^2 \psi_{2,\varepsilon}, \tag{1.8}$$

it was shown in [15] the following second order approximation result

$$\Big\| u - \big(\varepsilon^{it/\varepsilon^2}(v_+ + \varepsilon w_+) + \varepsilon^{-it/\varepsilon^2}(\overline{v_-} + \varepsilon \overline{w_-}\big) \Big\|_{L^\infty(0,T;H^1(\mathbb{R}^d))} = o(\varepsilon), \tag{1.9}$$

where $w = (w_+, w_-)$ is the solution to the following Cauchy problem of a linear Schröginder equation

$$2iw_t - \Delta w + D\widetilde{f}(v).w = 0, \quad w(0) := (\varphi_1 - i\psi_1, \bar{\varphi}_1 + i\bar{\psi}_1),$$

where we use the notation in [15]:

$$D\widetilde{f}(v).w := \partial_{z_1}\tilde{f}(v)w_+ + \partial_{z_2}\tilde{f}(v)w_- + \partial_{\bar{z}_1}\tilde{f}(v)\bar{w}_+ + \partial_{\bar{z}_2}\tilde{f}(v)\bar{w}_-.$$

We point out that in [15], the authors study the problem mainly in energy spaces and the convergence results are obtained by using Strichartz estimates and Bourgain spaces. In this paper, we adopt a different point of view by treating the non-relativistic limit problem as the stability problem in the framework of *geometric optics*—the study of highly oscillating solutions to hyperbolic systems. We show that the Klein-Gordon equation with quadratic nonlinearity can be well approximated by a linear Schrödinger equation over a long time interval of order $O(\varepsilon^{-1})$. For general nonlinearities satisfying some regularity assumptions, we show convergence results over a time interval of order $O(1)$, as in [15]. However compared to the results in [15], we obtain better convergence rates: $O(\varepsilon)$ in (1.7) and $O(\varepsilon^2)$ in (1.9).

Our results also give uniform estimates for $\|\varepsilon^2\partial_t u\|_{H^\mu}$ for any $\mu > d/2$. Indeed, we are working in the new unknown U for which $\varepsilon^2\partial_t u$ is one component (see Section 2.1). Then the uniform estimates for U give uniform estimates for $\varepsilon^2\partial_t u$. This justifies the technical assumption (A) in [1], which is the key to design a uniformly convergent numerical scheme (see Theorem 4.1 in [1]).

1.2 Quadratic nonlinearity and long time approximation

The convergence results in previous works [22, 18, 12, 13, 15] are obtained in time of order $O(1)$. A natural question is whether one can obtain the asymptotic behavior of the solution over longer time. By employing the methods in the study of geometric optics, we find that for the Klein-Gordon equation with $f(u) = \lambda u^2$ and real-valued u, the solution can be uniformly approximated by a linear Schrödinger equation over a long time interval of order $O(\varepsilon^{-1})$.

To obtain the convergence results beyond the time $O(1)$ up to $O(\varepsilon^{-1})$, one needs to develop more structure of the equation. To this end, in Section 2, we rewrite (1.1) into a symmetric hyperbolic system. We then use the WKB method to construct an approximate solution which is global-in-time well defined. Moreover, the leading terms of this approximate solution satisfy linear Schrödinger equations. Now the question becomes the stability of such approximate solution. Thus we study the *purebred system* which is the system in the difference of the exact solution and the approximate solution.

Some compatible conditions are introduced by Joly, Métivier and Rauch in [7], which are called *strong transparency conditions* using the terminology there. These conditions allow us to use the so-called normal form reduction method to eliminate the $O(1)$ source term in the perturbed system (see equation (4.10)) up to a remainder term of order $O(\varepsilon)$. Then we can apply the classical theory for symmetric hyperbolic systems to obtain a long existence time of order $O(\varepsilon^{-1})$, even with $O(1)$ amplitude initial data.

The transparency conditions are analogous to the null conditions introduced by Klainerman [8]. The normal form reduction method extended from the Poincaré's theory of normal forms for the ordinary differential equations is essentially a proper change of unknown, and is analogous to the analysis of Shatah [19].

Unfortunately, the strong transparency conditions are not satisfied in our setting. We cannot simply use the normal form reduction method to obtain the long time existence. To overcome this difficulty, we decompose the $O(1)$ linear source term B_1 in the perturbed system into the transparent part B_1^t (the part that satisfies the strong transparency conditions) and the non-transparent part B_1^{nt}. We use the normal form reduction method to eliminate the transparent part. Then we carry out a singular localization to the non-transparent part B_1^{nt}. This localization is done by introducing a cut-off function χ around the resonance and decomposing B_1^{nt} into two parts, $\chi(D_x)B_1^{nt}$ and $(1-\chi)(D_x)B_1^{nt}$. We show $\chi(D_x)B_1^{nt}$ is of order $O(\varepsilon)$ by observing a *partially strong transparency condition* (see (4.28) later on). We use again the normal form reduction to eliminate $(1-\chi)(D_x)B_1^{nt}$ with a $O(\varepsilon)$ remainder. This localization is said to be singular due to our semiclassical setting where a localization of the form $\chi(\varepsilon D_x)$ is compatible. Indeed, this localization by using $\chi(D_x)$ is in fact localized in a shrinking neighborhood of the resonance. In the limit $\varepsilon \to 0$, this neighborhood converges to the resonance points. Then in the normal form reduction for the part $(1-\chi)(D_x)B_1^{nt}$ localized outside of this shrinking neighborhood, there arises a factor ε^{-1} which may cause some troubles (see Section 4.3 for more details). This idea of singular localization is inspired by the shrinking cut-off method introduced by Germain, Masmoudi and Shatah [4].

Our main result is stated as follows.

Theorem 1.1. *Assume that the real initial datum* $(u_{0,\varepsilon}, u_{1,\varepsilon})$ *has the form in* (1.3) *with*

$$\begin{aligned} &(\varphi_0, \psi_0) \in (H^s)^2 \quad \text{independent of } \varepsilon, \\ &\big\{(\varphi_\varepsilon, \psi_\varepsilon, \varepsilon\nabla\varphi_\varepsilon)\big\}_{0<\varepsilon<0} \text{ uniformly bounded in } (H^{s-4})^{d+2} \end{aligned} \tag{1.10}$$

for some $s > d/2+4$*. Then there exists* $\varepsilon_0 > 0$ *such that for any* $0 < \varepsilon < \varepsilon_0$ *the Cauchy problem* (1.1)–(1.2) *with* $f(u) = \lambda u^2$, $\lambda \in \mathbb{R}$ *admits a unique solution* $u \in L^\infty\big(0, \frac{T}{\varepsilon}; H^{s-4}\big)$ *for some* $T > 0$ *independent of* ε*. Moreover, there exists a constant* C *independent of* ε *such that*

$$\Big\| u - \Big(e^{it/\varepsilon^2} v + e^{-it/\varepsilon^2}\bar{v}\Big)\Big\|_{L^\infty\left(0,\frac{T}{\varepsilon};H^{s-4}\right)} \le C\,\varepsilon,$$

where $v \in L^\infty(0,\infty; H^s)$ *is the solution to the following Cauchy problem of linear Schrödinger equation*

$$2iv_t - \Delta v = 0, \quad v(0) = \frac{\varphi_0 - i\psi_0}{2}. \tag{1.11}$$

We give a remark to explain why we have a linear Schrödinger equation instead of a nonlinear one in Theorem 1.1.

Remark 1.2. *The nonlinear terms* $\tilde{f}_+$ *and* $\tilde{f}_-$ *in* (1.5) *and* (1.6) *are respectively the Fourier coefficients of order* 1 *and* -1 *to the Fourier series of* $f(u_0)$ *in* θ *where* $u_0(\theta) := e^{i\theta}v_+ + e^{-i\theta}\bar{v}_-$*:*

$$f(u_0(\theta)) = \sum_{k\in\mathbb{Z}} e^{ik\theta} f_k, \quad f_k := \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\theta} f(u_0(\theta))d\theta. \tag{1.12}$$

Indeed, under the gauge invariance assumption as in [15] which means that $f(e^{is}u) = e^{is}f(u)$*, (the nonlinearities of the typical form* $g(|u|^2)u$ *satisfy such gauge invariance assumption), for any* $s \in \mathbb{R}$*, direct calculation implies*

$$\tilde{f}_+ = f_1, \quad \tilde{f}_- = f_{-1}.$$

Under the quadratic nonlinearity $f(u) = \lambda u^2$ *in Theorem 1.1, we have*

$$f(u_0(\theta)) = \lambda(e^{i\theta}v_+ + e^{-i\theta}\bar{v}_-)^2 = \lambda\left(e^{2i\theta}v_+^2 + 2v_+\bar{v}_- + e^{-2i\theta}\bar{v}_-^2\right). \tag{1.13}$$

Clearly, the order one and minus one coefficients f_1 *and* f_{-1} *of the Fourier series in* (1.13) *are both zero. Hence, there is no nonlinear term in the approximate Schrödinger equation.*

1.3 General nonlinearity and local-in-time approximation

To complete our study, we consider the case with general nonlinearities $f(u)$, especially the nonlinearities of the form

$$f(u) = \lambda u^{q+1}, \quad q \geq 0,\ q \in \mathbb{Z}; \qquad f(u) = \lambda|u|^q u, \quad q \geq 0. \tag{1.14}$$

First of all, we point out that unlike the result for the case with quadratic nonlinearity where we obtained a long time of order $O(\varepsilon^{-1})$ approximation, for general nonlinearities we only have the approximation of the Klein-Gordon equations by Schrödinger equations over time of order $O(1)$. Another main difference is that the Klein-Gordon equations with general nonlinearities are approximated by *nonlinear* Schrödinger equations instead of linear ones.

We also point out that compared to [15], here we are working in more regular Sobolev spaces. Thus we need more regularity for f, but we do not need to control the growth of $f(u)$ with respect to u, thanks to the L^∞ norm of u by Sobolev embedding. This implies that we can handle the nonlinearities in (1.14) with q arbitrarily large, while in [15] it has to be assumed that $q < 4/(d-2)$.

We finally point out that we obtain better convergence rates for the error estimates than the ones in [15]. For initial data in H^s, $s > d/2+4$ and nonlinearities

$f \in C^m,\ m > s$, we improve the error in (1.7) from $o(\varepsilon^{1/2})$ to $O(\varepsilon^1)$. If f enjoys more regularity in $C^m,\ m > s+1$, we can improve the error in (1.9) from $o(\varepsilon^1)$ to $O(\varepsilon^2)$. The error estimates are obtained in the Sobolev space H^{s-4}. To prove such results, we employ again the techniques in geometric optics.

The first result concerns a first order approximation. The result also gives an extension of the error estimate (1.4)-(1.7) obtained in [15]. Our basic Schrödinger equation is

$$2iv_t - \Delta v + \tilde{f}(v) = 0, \tag{1.15}$$

where

$$\tilde{f}(v) := \frac{1}{2\pi} \int_0^{2\pi} e^{-i\theta} f(e^{-i\theta}\bar{v} + e^{i\theta} v) d\theta. \tag{1.16}$$

As in (1.11), the initial datum is chosen as

$$v(0) = \frac{\varphi_0 - i\psi_0}{2}. \tag{1.17}$$

By the classical theory for the local well-posedness of nonlinear Schrödinger equations (see for instance Chapter 8 of [16]), if $f \in C^m,\ m > s > d/2+4$, the Cauchy problem (1.15)-(1.17) admits a unique solution $v \in C\left([0, T_0^*); H^s\right)$ with $T_0^* > 0$ the maximal existence time. Then we have

Theorem 1.3. *For real initial data satisfying* (1.3) *and* (1.10) *with* $s > d/2 + 4$ *and nonlinearity* $f \in C^m,\ m > s$, *the Cauchy problem* (1.1)–(1.2) *admits a unique solution* $u \in C\left([0, T_\varepsilon^*); H^{s-4}\right)$ *where* $T_\varepsilon^* > 0$ *is the maximal existence time. Moreover, we have*

$$\liminf_{\varepsilon \to 0} T_\varepsilon^* \geq T_0^*, \tag{1.18}$$

and for any $T < \min\{T_\varepsilon^*, T_0^*\}$, *there exists a constant* $C(T)$ *independent of* ε *such that*

$$\left\| u - \left(e^{it/\varepsilon^2} v + e^{-it/\varepsilon^2} \bar{v} \right) \right\|_{L^\infty(0,T;H^{s-4})} \leq C(T)\,\varepsilon, \tag{1.19}$$

where v *is the solution to* (1.15)-(1.17).

The second result concerns a second order approximation. This also gives an extension of the convergence result (1.9) obtained in [15].

Theorem 1.4. *Under the assumptions in Theorem 1.3, if in addition* $f \in C^m,\ m > s+1$, *and the initial datum is of the form* (1.8) *satisfying*

$$\begin{aligned} &(\varphi_1, \psi_1) \in (H^s)^2 \quad \textit{independent of } \varepsilon, \\ &\left\{(\varphi_{2,\varepsilon}, \psi_{2,\varepsilon}, \varepsilon\nabla\varphi_{2,\varepsilon})\right\}_{0<\varepsilon<1} \textit{ uniformly bounded in } (H^{s-4})^{d+2}, \end{aligned} \tag{1.20}$$

then for any $T < \min\{T_\varepsilon^*, T_0^*\}$*, there exists a constant* $C(T)$ *independent of* ε *such that*

$$\left\| u - \left(e^{it/\varepsilon^2}(v + \varepsilon w) + e^{-it/\varepsilon^2}(\bar{v} + \varepsilon \bar{w}) \right) \right\|_{L^\infty(0,T;H^{s-4})} \le C(T)\,\varepsilon^2, \tag{1.21}$$

where v *is the solution to* (1.15)-(1.17) *and* w *is the solution to the Cauchy problem*

$$2iw_t - \Delta w = \tilde{\tilde{f}}(w), \quad w(0) = \frac{\varphi_1 - i\psi_1}{2} \tag{1.22}$$

with

$$\tilde{\tilde{f}}(w) := \frac{1}{2\pi} \int_0^{2\pi} e^{-i\theta} f'(e^{-i\theta}\bar{v} + e^{i\theta} v)(e^{-i\theta}\bar{w} + e^{i\theta} w)d\theta. \tag{1.23}$$

We give several remarks on our results for general nonlinearities.

Remarks 1.5.

- *Under the assumptions in Theorem 1.4, the Cauchy problem* (1.22) *admits a unique solution* $w \in C([0, T_0^*); H^s)$*. The maximal existence time is the same as that of* $v \in C([0, T_0^*); H^s)$ *because* $\tilde{\tilde{f}}(w)$ *in* (1.23) *is linear in* w*.*
- *For* $f(u) = \lambda u^{q+1}$*,* $q \ge 0$*,* $q \in \mathbb{Z}$*, we have* $f \in C^\infty$*. Thus our results apply to such nonlinearities. For general* $f(u) = \lambda |u|^q u$*, to make sure* $f \in C^m$*,* $m > s > d/2 + 4$*, we need to assume* $q > d/2 + 4$*.*
- *For the typical cubic nonlinearity* $f(u) = \lambda u^3$*, the nonlinearity* (1.16) *of the approximate Schrödinger equation is also cubic* $\tilde{f}(v) = 3\lambda v^3$*.*
- *By* (1.18)*, there exists* $\varepsilon_0 > 0$ *such that for any* $0 < \varepsilon < \varepsilon_0$ *there holds* $T_\varepsilon^* \ge T_0^*$*. Then for* $\varepsilon < \varepsilon_0$*, the error estimates* (1.19) *and* (1.21) *hold for any* $T < T_0^*$*.*
- *Theorem 1.3 and Theorem 1.4 hold true if initial data* $u_{0,\varepsilon}$ *and* $u_{1,\varepsilon}$ *in* (1.1) *are independent of* ε*, i.e.* $\varphi_\varepsilon = \psi_\varepsilon = 0$ *in* (1.3)*.*
- *The results in Theorem 1.3 and Theorem 1.4 can be generalized to the Klein-Gordon equation with complex-valued unknown* $u \in \mathbb{C}$*. The proof is rather similar.*

This paper is organized as follows. From Section 2 to Section 4, we give a proof for Theorem 1.1. Section 5 is devoted to the proof of Theorem 1.3 and Theorem 1.4.

In the sequel, if there is no specification, C denotes a constant independent of ε. Precisely, from Section 2 to Section 4, we have $C = C(s, d, D_0)$ with

$$D_0 := \|(\varphi_0, \psi_0)\|_{H^s} + \sup_{0<\varepsilon<1} \|(\varphi_\varepsilon, \psi_\varepsilon, \varepsilon\nabla\varphi_\varepsilon)\|_{H^{s-4}}.$$

In Section 5, the dependency of C is the same as above for the argument associate with the proof of Theorem 1.3. Associate with the proof of Theorem 1.4, we have $C = (s, d, D_1)$ with

$$D_1 := \|(\varphi_0, \psi_0)\|_{H^s} + \|(\varphi_1, \psi_1)\|_{H^s} + \sup_{0<\varepsilon<1} \|(\varphi_{2,\varepsilon}, \psi_{2,\varepsilon}, \varepsilon\nabla\varphi_{2,\varepsilon})\|_{H^{s-4}}.$$

However, the value of C may differ from line to line.

2 Proof of Theorem 1.1

We give a proof of Theorem 1.1 in this Section. We reformulate this nonrelativistic limit problem as a stability problem in geometric optics and prove Theorem 1.1 by proving the stability of WKB approximate solutions.

2.1 The equivalent symmetric hyperbolic system

We rewrite the Klein-Gordon equation into a symmetric hyperbolic system by introducing

$$U := (w, v, u) := \left(\varepsilon\nabla^T u, \varepsilon^2\partial_t u, u\right)^T := \left(\varepsilon(\partial_{x_1} u, \cdots, \partial_{x_d} u), \varepsilon^2\partial_t u, u\right)^T.$$

Then the equation (1.1) is equivalent to

$$\partial_t U - \frac{1}{\varepsilon}A(\partial_x)U + \frac{1}{\varepsilon^2}A_0 U = F(U), \tag{2.1}$$

where

$$A(\partial_x) := \begin{pmatrix} 0_{d\times d} & \nabla & 0 \\ \nabla^T & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_0 := \begin{pmatrix} 0_{d\times d} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad F(U) = -\begin{pmatrix} 0 \\ f(u) \\ 0 \end{pmatrix}. \tag{2.2}$$

Since we are now considering quadratic nonlinearity $f(u) = \lambda u^2$, we can write

$$F(U) = B(U, U) \tag{2.3}$$

with B a symmetric bilinear form defined as

$$B(U_1, U_2) = -\lambda \begin{pmatrix} 0 \\ u_1 u_2 \\ 0 \end{pmatrix}, \quad \text{for any } U_j = \begin{pmatrix} w_j \\ v_j \\ u_j \end{pmatrix}, \ j \in \{1, 2\}. \tag{2.4}$$

Here the notation $\nabla := (\partial_{x_1}, \cdots, \partial_{x_d})^T$, and $0_{d\times d}$ denotes zero matrix of order $d \times d$. In what follows, we will use 0_d to denote the zero column vector of dimension d.

Under the assumptions on the initial data in (1.2), (1.3) and (1.10), we have

$$U(0) = \left(\varepsilon\nabla^T(\varphi_0 + \varepsilon\varphi_\varepsilon), \psi_0 + \varepsilon\psi_\varepsilon, \varphi_0 + \varepsilon\varphi_\varepsilon\right)^T \tag{2.5}$$

which is uniformly bounded in Sobolev space H^{s-4} with respect to ε.

The differential operator on the left-hand side of (2.1) is symmetric hyperbolic with constant coefficients. In spite of the large prefactors $1/\varepsilon$ and $1/\varepsilon^2$ in front of $A(\partial_x)$ and A_0, the H^{s-4} estimate is uniform and independent of ε because $A(\partial_x)$ and A_0 are both anti-adjoint operators. The well-posedness of Cauchy problem (2.1)-(2.5) in $C\left([0,T_\varepsilon^*); H^{s-4}\right)$ is classical (see for instance Chapter 2 of Majda [14] or Chapter 7 of Métivier [16]). Moreover, by the form of the quadratic nonlinearity $B(U)$, the classical existence time satisfies

$$T_\varepsilon^* \propto \frac{1}{\|u\|_{L^\infty}}.$$

This means, the classical existence time is $O(1)$ in the nonrelativistic limit regime.

To study the solution of (2.1)-(2.5) beyond the classical time $O(1)$, we turn to study the stability of some approximate solutions called WKB solutions. First of all, we construct an approximate solution over long time (here we actually construct a global-in-time approximate solution) by WKB expansion, where the leading terms of the approximate solution satisfy linear Schrödinger equations. We then consider the perturbed system associate with this approximate solution (see (4.2) later on). Compared to the original system (2.1), the perturbed system is *less nonlinear* in the sense that the nonlinearity becomes small of order $O(\varepsilon)$. This makes the well-posedness analysis easier. Indeed, we can show the existence and uniform bound of the solution to the perturbed system over a long time interval of order $O(\varepsilon^{-1})$. Together with the global uniform estimate for the approximate solution, we obtained the following long time asymptotic behavior of the solution to (2.1)-(2.5) in the limit $\varepsilon \to 0$:

Theorem 2.1. *There exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, the Cauchy problem (2.1)-(2.5) admits a unique solution $U \in L^\infty\left(0, \frac{T}{\varepsilon}; H^{s-4}\right)$ for some $T > 0$ independent of ε. Moreover, there holds*

$$\|U - U_a\|_{L^\infty\left(0,\frac{T}{\varepsilon};H^{s-4}\right)} \leq C\,\varepsilon,$$

where U_a is the approximate solution obtained in Proposition 2.2 in the next page.

2.2 WKB expansion and approximate solution

We look for an approximate solution to (2.1) by using WKB expansion which is a typical technique in geometric optics. The main idea is as follows.

We make a formal power series expansion in ε for the solution and each term in the series is a trigonometric polynomial in $\theta := t/\varepsilon^2$:

$$U_a = \sum_{n=0}^{K_a+1} \varepsilon^n U_n, \qquad U_n = \sum_{p\in\mathcal{H}_n} e^{ip\theta} U_{n,p}, \qquad K_a \in \mathbb{Z}_+,\ \mathcal{H}_n \subset \mathbb{Z}. \tag{2.6}$$

The amplitudes $U_{n,p}(t,x)$ are not highly-oscillating (independent of θ) and satisfies $U_{n,-p} = \overline{U}_{n,p}$ due to the reality of U_a. Here $\mathcal{H}_n$ is the n-th order harmonics set and will be determined in the construction of U_a. The *zero-order* or *fundamental* harmonics set $\mathcal{H}_0$ is defined as $\mathcal{H}_0 := \{p \in \mathbb{Z} :\ \det(ip + A_0) = 0\}$. Since $A_0 \neq 0$, the set $\mathcal{H}_0$ is usually finite. Indeed, with A_0 given in (2.2), we have

$$\mathcal{H}_0 = \{-1, 0, 1\}.$$

Higher order harmonics are generated by the fundamental harmonics and the nonlinearity of the system. In general there holds the inclusion $\mathcal{H}_n \subset \mathcal{H}_{n+1}$.

We plug (2.6) into (2.1) and deduce the system of order $O(\varepsilon^n)$:

$$\begin{aligned} \Phi_{n,p} := {}& \partial_t U_{n,p} - A(\partial_x) U_{n+1,p} + \big(ip + A_0\big) U_{n+2,p} \\ & - \sum_{\substack{n_1+n_2=n \\ p_1+p_2=p}} B(U_{n_1,p_1}, U_{n_2,p_2}) = 0, \quad n \in \mathbb{Z},\ n \geq -2 \text{ and } p \in \mathbb{Z}. \end{aligned} \tag{2.7}$$

In (2.7), we imposed $U_n = 0$ for any $n \leq -1$. Then to solve (2.1), it is sufficient to solve $\Phi_{n,p} = 0$ for all $(n,p) \in \mathbb{Z}^2$. This is in general not possible because there are infinity of n. However, we can solve (2.1) approximately by solving $\Phi_{n,p} = 0$ up to some nonnegative order $-2 \leq n \leq K_a - 1$ with $K_a \geq 1$, then U_a solves (2.1) with a remainder of order $O(\varepsilon^{K_a})$ which is small and goes to zero in the limit $\varepsilon \to 0$. More precisely, we look for an approximate solution U_a of the form (2.6) satisfying

$$\begin{cases} \partial_t U_a - \dfrac{1}{\varepsilon} A(\partial_x) U_a + \dfrac{1}{\varepsilon^2} A_0 U_a = B(U_a, U_a) - \varepsilon^{K_a} R^\varepsilon, \\ U_a(0,x) = U(0,x) - \varepsilon^K \Psi^\varepsilon(x), \end{cases} \tag{2.8}$$

where $|R^\varepsilon|_{L^\infty} + |\Psi^\varepsilon|_{L^\infty}$ is bounded uniformly in ε. Parameters K_a and K describe the level of precision of the approximate solution U_a.

We state the result in constructing the approximate solution:

Proposition 2.2. *There exists $U_a \in L^\infty\big(0,\infty; H^{s-4}\big)$ solving (2.8) for all $(t,x) \in (0,\infty)\times\mathbb{R}^d$ with $K_a = 2,\ K = 1$ and there holds the estimate*

$$\sup_{0<\varepsilon<1} \big(\|R^\varepsilon\|_{L^\infty(0,\infty;H^{s-4})} + \|\Psi^\varepsilon\|_{H^{s-4}}\big) < +\infty. \tag{2.9}$$

Moreover, U_a is of the form (2.6) with $U_n \in L^\infty(0,\infty; H^{s-2}),\ 0 \leq n \leq K_a + 1$, and the leading term U_0 is given by

$$U_0 = e^{-it/\varepsilon^2} U_{0,-1} + e^{it/\varepsilon^2} U_{0,1}, \tag{2.10}$$

where

$$U_{0,1} = g_0 e_+, \quad U_{0,-1} = \bar{g}_0 e_- \qquad e_\pm := (0_d^T, \pm i, 1)^T \tag{2.11}$$

with g_0 *the unique solution to*

$$2i\partial_t g_0 - \Delta g_0 = 0, \quad g_0(0) = \frac{\varphi_0 - i\psi_0}{2}. \tag{2.12}$$

2.3 Proof of Theorem 1.1

Let us admit Theorem 2.1 and Proposition 2.2. Then we have that the main result Theorem 1.1 is a corollary of Theorem 2.1 and Proposition 2.2. Indeed, by Proposition 2.2, we have

$$\|U_a - U_0\|_{L^\infty\left(0,\frac{T}{\varepsilon};H^{s-4}\right)} \leq C\,\varepsilon,$$

where U_0 satisfies (2.10)-(2.12). Then going back to the original unknown u, by Theorem 2.1, we obtain the results in Theorem 1.1.

Hence, it is left to prove Proposition 2.2 and Theorem 2.1. This is done in Section 3 and Section 4 respectively.

3 Proof of Proposition 2.2

We now carry out the idea in Section 2.2 to construct an approximate solution satisfying the properties stated in Proposition 2.2.

3.1 WKB cascade

We start from considering $\Phi_{-2,p} = 0$ corresponding to the equations in the terms of order $O(\varepsilon^{-2})$. We reproduce such equations as follows

$$(ip + A_0)U_{0,p} = 0, \quad \text{for all } p. \tag{3.1}$$

It is easy to find that $(ip + A_0)$ are invertible except $p \in \mathcal{H}_0 = \{-1, 0, 1\}$. We then deduce from (3.1) that

$$U_{0,p} = 0, \quad \text{for all } p \text{ such that } |p| \geq 2. \tag{3.2}$$

This is in fact how we determine $\mathcal{H}_0$: for any $p \notin \mathcal{H}_0$, necessarily $U_{0,p} = 0$.

As in [15], we do not need to include the mean mode $U_{0,0}$ in the approximation. Hence, for simplicity, we take

$$U_{0,0} = 0. \tag{3.3}$$

For $p = 1$, (3.1) is equivalent to the so called polarization condition $U_{0,p} \in \ker(ip + A)$. This implies

$$U_{0,1} = g_0 e_+, \quad e_+ := (0_d^T, i, 1)^T, \quad g_0 \text{ is a scalar function.} \tag{3.4}$$

For $p = -1$, reality implies

$$U_{0,-1} = \overline{U}_{0,1} = \bar{g}_0 e_-, \quad e_- := \bar{e}_+ = (0_d^T, -i, 1)^T. \tag{3.5}$$

We continue to consider the equations in the terms of order $O(\varepsilon^{-1})$ which are $\Phi_{-1,p} = 0$:

$$-A(\partial_x)U_{0,p} + (ip + A_0)U_{1,p} = 0, \quad \text{for all } p. \tag{3.6}$$

When $p = 0$, by the choice of the leading mean mode in (3.3), equation (3.6) becomes

$$A_0 U_{1,0} = 0$$

which is equivalent to

$$U_{1,0} = (h_1^T, 0, 0)^T \quad \text{for some vector valued function } h_1 \in \mathbb{R}^d. \tag{3.7}$$

When $p = 1$, by (3.4), equation (3.6) is equivalent to

$$U_{1,1} = g_1 e_+ + (\nabla^T g_0, 0, 0)^T \quad \text{for some scalar function } g_1. \tag{3.8}$$

When $|p| \geq 2$, the invertibility of $(ip + A_0)$ and (3.2) imply

$$U_{1,p} = 0, \quad \text{for all } p \text{ such that } |p| \geq 2. \tag{3.9}$$

The equations in the terms of order $O(\varepsilon^0)$ are $\Phi_{0,p} = 0$ as follows:

$$\partial_t U_{0,p} - A(\partial_x)U_{1,p} + (ip + A_0)U_{2,p} = \sum_{p_1+p_2=p} B(U_{0,p_1}, U_{0,p_2}), \quad \text{for all } p. \tag{3.10}$$

When $p = 0$, by (2.4), (3.2)–(3.5), equation (3.10) becomes

$$-A(\partial_x)U_{1,0} + A_0 U_{2,0} = 2B(U_{0,1}, U_{0,-1}) = -2\lambda(0_d^T, |g_0|^2, 0)^T$$

which is equivalent to (by employing (2.2) and (3.7))

$$U_{2,0} = (h_2^T, 0, \mathrm{div} h_1 - 2\lambda |g_0|^2)^T \quad \text{for some vector valued function } h_2 \in \mathbb{R}^d. \tag{3.11}$$

When $p = 1$, by (2.2), (2.4), (3.2) and (3.3), equation (3.10) becomes

$$\partial_t U_{0,1} - A(\partial_x)U_{1,1} + (i + A_0)U_{2,1} = 0. \tag{3.12}$$

By (3.4) and (3.8), equation (3.12) is equivalent to

$$\begin{cases} 2i\partial_t g_0 - \Delta g_0 = 0, \\ U_{2,1} = g_2 e_+ + (\nabla^T g_1, \partial_t g_0, 0)^T, \quad \text{for some scalar function } g_2. \end{cases} \tag{3.13}$$

This is how we obtain the linear Schrödinger equation (1.11) and (2.12). The initial datum of g_0 is determined in such a way that $U_0(0) = (0_d^T, \psi_0, \varphi_0)^T$ which is the leading term of initial data $U(0)$ (see (2.5)). This imposes

$$g_0(0) = \frac{\varphi_0 - i\psi_0}{2}. \tag{3.14}$$

When $p = 2$, by (2.2), (2.4), (3.2)–(3.4), (3.9), equation (3.10) becomes

$$(2i + A_0)U_{2,2} = B(U_{0,1}, U_{0,1}) = -2\lambda(0_d^T, g_0^2, 0)^T$$

which is equivalent to

$$U_{2,2} = \frac{\lambda}{3}\left(0_d^T, 2ig_0^2, g_0^2\right)^T. \tag{3.15}$$

When $|p| \geq 3$, equation (3.10) implies

$$U_{2,p} = 0, \quad \text{for all } p \text{ such that } |p| \geq 3.$$

We finally consider the equations of order $O(\varepsilon)$, that are $\Phi_{1,p} = 0$:

$$\partial_t U_{1,p} - A(\partial_x)U_{2,p} + (ip + A_0)U_{3,p} = 2\sum_{p_1+p_2=p} B(U_{0,p_1}, U_{1,p_2}), \quad \text{for all } p. \tag{3.16}$$

When $p = 0$, by (2.2), (2.4), (3.2)–(3.5), (3.8), (3.9), equation (3.16) becomes

$$\partial_t U_{1,0} - A(\partial_x)U_{2,0} + A_0 U_{3,0} = 4\Re B(U_{0,1}, U_{1,-1}) = -4\lambda(0_d^T, \Re(g_0\bar{g}_1), 0)^T$$

which is equivalent to (by (3.7) and (3.11))

$$\partial_t h_1 = 0,\ U_{3,0} = (h_3^T, 0, \mathrm{div} h_2 - 4\lambda\Re(g_0\bar{g}_1))^T, \tag{3.17}$$

for some vector valued function $h_3 \in \mathbb{R}^d$. The notation $\Re a$ stands for the real part of a.

Here we take a trivial solution $h_1 = 0$ to the equation $\partial_t h_1 = 0$ in (3.17). By (3.7), this means

$$U_{1,0} = 0. \tag{3.18}$$

When $p = 1$, by (2.4), (3.2), (3.3), (3.9) and (3.18), equation (3.16) becomes

$$\partial_t U_{1,1} - A(\partial_x)U_{2,1} + (i + A_0)U_{3,1} = 0$$

which is equivalent to

$$\begin{cases} 2i\partial_t g_1 - \Delta g_1 = 0, \\ U_{3,1} = g_3 e_+ + (\nabla^T g_2, \partial_t g_1, 0)^T, \quad \text{for some scalar function } g_3. \end{cases}$$

Here we used (3.8) and (3.13).

We find that g_1 satisfies the same linear Schrödinger equation as g_0. Since we do not need to include initial data of g_1 (this is needed sometimes in order to have a better initial approximation), we will take a trivial solution $g_1 = 0$.

When $p = 2$, by (2.2), (2.4), (3.4), (3.2), (3.8), (3.9) and (3.18), equation (3.16) becomes

$$-A(\partial_x)U_{2,2} + (2i + A_0)U_{3,2} = 2B(U_{0,1}, U_{1,1}) = -2\lambda(0_d^T, g_0 g_1, 0)^T$$

which is equivalent to (by (3.15))

$$U_{3,2} = \frac{2\lambda}{3}\left(g_0 \nabla^T g_0, 2ig_0 g_1, g_0 g_1\right)^T.$$

When $|p| \geq 3$, (3.16) is equivalent to

$$U_{3,p} = 0, \quad \text{for all } p \text{ such that } |p| \geq 3.$$

Now we are ready to prove Proposition 2.2.

3.2 Approximate solution and end of the proof

By (1.10), we have $g_0(0) \in H^s$ with $s > d/2 + 4$. Then classically there exists a unique global-in-time solution g_0 to the Cauchy problem $(3.13)_1$-(3.14) in Sobolev space H^s. Moreover we have the estimates

$$\|\partial_t g_0\|_{L^\infty(0,\infty;H^{s-2})} \leq C\|g_0\|_{L^\infty(0,\infty;H^s)} \leq C\|(\phi_0, \psi_0)\|_{H^s}. \tag{3.19}$$

To construct an approximate solution, we need to determine g_j and h_j, $j \in \{1, 2, 3\}$, appeared in Section 3.1. To prove Proposition 2.2, it suffices to take

$$g_1 = g_2 = g_3 = h_1 = h_2 = h_3 = 0.$$

This gives, by employing the argument in Section 3.1, that

$$\begin{aligned} &U_{0,1} = g_0 e_+, \quad U_{1,1} = \begin{pmatrix} \nabla g_0 \\ 0 \\ 0 \end{pmatrix}, \quad U_{2,0} = -2\lambda \begin{pmatrix} 0_d \\ 0 \\ |g_0|^2 \end{pmatrix}, \\ &U_{2,1} = \begin{pmatrix} 0_d \\ \partial_t g_0 \\ 0 \end{pmatrix}, \quad U_{2,2} = \frac{\lambda}{3}\begin{pmatrix} 0_d \\ 2ig_0^2 \\ g_0^2 \end{pmatrix}, \quad U_{3,2} = \begin{pmatrix} g_0 \nabla g_0 \\ 0 \\ 0 \end{pmatrix}, \end{aligned} \tag{3.20}$$

and $U_{n,p}=0$ for all other $(n,p)\in\mathbb{Z}^2,\ p\geq 0$, and $U_{n,p}=\overline{U}_{n,-p}$ for $p<0$.

We observe that all the components in (3.20) are determined by the leading amplitude g_0. By the estimate of g_0 in (3.19), we have for any $(n,p)\in\mathbb{Z}^2$:

$$U_{n,p}\in L^\infty(0,\infty;H^{s-2}),\quad \partial_t U_{n,p}\in L^\infty(0,\infty;H^{s-4}). \tag{3.21}$$

Plugging all such $U_{n,p}$ into (2.6) gives an approximate solution U_a of the form

$$U_a=U_0+\varepsilon U_1+\varepsilon^2U_2+\varepsilon^3U_3$$

which solves the following Cauchy problem globally in time

$$\begin{cases}\partial_t U_a-\dfrac{1}{\varepsilon}A(\partial_x)U_a+\dfrac{1}{\varepsilon^2}A_0U_a=B(U_a,U_a)-\varepsilon^2R^\varepsilon,\\ U_a(0)=(\varepsilon\nabla^T\varphi_0,\psi_0,\varphi_0)^T+\varepsilon^2U_2(0)+\varepsilon^3U_3(0),\end{cases} \tag{3.22}$$

where

$$\begin{aligned}R^\varepsilon:=&\,2B(U_0,U_2)+B(U_1,U_1)+2\varepsilon B(U_1,U_2)+\varepsilon^2B(U_2,U_2)\\ &-\sum_{n=2}^{3}\varepsilon^{n-2}\sum_p e^{ipt/\varepsilon^2}\partial_tU_{n,p}-\sum_p e^{ipt/\varepsilon^2}A(\partial_x)U_{3,p}.\end{aligned} \tag{3.23}$$

Now, Proposition 2.2 follows directly from (1.10), (3.20)-(3.23).

4 Proof of Theorem 2.1

This section is devoted to prove Theorem 2.1. This is the stability of the approximate WKB solution obtained in Proposition 2.2 over long time of order $O(\varepsilon^{-1})$.

Associated with the approximate solution U_a in Proposition 2.2, we define the perturbation

$$\dot U:=\frac{U-U_a}{\varepsilon}, \tag{4.1}$$

where $U\in C([0,T^*_\varepsilon);H^s)$ is the local-in-time solution to original Cauchy problem (2.1)-(2.5). Then at least over time interval $[0,T^*_\varepsilon)$, the perturbation $\dot U$ solves

$$\begin{cases}\partial_t\dot U-\dfrac{1}{\varepsilon}A(\partial_x)\dot U+\dfrac{1}{\varepsilon^2}A_0\dot U=2B(U_a)\dot U+\varepsilon B(\dot U,\dot U)+\varepsilon R^\varepsilon,\\ \dot U(0)=\Psi^\varepsilon,\end{cases} \tag{4.2}$$

where the linear operator $B(V)$ for some $V\in\mathbb{C}^{d+2}$ is defined as

$$B(V)W:=B(V,W),\quad \text{for any } W\in\mathbb{C}^{d+2},$$

where B is defined by (2.4). The remainder $(R^\varepsilon,\Psi^\varepsilon)$ satisfies the uniform estimate given in (2.9).

To prove Theorem 2.1, it is sufficient to show the well-posedness of (4.2) in the Sobolev space H^{s-4} over some time interval $[0,\frac{T}{\varepsilon}]$ with $T>0$ independent of ε.

4.1 Preparation

The perturbed system (4.2) has small nonlinearity of order $O(\varepsilon)$. By careful, rather classical analysis (L^2 estimate and Grownwall's inequality), it can be shown that the maximal existence time, denoted by $\dot{T}_\varepsilon^*$, to Cauchy problem (4.2) satisfies

$$\lim_{\varepsilon \to 0} \dot{T}_\varepsilon^* = \infty.$$

By employing the arguments in [3], we can even show the existence up to the time of the logarithmic order:

$$\dot{T}_\varepsilon^* \geq T_0 |\ln \varepsilon|, \quad \text{for some } T_0 > 0 \text{ independent of } \varepsilon.$$

To achieve an even larger scale of the maximal existence time as

$$\dot{T}_\varepsilon^* \geq \frac{T}{\varepsilon}, \quad \text{for some } T > 0 \text{ independent of } \varepsilon,$$

as well as the uniform estimate in $L^\infty(0, \frac{T}{\varepsilon}; H^{s-4})$, we need to make use of more structure of the system (4.2).

4.1.1 Classical and semiclassical Fourier multipliers

We introduce some concepts about Fourier multipliers. This will be needed in the coming sections.

We say a smooth scalar, vector or matrix valued function $\sigma(\xi)$ to be a classical symbol of order m provided

$$|\partial_\xi^\alpha \sigma(\xi)| \leq C_\alpha \langle \xi \rangle^{m-\alpha}, \quad \langle \xi \rangle := \left(1 + |\xi|^2\right)^{\frac{1}{2}}, \quad \text{for any } \alpha \in \mathbb{N}^d.$$

We use S^m to denote the set of all classical symbols of order m. The classical Fourier multiplier associated with a symbol $\sigma(\xi)$ is denoted by $\sigma(D_x)$, and is defined as

$$\sigma(D_x) u := \mathfrak{F}^{-1}[\sigma(\xi)\hat{u}(\xi)] = \mathfrak{F}^{-1}[\sigma] * u, \tag{4.3}$$

where $\hat{u}(\xi) = \mathfrak{F}[u](\xi)$ is the Fourier transform of u and $\mathfrak{F}^{-1}$ denotes the inverse of Fourier transform.

The semiclassical Fourier multiplier associated with a symbol $\sigma(\xi)$ is denoted by $\sigma(\varepsilon D_x)$, and is defined as

$$\sigma(\varepsilon D_x) u := \mathfrak{F}^{-1}[\sigma(\varepsilon\xi)\hat{u}(\xi)] = \mathfrak{F}^{-1}[\sigma(\varepsilon \cdot)] * u = \varepsilon^{-d} \mathfrak{F}^{-1}[\sigma] \left(\frac{\cdot}{\varepsilon}\right) * u. \tag{4.4}$$

The definitions in (4.3) and (4.4) can be generated to any σ as long as the definitions make sense.

We now give two properties that we will use in this paper for classical and semiclassical Fourier multipliers. The first one is rather direct:

Lemma 4.1. *Let $\sigma \in L^\infty$, then for any $s \in \mathbb{R}$:*

$$\|\sigma(D_x)u\|_{H^s} + \|\sigma(\varepsilon D_x)u\|_{H^s} \leq \|\sigma(\cdot)\|_{L^\infty}\|u\|_{H^s}.$$

The second one is about the commutator estimates.

Lemma 4.2. *Let $\sigma \in C^1$ such that $\|\nabla_\xi \sigma\|_{L^\infty} < \infty$ and $g(x) \in H^{d/2+1+\eta_0}$ a scalar function for some $\eta_0 > 0$. Then there holds for any $s \geq 0$:*

$$\|[\sigma(\varepsilon D_x), g(x)]u\|_{H^s} \leq \varepsilon\, C_{\eta_0}\, 2^s \|\nabla_\xi \sigma\|_{L^\infty} \left(\|g\|_{H^{\frac{d}{2}+1+\eta_0}} \|u\|_{H^s} + \|g\|_{H^{s+1}} \|u\|_{H^{\frac{d}{2}+\eta_0}} \right).$$

The point of Lemma 4.2 is that the commutator of a semiclassical Fourier multiplier and a regular scalar function is of order ε.

Proof of Lemma 4.2. Let

$$I(\xi) := \mathfrak{F}\big[[\sigma(\varepsilon D_x), g(x)]u\big](\xi).$$

Then

$$\|[\sigma(\varepsilon D_x), g(x)]u\|_{H^s} = \|\langle\xi\rangle^s I(\xi)\|_{L^2}.$$

By the definition of semiclassical Fourier multiplier, we have

$$\begin{aligned}
I(\xi) &= \mathfrak{F}[\sigma(\varepsilon D_x)(gu)] - \mathfrak{F}[g\sigma(\varepsilon D_x)(u)] = \sigma(\varepsilon\xi)\mathfrak{F}[(gu)] - \mathfrak{F}[g\sigma(\varepsilon D_x)(u)] \\
&= \sigma(\varepsilon\xi)(\hat{g} * \hat{u})(\xi) - \big(\hat{g} * (\sigma(\varepsilon\cdot)\hat{u}\big)(\xi) \\
&= \sigma(\varepsilon\xi)\int_{\mathbb{R}^d} \hat{g}(\eta)\hat{u}(\xi-\eta)d\eta - \int_{\mathbb{R}^d} \hat{g}(\eta)\sigma(\varepsilon\xi - \varepsilon\eta)\hat{u}(\xi-\eta)d\eta \\
&= \int_{\mathbb{R}^d} \hat{g}(\eta)\left(\sigma(\varepsilon\xi) - \sigma(\varepsilon\xi - \varepsilon\eta)\right)\hat{u}(\xi-\eta)d\eta \\
&= \int_{\mathbb{R}^d} \hat{g}(\eta) \int_0^1 \varepsilon\eta \cdot (\nabla_\xi\sigma)(\varepsilon\xi - \varepsilon(1-t)\eta)dt\, \hat{u}(\xi-\eta)\, d\eta.
\end{aligned}$$

Then

$$\begin{aligned}
|\langle\xi\rangle^s I(\xi)| &\leq \varepsilon\|\nabla_\xi\sigma\|_{L^\infty} \int_{\mathbb{R}^d} \langle\xi\rangle^s |\eta||\hat{g}(\eta)|\,|\hat{u}(\xi-\eta)|\,d\eta \\
&\leq \varepsilon\|\nabla_\xi\sigma\|_{L^\infty} \Big(\int_{|\eta|>\frac{|\xi|}{2}} \langle\xi\rangle^s |\eta||\hat{g}(\eta)|\,|\hat{u}(\xi-\eta)|\,d\eta \\
&\qquad + \int_{|\eta|\leq\frac{|\xi|}{2}} \langle\xi\rangle^s |\eta||\hat{g}(\eta)|\,|\hat{u}(\xi-\eta)|\,d\eta \Big) \\
&\leq \varepsilon 2^s \|\nabla_\xi\sigma\|_{L^\infty} \Big(\int_{|\eta|>\frac{|\xi|}{2}} \langle\eta\rangle^s |\eta||\hat{g}(\eta)|\,|\hat{u}(\xi-\eta)|\,d\eta \\
&\qquad + \int_{|\eta|\leq\frac{|\xi|}{2}} \langle\xi-\eta\rangle^s |\eta||\hat{g}(\eta)|\,|\hat{u}(\xi-\eta)|\,d\eta \Big) \\
&\leq \varepsilon 2^s \|\nabla_\xi\sigma\|_{L^\infty} \left(|\langle\xi\rangle^{s+1}\hat{g}(\xi)| * |\hat{u}(\xi)| + |\xi\hat{g}(\xi)| * |\langle\xi\rangle^s\hat{u}(\xi)| \right).
\end{aligned}$$

Young's inequality yields

$$|\langle\xi\rangle^s I(\xi)|_{L^2} \le \varepsilon 2^s \|\nabla_\xi \sigma\|_{L^\infty} \left(\|\langle\xi\rangle^{s+1}\hat{g}(\xi)\|_{L^2}\|\hat{u}(\xi)\|_{L^1} + \|\xi\hat{g}(\xi)\|_{L^1}\|\langle\xi\rangle^s\hat{u}(\xi)\|_{L^2} \right).$$

Hölder's inequality implies

$$\|\xi\hat{g}(\xi)\|_{L^1} \le C_{\eta_0}\|\langle\xi\rangle^{d/2+1+\eta_0}\hat{g}(\xi)\|_{L^2}, \quad \|\hat{u}(\xi)\|_{L^1} \le C_{\eta_0}\|\langle\xi\rangle^{d/2+\eta_0}\hat{u}(\xi)\|_{L^2}.$$

Finally, we obtain

$$|\langle\xi\rangle^s I(\xi)|_{L^2} \le \varepsilon C_{\eta_0} 2^s \|\nabla_\xi \sigma\|_{L^\infty} \left(\|g\|_{H^{\frac{d}{2}+1+\eta_0}}\|u\|_{H^s} + \|g\|_{H^{s+1}}\|u\|_{H^{\frac{d}{2}+\eta_0}} \right).$$

This completes the proof of Lemma 4.2. □

4.1.2 Spectral decomposition

We rewrite the linear differential operator on the left-hand side of (4.2) as

$$\partial_t + \frac{i}{\varepsilon^2}\left(-A(\varepsilon D_x) + A_0/i\right), \quad D_x := \partial_x/i.$$

The symbol of the semiclassical Fourier multiplier $(-A(\varepsilon D_x) + A_0/i)$ is

$$-A(\xi) + A_0/i$$

which is a symmetric matrix for any $\xi \in \mathbb{R}^d$. Direct calculation gives the following smooth spectral decomposition

$$-A(\xi) + A_0/i = \lambda_1(\xi)\Pi_1(\xi) + \lambda_2(\xi)\Pi_2(\xi) + \lambda_3(\xi)\Pi_3(\xi) \tag{4.5}$$

with the eigenvalues

$$\lambda_1(\xi) = \sqrt{1+|\xi|^2} = \langle\xi\rangle, \quad \lambda_2(\xi) = -\sqrt{1+|\xi|^2} = -\langle\xi\rangle, \quad \lambda_3(\xi) \equiv 0 \tag{4.6}$$

and eigenprojections

$$\Pi_j(\xi) = \frac{1}{2}\begin{pmatrix} \frac{\xi\xi^T}{\lambda_j^2} & \frac{\xi}{\lambda_j} & \frac{-i\xi}{\lambda_j^2} \\ \frac{\xi^T}{\lambda_j} & 1 & \frac{-i}{\lambda_j} \\ \frac{i\xi^T}{\lambda_j^2} & \frac{i}{\lambda_j} & \frac{1}{\lambda_j^2} \end{pmatrix}, \quad \Pi_3(\xi) = \frac{1}{d+|\xi|^2}\begin{pmatrix} \mathrm{Id}_d & 0 & -i\xi \\ 0 & 0 & 0 \\ i\xi^T & 0 & |\xi|^2 \end{pmatrix}, \tag{4.7}$$

where $j \in \{1,2\}$, $\xi = (\xi_1, \cdots, \xi_d)^T$ is a column vector and Id_d denotes the unit matrix of order d. It is direct to check that $\lambda_j \in S^1$ and $\Pi_j \in S^0$ for any $j \in \{1,2,3\}$. According to (4.5), we can write

$$-A(\varepsilon D_x) + A_0/i = \lambda_1(\varepsilon D_x)\Pi_1(\varepsilon D_x) + \lambda_2(\varepsilon D_x)\Pi_2(\varepsilon D_x).$$

4.1.3 Diagonalization

We want to go deep to the structure of the system in (4.2). Hence, we consider the system mode by mode, though the following change of variable

$$\dot{U}_1 = \begin{pmatrix} \dot{U}_1^1 \\ \dot{U}_1^2 \\ \dot{U}_1^3 \end{pmatrix} := \begin{pmatrix} \Pi_1(\varepsilon D_x)\dot{U} \\ \Pi_2(\varepsilon D_x)\dot{U} \\ \Pi_3(\varepsilon D_x)\dot{U} \end{pmatrix} \in \mathbb{R}^{d+2} \times \mathbb{R}^{d+2} \times \mathbb{R}^{d+2}.$$

We remark that, by Lemma 4.1, $\Pi_j(\varepsilon D_x),\ j \in \{1,2,3\}$ are linear operators bounded from H^s to H^s for any $s \in \mathbb{R}$. Hence

$$\|\dot{U}_1(t,\cdot)\|_{H^s} \leq C\|\dot{U}(t,\cdot)\|_{H^s}, \quad \text{for any } s \in \mathbb{R} \text{ and any } t \geq 0.$$

Inversely, we can reconstruct $\dot{U}$ via $\dot{U}_1$ due to the fact

$$\Pi_1 + \Pi_2 + \Pi_3 = \mathrm{Id}.$$

We observe that

$$B(U_a) = B(U_0) + \varepsilon B(U_r), \quad U_r := U_1 + \varepsilon U_2 + \varepsilon^2 U_3. \tag{4.8}$$

Then by (4.2), the equation in $\dot{U}_1$ is of the form

$$\partial_t \dot{U}_1 + \frac{i}{\varepsilon^2} A_1(\varepsilon D_x)\dot{U}_1 = B_1\dot{U}_1 + \varepsilon P_1\dot{U}_1 + \varepsilon F_1(\dot{U}_1,\dot{U}_1) + \varepsilon R_1. \tag{4.9}$$

The propagator A_1 on the left-hand side is a diagonal matrix valued semiclassical Fourier multiplier

$$A_1(\varepsilon D_x) := \mathrm{diag}\,\{\lambda_1(\varepsilon D_x), \lambda_2(\varepsilon D_x), 0\}.$$

The leading linear operator B_1 on the right-hand side is

$$B_1 := 2\left(\Pi_i(\varepsilon D_x)B(U_0)\Pi_j(\varepsilon D_x)\right)_{1\leq i,j\leq 3},$$

which is of matrix form and is associate with the leading term U_0.

The remainder linear operator P_1 is

$$P_1 := 2\left(\Pi_i(\varepsilon D_x)B(U_r)\Pi_j(\varepsilon D_x)\right)_{1\leq i,j\leq 3},$$

which is associated with the remainder term U_r defined in (4.8).

The nonlinear term F_1 is

$$F_1 := \begin{pmatrix} \Pi_1(\varepsilon D_x)B(\dot{U},\dot{U}) \\ \Pi_2(\varepsilon D_x)B(\dot{U},\dot{U}) \\ \Pi_3(\varepsilon D_x)B(\dot{U},\dot{U}) \end{pmatrix}, \quad \dot{U} = \dot{U}_1^1 + \dot{U}_1^2 + \dot{U}_1^3.$$

Finally the remainder R_1 is

$$R_1 := \begin{pmatrix} \Pi_1(\varepsilon D_x)R^\varepsilon \\ \Pi_2(\varepsilon D_x)R^\varepsilon \\ \Pi_3(\varepsilon D_x)R^\varepsilon \end{pmatrix}.$$

To avoid notational complexity, we rewrite (4.9) in the following more compact form

$$\partial_t \dot{U}_1 + \frac{i}{\varepsilon^2} A_1(\varepsilon D_x)\dot{U}_1 = B_1 \dot{U}_1 + \varepsilon \mathcal{R}_1, \tag{4.10}$$

where $\mathcal{R}_1$ is the sum of all the $O(\varepsilon)$ terms. By Proposition 2.2 about the approximate solution, Lemma 4.1 and Lemma 4.2 about the actions of Fourier multipliers, we have the estimate

$$\|\mathcal{R}_1(t,\cdot)\|_{H^\mu} \leq C\left(1 + \|\dot{U}(t,\cdot)\|_{L^\infty}\right)\|\dot{U}(t,\cdot)\|_{H^\mu}, \quad \text{for all } 0 \leq \mu \leq s-4,$$

where $s > d/2 + 4$ is from the regularity assumption on the initial data (1.10). The initial datum of $\dot{U}_1$ is

$$\dot{U}_1(0) = \begin{pmatrix} \Pi_1(\varepsilon D_x)\Psi^\varepsilon \\ \Pi_2(\varepsilon D_x)\Psi^\varepsilon \\ \Pi_3(\varepsilon D_x)\Psi^\varepsilon \end{pmatrix}. \tag{4.11}$$

To show long time well-posedness for (4.10) with $O(1)$ initial datum (4.11), the idea here, as well as in [7, 21, 9, 11, 10], is to eliminate the $O(1)$ term B_1 on the right-hand side of (4.10) up to a $O(\varepsilon)$ remainder. This implies a small right-hand side of order $O(\varepsilon)$. However, the strong transparency conditions are not satisfied in our setting, so we cannot use the normal form reduction method to achieve this.

To this end, we first decompose B_1 into the transparent part B_1^t and the non-transparent part B_1^{nt}. We use normal form reduction method to eliminate the transparent part (see Section 4.2 for more details).

Then we carry out a singular localization to the non-transparent part B_1^{nt}. Together with another normal form reduction, we obtain an $O(\varepsilon)$ remainder. This is done in Section 4.3 and Section 4.4.

Thus, we improve the right-hand side from $O(1)$ to $O(\varepsilon)$. The classical theory in the well-posedness for symmetric hyperbolic systems implies the existence up to time of order $O(\varepsilon^{-1})$.

4.2 First normal form reduction

We decompose B_1 into the transparent part and the nontransparent part as

$$B_1 = B_1^t + B_1^{nt}$$

with

$$B_1^t = 2\begin{pmatrix} \Pi_1 B(U_0)\Pi_1 & \Pi_1 B(U_0)\Pi_2 & 0 \\ \Pi_2 B(U_0)\Pi_1 & \Pi_2 B(U_0)\Pi_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 & e^{it/\varepsilon^2}\Pi_1 B(U_{0,1})\Pi_3 \\ 0 & 0 & e^{-it/\varepsilon^2}\Pi_2 B(U_{0,-1})\Pi_3 \\ 0 & 0 & 0 \end{pmatrix},$$

$$B_1^{nt} = 2\begin{pmatrix} 0 & 0 & e^{-it/\varepsilon^2}\Pi_1 B(U_{0,-1})\Pi_3 \\ 0 & 0 & e^{it/\varepsilon^2}\Pi_2 B(U_{0,1})\Pi_3 \\ 0 & 0 & 0 \end{pmatrix},$$

where we used the fact

$$U_0 = e^{it/\varepsilon^2}U_{0,1} + e^{-it/\varepsilon^2}U_{0,-1}, \quad \Pi_3 B(U_0)\Pi_j = 0, \text{ for any } j \in \{1,2,3\}$$

and the simplified notation (and we will use this simplified notation in the sequel unless there is a specification)

$$\Pi_j = \Pi_j(\varepsilon D_x), \quad \text{for any } j \in \{1,2,3\}.$$

We introduce the following formal change of variable

$$\dot{U}_2 = \left(\mathrm{Id} + \varepsilon^2 M\right)^{-1}\dot{U}_1, \tag{4.12}$$

where M is of the following form

$$M = \sum_{p=\pm 1} e^{ipt/\varepsilon^2}\begin{pmatrix} M_{11}^{(p)} & M_{12}^{(p)} & 0 \\ M_{21}^{(p)} & M_{22}^{(p)} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & e^{it/\varepsilon^2}M_{13}^{(1)} \\ 0 & 0 & e^{-it/\varepsilon^2}M_{23}^{(-1)} \\ 0 & 0 & 0 \end{pmatrix} \tag{4.13}$$

with $M_{ij}^{(p)}$ to be determined.

Then, by (4.10), the system in $\dot{U}_2$ has the form

(4.14)

$$\partial_t \dot{U}_2 + \frac{i}{\varepsilon^2}A_1(\varepsilon D_x)\dot{U}_2 = B_1^{nt}\dot{U}_2 + \left(\mathrm{Id} + \varepsilon^2 M\right)^{-1}\left(B_1^t - i[A_1(\varepsilon D_x), M] - \varepsilon^2\partial_t M\right)\dot{U}_2$$
$$+ \left(\mathrm{Id} + \varepsilon^2 M\right)^{-1}\left(\varepsilon^2 B_1^t M\dot{U}_2 + \varepsilon^2[B_1^{nt}, M]\dot{U}_2 + \varepsilon\mathcal{R}_1\right).$$

The idea is to determine the operator M such that the second $O(1)$ term on the right-hand side of (4.14) is eliminated with a $O(\varepsilon)$ remainder. This is done in the following proposition.

Proposition 4.3. *Recall g_0 from* (3.4) *and* (3.13)-(3.14)*, and we introduce the notations*

$$g_0^{(1)} := g_0, \quad g_0^{(-1)} := \bar{g}_0.$$

There exist $\widetilde{M}_{ij}^{(p)} \in S^0$, $1 \le i,j \le 3$, $p \in \{-1,1\}$, such that M defined in (4.13) *with $M_{ij}^{(p)} := g_0^{(p)}\widetilde{M}_{ij}^{(p)}(\varepsilon D_x)$ satisfies*

$$B_1^t - i[A_1(\varepsilon D_x), M] - \varepsilon^2\partial_t M = \varepsilon M_r,$$

where M_r is a linear operator satisfying

$$\|M_r u\|_{H^\mu} \le C\|u\|_{H^\mu}, \quad \textit{for any } d/2 < \mu \le s-2 \ .$$

Proof. Given M of the form (4.13), we calculate

$$\begin{aligned}
[A_1(\varepsilon D_x), M] &= A_1(\varepsilon D_x)M - MA_1(\varepsilon D_x) \\
&= \sum_{p=\pm 1} e^{ipt/\varepsilon^2} \begin{pmatrix} \lambda_1 M_{11}^{(p)} - M_{11}^{(p)}\lambda_1 & \lambda_1 M_{12}^{(p)} - M_{12}^{(p)}\lambda_2 & 0 \\ \lambda_2 M_{21}^{(p)} - M_{21}^{(p)}\lambda_1 & \lambda_2 M_{22}^{(p)} - M_{22}^{(p)}\lambda_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & 0 & e^{it/\varepsilon^2}\lambda_1 M_{13}^{(1)} \\ 0 & 0 & e^{-it/\varepsilon^2}\lambda_2 M_{23}^{(-1)} \\ 0 & 0 & 0 \end{pmatrix},
\end{aligned}$$

where we used the fact $\lambda_3 = 0$ and the simplified notation $\lambda_j := \lambda_j(\varepsilon D_x)$, $j \in \{1,2,3\}$.

We then calculate

$$\varepsilon^2 \partial_t M = \sum_{p=\pm 1} e^{ipt/\varepsilon^2}(ip) \begin{pmatrix} M_{11}^{(p)} & M_{12}^{(p)} & 0 \\ M_{21}^{(p)} & M_{22}^{(p)} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & e^{it/\varepsilon^2}(i)M_{13}^{(1)} \\ 0 & 0 & e^{-it/\varepsilon^2}(-i)M_{23}^{(-1)} \\ 0 & 0 & 0 \end{pmatrix} + \varepsilon^2 M_r^{(1)},$$

where

$$M_r^{(1)} := \sum_{p=\pm 1} e^{ipt/\varepsilon^2}\partial_t \begin{pmatrix} M_{11}^{(p)} & M_{12}^{(p)} & 0 \\ M_{21}^{(p)} & M_{22}^{(p)} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & e^{it/\varepsilon^2}\partial_t M_{13}^{(1)} \\ 0 & 0 & e^{-it/\varepsilon^2}\partial_t M_{23}^{(-1)} \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.15}$$

Then

$$\begin{aligned}
&i[A_1(\varepsilon D_x), M] + \varepsilon^2\partial_t M \\
&= i\sum_{p=\pm 1} e^{ipt/\varepsilon^2} \begin{pmatrix} M_{11}^{(p)}(\lambda_1 - \lambda_1 + p) & M_{12}^{(p)}(\lambda_1 - \lambda_2 + p) & 0 \\ M_{21}^{(p)}(\lambda_2 - \lambda_1 + p) & M_{22}^{(p)}(\lambda_2 - \lambda_2 + p) & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&\quad + i\begin{pmatrix} 0 & 0 & e^{it/\varepsilon^2}M_{13}^{(1)}(\lambda_1 + 1) \\ 0 & 0 & e^{-it/\varepsilon^2}M_{23}^{(-1)}(\lambda_2 - 1) \\ 0 & 0 & 0 \end{pmatrix} + \varepsilon^2 M_r^{(1)} + \varepsilon M_r^{(2)},
\end{aligned}$$

where

$$\begin{aligned}
M_r^{(2)} :=& \frac{1}{\varepsilon}\sum_{p=\pm 1} e^{ipt/\varepsilon^2} \begin{pmatrix} [\lambda_1, M_{11}^{(p)}] & [\lambda_1, M_{12}^{(p)}] & 0 \\ [\lambda_2, M_{21}^{(p)}] & [\lambda_2, M_{22}^{(p)}] & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&+ \frac{1}{\varepsilon}\begin{pmatrix} 0 & 0 & e^{it/\varepsilon^2}[\lambda_1, M_{13}^{(1)}] \\ 0 & 0 & e^{-it/\varepsilon^2}[\lambda_2, M_{23}^{(-1)}] \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{4.16}$$

Now we are ready to give the definitions of $\widetilde{M}_{ij}^{(p)}(\xi)$:

(4.17)

$$\widetilde{M}_{ij}^{(p)}(\xi) := -2i(\lambda_i(\xi) - \lambda_j(\xi) + p)^{-1}\Pi_i(\xi)B(e_p)\Pi_j(\xi),\ 1 \le i, j \le 2,\, p = \pm 1,$$
$$\widetilde{M}_{13}^{(1)}(\xi) := -2i(\lambda_1(\xi) + 1)^{-1}\Pi_1(\xi)B(e_1)\Pi_3(\xi),$$
$$\widetilde{M}_{23}^{(-1)}(\xi) := -2i(\lambda_2(\xi) - 1)^{-1}\Pi_2(\xi)B(e_{-1})\Pi_3(\xi).$$

Here the notation $e_p,\ p \in \{-1, 1\}$ is defined as

$$e_1 := e_+, \quad e_{-1} := e_-, \tag{4.18}$$

where e_+ and e_- are given in (3.4) and (3.5).

By the expression of the eigenvalues λ_j in (4.6), we have for all $\xi \in \mathbb{R}^d$:

$$\begin{aligned} &|\lambda_i(\xi) - \lambda_j(\xi) + p| \ge 1, \quad 1 \le i, j \le 2,\, p = \pm 1, \\ &|\lambda_1(\xi) + 1| = |\lambda_2(\xi) - 1| \ge 2. \end{aligned} \tag{4.19}$$

Then it is direct to prove all the $\widetilde{M}_{ij}^{(p)}$ given in (4.17) belong to the symbol class S^0.

Let M be defined as in (4.13) with $M_{ij}^{(p)} = g_0^{(p)}\widetilde{M}_{ij}^{(p)}(\varepsilon D_x)$. By (3.19), we have

$$\|Mu\|_{H^\mu} \le C\|g_0\|_{L^\infty(0,\infty;H^s)}\|u\|_{H^\mu} \le C\|u\|_{H^\mu}, \quad \text{for any } d/2 < \mu \le s - 2\ . \tag{4.20}$$

We recall

$$\Pi_i B(U_0)\Pi_j = \sum_{p=\pm 1} e^{ipt/\varepsilon^2}\Pi_i B(g_0^{(p)}e_p)\Pi_j = \sum_{p=\pm 1} e^{ipt/\varepsilon^2} g_0^{(p)}\Pi_i B(e_p)\Pi_j + \varepsilon M_r^{(3)}$$

with

$$M_r^{(3)} := \frac{1}{\varepsilon}\sum_{p=\pm 1} e^{ipt/\varepsilon^2}[\Pi_i, g_0^{(p)}]B(e_p)\Pi_j. \tag{4.21}$$

Then

$$B_1^t - i[A_1(\varepsilon D_x), M] - \varepsilon^2\partial_t M = \varepsilon M_r$$

with

$$M_r = -\varepsilon M_r^{(1)} - M_r^{(2)} + M_r^{(3)}.$$

It is left to show the uniform bound for the operator M_r. By Lemma 4.2 about the commutator estimate, we have for any $j \in \{1, 2, 3\}$:

$$\|M_r^{(j)}u\|_{H^\mu} \le C\|g_0\|_{L^\infty(0,\infty;H^s)}\|u\|_{H^\mu} \le C\|u\|_{H^\mu}, \quad \text{for any } d/2 < \mu \le s - 2\ .$$

We complete the proof.

□

Let M be the operator determined in Proposition 4.3. By the uniform operator norm for M obtained in (4.20), there exist $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, the linear operator $\mathrm{Id} + \varepsilon^2 M$ is uniformly bounded and invertible. Then the change of variable (4.12) is well defined, and the system (4.14) in $\dot{U}_2$ becomes

$$\partial_t \dot{U}_2 + \frac{i}{\varepsilon^2} A_1(\varepsilon D_x)\dot{U}_2 = B_1^{nt}\dot{U}_2 + \varepsilon \mathcal{R}_2, \tag{4.22}$$

where there hods the estimate

$$\|\mathcal{R}_2(t,\cdot)\|_{H^\mu} \leq C(1 + \|\dot{U}_2(t,\cdot)\|_{H^\mu})\|\dot{U}_2(t,\cdot)\|_{H^\mu}, \quad \text{for any } d/2 < \mu \leq s-4 \ .$$

For a better understanding of the normal form method, we introduce the definitions and properties for resonances and transparencies in the following remark.

Remark 4.4. *In the proof of Proposition 4.3, a key observation is that the factors* $\lambda_i - \lambda_j + p$ *presented in* (4.17) *are away from zero (see* (4.19)*). This makes sure that the operators* $\widetilde{M}_{ij}^{(p)}$ *in* (4.17) *are well defined.*

We call resonances *the frequencies* ξ *satisfying* $\lambda_i(\xi) - \lambda_j(\xi) + p = 0$ *for some* (i,j,p)*. We say the strong transparency conditions are satisfied provided for all* ξ *and all* (i,j,p) *there holds*

$$|\Pi_i(\xi)B(e_p)\Pi_j(\xi)| \leq C|\lambda_i(\xi) - \lambda_j(\xi) + p|. \tag{4.23}$$

The terms on the left-hand side of (4.23) *are called interaction coefficients and the factors on the right-hand side are called resonance equations. If* (4.23) *is satisfied for some* (i,j,p)*, we say the* (i,j,p)*-interaction coefficient is strongly transparent.*

If there is no resonance, under some regularity assumption for the eigenvalues and eigenprojections, for instance $\lambda_j(\xi) \in S^1$, $\Pi_j(\xi) \in S^0$ *corresponding to our setting, strong transparency conditions are satisfied.*

In our setting, it is direct to check that the coefficients in B_1^t *are strongly transparent and the coefficients in* B_1^{nt} *are not. Indeed, we already showed in the proof of Proposition 4.3 that there is no resonance associated with the interaction coefficients in* B_1^t *(see* (4.19)*); this implies the coefficients in* B_1^t *are strongly transparent. Associated with the two coefficients in* B_1^{nt}*, the resonance sets are*

$$R_{13} := \{\xi : \lambda_1(\xi) - 1 = 0\} = \{0_d\}, \quad R_{23} := \{\xi : \lambda_2(\xi) + 1 = 0\} = \{0_d\}.$$

We now calculate the interaction coefficients $\Pi_1(\xi)B(e_-)\Pi_3(\xi)$ *and* $\Pi_2(\xi)B(e_+)\Pi_3(\xi)$ *and check the strong transparency conditions.*

Direct calculation gives

$$B(e_-) = B(e_+) = -\lambda \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

On one hand, together with (4.7)*, we obtain the interaction coefficients are*

$$\Pi_1(\xi)B(e_-)\Pi_3(\xi) = \frac{-\lambda}{2(d+|\xi|^2)}\begin{pmatrix} \frac{i\xi\xi^T}{\lambda_1} & 0 & \frac{\xi|\xi|^2}{\lambda_1} \\ \frac{i\xi^T}{\lambda_1} & 0 & \frac{|\xi|^2}{\lambda_1} \\ \frac{-\xi^T}{\lambda_1} & 0 & \frac{i|\xi|^2}{\lambda_1} \end{pmatrix},$$

(4.24)

$$\Pi_2(\xi)B(e_+)\Pi_3(\xi) = \frac{-\lambda}{2(d+|\xi|^2)}\begin{pmatrix} \frac{i\xi\xi^T}{\lambda_2} & 0 & \frac{\xi|\xi|^2}{\lambda_2} \\ \frac{i\xi^T}{\lambda_2} & 0 & \frac{|\xi|^2}{\lambda_2} \\ \frac{-\xi^T}{\lambda_2} & 0 & \frac{i|\xi|^2}{\lambda_2} \end{pmatrix}.$$

On the other hand, the resonance equations satisfy

$$|\lambda_1(\xi)-1|^{-1} = |\lambda_2(\xi)-1|^{-1} = \frac{1}{\sqrt{1+|\xi|^2}-1} = \frac{\sqrt{1+|\xi|^2}+1}{|\xi|^2}. \tag{4.25}$$

We find that $|\Pi_1(\xi)B(e_-)\Pi_3(\xi)| \cdot |\lambda_1(\xi)-1|^{-1}$ *and* $|\Pi_2(\xi)B(e_+)\Pi_3(\xi)| \cdot |\lambda_2(\xi)+1|^{-1}$ *are unbounded near* $\xi = 0_d$*. This implies that the strong transparency conditions* (4.23) *are not satisfied when* $(i,j,p) = (1,3,-1)$ *or* $(i,j,p) = (2,3,1)$*.*

4.3 A singular localization

This section together with Section 4.4 are devoted to improve the non-transparent part B_1^{nt} from $O(1)$ to $O(\varepsilon)$ such that we can obtain the existence in long time of order $O(\varepsilon^{-1})$ for (4.22). As observed in Remark 4.4, the interaction coefficients in B_1^{nt} do not satisfy the strong transparency conditions. We cannot employ the normal form method as in Proposition 4.3 to eliminate B_1^{nt}.

We introduce a cut-off function near the resonance $\xi = 0_d$:

$$\chi \in C_c^\infty(B(0_d,2)), \quad \chi = 1 \text{ on } B(0_d,1). \tag{4.26}$$

We then consider the decomposition

$$B_1^{nt} = \chi(D_x)B_1^{nt} + (1-\chi)(D_x)B_1^{nt} =: B_{in} + B_{out}. \tag{4.27}$$

We first show that B_{in} is small of order $O(\varepsilon)$. This can be achieved because of an observation from (4.24)-(4.25) that some *transparency type conditions* are satisfied:

(4.28)

$$|\Pi_1(\xi)B(U_{0,-1})\Pi_3(\xi)| \leq C|\lambda_1(\xi)-1|^{1/2}, \quad |\Pi_2(\xi)B(U_{0,1})\Pi_3(\xi)| \leq C|\lambda_2(\xi)+1|^{1/2}.$$

Compared to (4.23), the conditions in (4.28) are weaker. Thus we may call such conditions as *partially strong transparency conditions*.

Proposition 4.5. *There holds*

$$\|B_{in}u\|_{H^\mu} \leq \varepsilon\, C\|g_0\|_{H^s}\|u\|_{H^\mu}, \quad \textit{for all } \; d/2 < \mu \leq s-1.$$

Proof. It is sufficient to show for all $d/2 < \mu \le s-1$ that

$$\begin{aligned} &\|\chi(D_x)\Pi_1 B(U_{0,-1})\Pi_3 u\|_{H^\mu} \le \varepsilon C \|g_0\|_{H^s}\|u\|_{H^\mu},\\ &\|\chi(D_x)\Pi_2 B(U_{0,1})\Pi_3 u\|_{H^\mu} \le \varepsilon C \|g_0\|_{H^s}\|u\|_{H^\mu}. \end{aligned} \tag{4.29}$$

By Lemma 4.2,

$$\Pi_1 B(U_{0,-1})\Pi_3 = \Pi_1 B(e_-)\Pi_3 \bar{g}_0 + \varepsilon \mathcal{R}_{13} \tag{4.30}$$

with $\mathcal{R}_{13}$ satisfying

$$\|\mathcal{R}_{13} u\|_{H^\mu} \le C\|g_0\|_{H^s}\|u\|_{H^\mu}, \quad \text{for any } \; d/2 < \mu \le s-1.$$

By (4.24), we find

$$B_{13}(\xi) := \frac{\Pi_1(\xi)B(e_-)\Pi_3(\xi)}{|\xi|} \in S^{-1}, \quad B_{23}(\xi) := \frac{\Pi_1(\xi)B(e_+)\Pi_3(\xi)}{|\xi|} \in S^{-1}. \tag{4.31}$$

Then

$$\chi(D_x)\Pi_1 B(e_-)\Pi_3 = \chi(D_x)|\varepsilon D_x| B_{13}(\varepsilon D_x) = \varepsilon\, \chi(D_x)|D_x|B_{13}(\varepsilon D_x).$$

Recall $s > d/2 + 4$, then for any $d/2 < \mu \le s-1$,

$$\|\chi(\varepsilon D_x)\Pi_1 B(e_-)\Pi_3(\bar{g}_0 u)\|_{H^\mu} \le \varepsilon \|\chi(\xi)|\xi|B_{13}(\varepsilon\xi)\|_{L^\infty}\|\bar{g}_0 u\|_{H^\mu} \le \varepsilon\, C\|g_0\|_{H^s}\|u\|_{H^\mu}. \tag{4.32}$$

The estimate $(4.29)_1$ follows directly from (4.30)-(4.32). The proof of $(4.29)_2$ can be done similarly.

□

4.4 Second normal form reduction

Since B_{out} is localized outside of the resonance $\xi = 0_d$, then the normal form reduction method as in the proof of Proposition 4.3 can be employed. Some troubles may arise because of the singular localization in B_{out}. Indeed, by (4.31), the quantities

$$\begin{aligned} \frac{(1-\chi)(\xi)\Pi_1(\varepsilon\xi)B(e_-)\Pi_3(\varepsilon\xi)}{\lambda_1(\varepsilon\xi)-1} &= \frac{(1-\chi)(\xi)}{\varepsilon\,|\xi|} B_{13}(\varepsilon\xi)(\lambda_1^2(\varepsilon\xi)+1),\\ \frac{(1-\chi)(\xi)\Pi_2(\varepsilon\xi)B(e_+)\Pi_3(\varepsilon\xi)}{\lambda_2(\varepsilon\xi)+1} &= \frac{(1-\chi)(\xi)}{-\varepsilon\,|\xi|} B_{23}(\varepsilon\xi)(\lambda_2^2(\varepsilon\xi)+1) \end{aligned} \tag{4.33}$$

are of order $O(\varepsilon^{-1})$. This implies the operator M as well as the remainder M_r defined through the normal form reduction method in the proof of Proposition 4.3 are no longer uniformly bounded (see (4.16), (4.17)).

This problem can be solved. Indeed, even if M is of order $O(\varepsilon^{-1})$, the change of variable in (4.12) is still valid because $\varepsilon^2 M$ is of order $O(\varepsilon)$. By the fact $\lambda_3(\xi) \equiv 0$ independent of ξ, we can find a way to avoid the $O(\varepsilon^{-1})$ remainder in (4.16) (see Remark 4.7). The other remainders defined as in (4.15) and (4.21) are uniformly bounded and is of order $O(1)$. The details are given as follows.

We consider another change of variable

$$\dot{U}_3 = \left(\mathrm{Id} + \varepsilon^2 N\right)^{-1} \dot{U}_2, \tag{4.34}$$

where N is of the form

$$N = \begin{pmatrix} 0 & 0 & e^{-it/\varepsilon^2} N_{13} \\ 0 & 0 & e^{it/\varepsilon^2} N_{23} \\ 0 & 0 & 0 \end{pmatrix} \tag{4.35}$$

with N_{13} and N_{23} to be determined.

By (4.22) and (4.27), the system in $\dot{U}_3$ has the form

$$\begin{aligned} \partial_t \dot{U}_3 + \frac{i}{\varepsilon^2} A_1(\varepsilon D_x) \dot{U}_3 = \left(\mathrm{Id} + \varepsilon^2 N\right)^{-1} \left(B_{out} - i[A_1(\varepsilon D_x), N] - \varepsilon^2 \partial_t N\right) \dot{U}_3 \\ + B_{in} \dot{U}_3 + \left(\mathrm{Id} + \varepsilon^2 N\right)^{-1} \left(\varepsilon^2 B_{out} N \dot{U}_3 + \varepsilon^2 [B_{in}, N] \dot{U}_3 + \varepsilon \mathcal{R}_2\right). \end{aligned} \tag{4.36}$$

Then we have:

Proposition 4.6. *There exist $\widetilde{N}_{13}(\xi)$ and $\widetilde{N}_{23}(\xi)$ in symbol class S^0 such that N defined as* (4.35) *with*

$$N_{13} := \frac{\widetilde{N}_{13}(D_x)\bar{g}_0}{\varepsilon}, \quad N_{23} := \frac{\widetilde{N}_{23}(D_x) g_0}{\varepsilon} \tag{4.37}$$

satisfies

$$B_{out} - i[A_1(\varepsilon D_x), N] - \varepsilon^2 \partial_t N = \varepsilon N_r \tag{4.38}$$

for some linear operator N_r satisfying

$$\|N_r u\|_{H^\mu} \leq C \|u\|_{H^\mu}, \quad \text{for any } d/2 < \mu \leq s - 2\ . \tag{4.39}$$

Proof. Direct calculation gives

$$i[A_1(\varepsilon D_x), N] + \varepsilon^2 \partial_t N = i \begin{pmatrix} 0 & 0 & e^{-it/\varepsilon^2} (\lambda_1 - 1) N_{13} \\ 0 & 0 & e^{it/\varepsilon^2} (\lambda_2 + 1) N_{23} \\ 0 & 0 & 0 \end{pmatrix} + \varepsilon^2 N_r^{(1)},$$

where

$$N_r^{(1)} := \begin{pmatrix} 0 & 0 & e^{-it/\varepsilon^2} \partial_t N_{13} \\ 0 & 0 & e^{it/\varepsilon^2} \partial_t N_{23} \\ 0 & 0 & 0 \end{pmatrix}.$$

We define

$$\widetilde{N}_{13}(\xi) := (-2i)\frac{(1-\chi)(\xi)}{|\xi|}B_{13}(\varepsilon\xi)(\lambda_1^2(\varepsilon\xi)+1),$$
$$\widetilde{N}_{23}(\xi) := (-2i)\frac{(1-\chi)(\xi)}{-|\xi|}B_{23}(\varepsilon\xi)(\lambda_2^2(\varepsilon\xi)+1).$$

By (4.6), (4.26), (4.24) and (4.31), we have

$$\widetilde{N}_{13}(\xi) \in S^0, \quad \widetilde{N}_{23}(\xi) \in S^0.$$

By (4.33), there holds

$$\varepsilon^{-1}\widetilde{N}_{13}(\xi) = -2i(1-\chi)(\xi)(\lambda_1(\varepsilon\xi)-1)^{-1}\Pi_1(\varepsilon\xi)B(e_-)\Pi_3(\varepsilon\xi),$$
$$\varepsilon^{-1}\widetilde{N}_{23}(\xi) := -2i(1-\chi)(\xi)(\lambda_2(\varepsilon\xi)+1)^{-1}\Pi_2(\varepsilon\xi)B(e_+)\Pi_3(\varepsilon\xi).$$

Then for N_{13} and N_{23} defined by (4.37), we have
(4.40)

$$i[A_1(\varepsilon D_x), N] + \varepsilon^2\partial_t N = 2(1-\chi)(D_x)\begin{pmatrix} 0 & 0 & e^{-it/\varepsilon^2}\Pi_1 B(e_-)\Pi_3\bar{g}_0 \\ 0 & 0 & e^{it/\varepsilon^2}\Pi_2 B(e_+)\Pi_3 g_0 \\ 0 & 0 & 0 \end{pmatrix} + \varepsilon N_r^{(2)},$$

where

$$N_r^{(2)} := \begin{pmatrix} 0 & 0 & e^{-it/\varepsilon^2}\widetilde{N}_{13}(D_x)\partial_t\bar{g}_0 \\ 0 & 0 & e^{it/\varepsilon^2}\widetilde{N}_{23}(D_x)\partial_t g_0 \\ 0 & 0 & 0 \end{pmatrix}$$

satisfies

(4.41) $$\|N_r^{(2)}u\|_{H^\mu} \le C\|g_0\|_{H^s}\|u\|_{H^\mu} \le C\|u\|_{H^\mu}, \quad \text{for all } \ d/2 < \mu \le s-2.$$

We observe

(4.42) $$B_{out} = 2(1-\chi)(D_x)\begin{pmatrix} 0 & 0 & e^{-it/\varepsilon^2}\Pi_1 B(e_-)\Pi_3\bar{g}_0 \\ 0 & 0 & e^{it/\varepsilon^2}\Pi_2 B(e_+)\Pi_3 g_0 \\ 0 & 0 & 0 \end{pmatrix} + \varepsilon N_r^{(3)},$$

where

$$N_r^{(3)} = \frac{2}{\varepsilon}(1-\chi)(D_x)\begin{pmatrix} 0 & 0 & e^{-it/\varepsilon^2}\Pi_1 B(e_-)[\bar{g}_0,\Pi_3] \\ 0 & 0 & e^{it/\varepsilon^2}\Pi_2 B(e_+)[g_0,\Pi_3] \\ 0 & 0 & 0 \end{pmatrix}.$$

By Lemma 4.2, $N_r^{(3)}$ satisfies

$$\|N_r^{(3)}u\|_{H^\mu} \le C\|g_0\|_{H^s}\|u\|_{H^\mu} \le C\|u\|_{H^\mu}, \quad \text{for all } \ d/2 < \mu \le s-1.$$

By (4.40), (4.41) and (4.42) we obtained (4.38) with $N_r := N_r^{(3)} - N_r^{(2)}$ which satisfies (4.39).

□

Remark 4.7. *Compared to the proof of Proposition 4.3, a key point in the proof of Proposition 4.6 is that we avoid the commutator as in* (4.16)*, which becomes of order $O(\varepsilon^{-1})$ instead of $O(1)$ in the setting of Proposition 4.6 where N_{13} and N_{23} are indeed of order $O(\varepsilon^{-1})$.*

We can avoid this commutator due to observing that the eigenvalue λ_3 associated with Π_3 is a constant (zero actually) such that the commutator $[\lambda_3, \Pi_3]$ is zero. We then define N_{13} and N_{23} is a different way compared to $M_{ij}^{(p)}$: we changed the order of the Fourier multipliers ($\widetilde{N}_{ij}$ and $\widetilde{M}_{ij}^{(p)}$) and the scalar multipliers (g_0 and $\bar{g}_0$). Then in the normal form reduction, there arises the commutator $[\lambda_3, \Pi_3]$ which is zero instead of $[\lambda_1, \Pi_1]$ and $[\lambda_2, \Pi_2]$ which are non-zero.

The operator N determined in Proposition 4.6 is of the form

$$N = \frac{1}{\varepsilon}\widetilde{N}, \quad \widetilde{N} := \begin{pmatrix} 0 & 0 & e^{-it/\varepsilon^2}\widetilde{N}_{13}\bar{g}_0 \\ 0 & 0 & e^{it/\varepsilon^2}\widetilde{N}_{23}g_0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\widetilde{N}$ is uniformly-in-ε bounded from H^μ to H^μ for any $d/2 < \mu \leq s-2$. Then there exist $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, the operator $(\mathrm{Id} + \varepsilon^2 N) = (\mathrm{Id} + \varepsilon\widetilde{N})$ as well as $(\mathrm{Id} + \varepsilon^2 N)^{-1} = (\mathrm{Id} + \varepsilon\widetilde{N})^{-1}$ are well defined and uniformly-in-ε bounded from H^μ to H^μ for any $d/2 < \mu \leq s-2$. Together with Proposition 4.5, we can rewrite (4.36) as

$$\partial_t \dot{U}_3 + \frac{i}{\varepsilon^2}A_1(\varepsilon D_x)\dot{U}_3 = \varepsilon \mathcal{R}_3, \tag{4.43}$$

where there holds the estimate

$$\|\mathcal{R}_3(t,\cdot)\|_{H^\mu} \leq C(1 + \|\dot{U}_3(t,\cdot)\|_{H^\mu})\|\dot{U}_3(t,\cdot)\|_{H^\mu}, \quad \text{for any } d/2 < \mu \leq s-4\ .$$

4.5 Long time well-posedness and end of the proof

By (4.11), (4.12) and (4.34), we have

$$\dot{U}_3(0) = (\mathrm{Id} + \varepsilon^2 N)^{-1}(\mathrm{Id} + \varepsilon^2 M)^{-1}\begin{pmatrix} \Pi_1(\varepsilon D_x)\Psi^\varepsilon \\ \Pi_2(\varepsilon D_x)\Psi^\varepsilon \\ \Pi_3(\varepsilon D_x)\Psi^\varepsilon \end{pmatrix} \tag{4.44}$$

for which the H^{s-4} norm is uniformly bounded in ε.

We consider another change of variable corresponding to a rescalling in time:

$$\dot{U}_4(t) = \dot{U}_3(\varepsilon^{-1}t).$$

Then the equation and initial datum for $\dot{U}_4$ are

$$\begin{cases} \partial_t \dot{U}_4 + \dfrac{i}{\varepsilon^3}A_1(\varepsilon D_x)\dot{U}_4 = \mathcal{R}_4, \\ \dot{U}_4(0) = \dot{U}_3(0), \end{cases} \tag{4.45}$$

where $\mathcal{R}_4(t) := \mathcal{R}_3(\varepsilon^{-1}t)$ satisfies

$$\|\mathcal{R}_4(t,\cdot)\|_{H^\mu} \le C(1 + \|\dot{U}_4(t,\cdot)\|_{H^\mu})\|\dot{U}_4(t,\cdot)\|_{H^\mu}, \text{ for any } d/2 < \mu \le s-4 \ .$$

Since $s > d/2+4$, then by the classical theory in the well-posedness of symmetric hyperbolic systems (see for instance Chapter 2 of [14] or Chapter 7 of [16]), there exists a unique local-in-time solution $\dot{U}_4 \in L^\infty(0,T;H^{s-4})$ to Cauchy problem (4.45) for some $T > 0$ independent of ε.

Equivalently, there exists a unique solution $\dot{U}_3 \in L^\infty(0,\frac{T}{\varepsilon};H^{s-4})$ to (4.43)-(4.44). We go back to $\dot{U}$ and obtain the well-posedness in $L^\infty(0,\frac{T}{\varepsilon};H^{s-4})$. Since the approximate solution U_a is globally well defined and uniformly bounded in $L^\infty(0,\infty;H^{s-4})$, we can reconstruct the solution U for (2.1)-(2.5) in $L^\infty(0,\frac{T}{\varepsilon};H^{s-4})$ through (4.1). We then complete the proof of Theorem 2.1.

5 Proof of Theorem 1.3 and Theorem 1.4

This final section is devoted to the proof of Theorem 1.3 and Theorem 1.4.

5.1 The equivalent symmetric hyperbolic system

As in Section 2.1, we rewrite the Klein-Gordon equation into the form (2.1)-(2.2). Armed with initial data (2.5) satisfying (1.10) with $s > d/2+4$, there exists a unique solution in $C\left([0,T_\varepsilon^*);H^{s-4}\right)$ with the maximal existence time T_ε^* satisfying $\liminf_{\varepsilon\to 0} T_\varepsilon^* > 0$. Since the nonlinearity $F(U)$ only depends on $f(u)$, the classical existence time satisfies (see for instance Chapter 2 of [14] or Chapter 7 of [16])

$$T_\varepsilon^* \propto \frac{1}{C(f,\|u\|_{L^\infty})},$$

and there is a criterion for the lifespan

$$T_\varepsilon^* < \infty \Longrightarrow \lim_{t\to T_\varepsilon^*} \|u\|_{L^\infty} = \infty.$$

To prove Theorem 1.3 and Theorem 1.4, we turn to prove the following two theorems.

Theorem 5.1. *Under the assumptions in Theorem 1.3, the Cauchy problem* (2.1)–(2.5) *admits a unique solution* $U \in C\left([0,T_\varepsilon^*);H^{s-4}\right)$ *where the maximal existence time satisfies*

$$\liminf_{\varepsilon\to 0} T_\varepsilon^* \ge T_0^*, \tag{5.1}$$

where T_0 is the maximal existence time of the solution to (1.15)-(1.17)*. Moreover, for any $T < \min\{T_\varepsilon^*, T_0^*\}$, there exists a constant $C(T)$ independent of ε such that*

$$\|U - U_0\|_{L^\infty(0,T;H^{s-4})} \le C(T)\,\varepsilon, \tag{5.2}$$

where

$$U_0 := e^{it/\varepsilon^2} v e_+ + e^{-it/\varepsilon^2} \bar{v} e_- \tag{5.3}$$

with v the solution to (1.15)-(1.17)*.*

Theorem 5.2. *Under the assumption in Theorem 1.4, for any $T < \min\{T^*_\varepsilon, T^*_0\}$, there exists a constant $C(T)$ independent of ε such that*

$$\|U - U_0 - \varepsilon U_1\|_{L^\infty(0,T;H^{s-4})} \leq C(T)\,\varepsilon^2, \tag{5.4}$$

where U_0 is given in (5.3) *and*

$$U_1 := e^{it/\varepsilon^2}\left(w e_+ + (\nabla^T v, 0, 0)^T\right) + e^{-it/\varepsilon^2}\left(\bar{w} e_- + (\nabla^T \bar{v}, 0, 0)^T\right) \tag{5.5}$$

with w the solution to (1.22)-(1.23)*.*

Remark 5.3. *It is direct to observe that Theorem 1.3 is a corollary of Theorem 5.1 and Theorem 1.4 is a corollary of Theorem 5.2. Hence, it is left and sufficient to prove Theorem 5.1 and Theorem 5.2.*

5.2 WKB expansion and approximate solutions

As in Section 2.2 and Section 3, we construct approximate solutions of the form (2.6) to (2.1) by using WKB expansion.

5.2.1 WKB cascade

In the WKB cascade, the equations of order $O(\varepsilon^{-2})$ and $O(\varepsilon^{-1})$ are the same as in Section 3.1, so we do not repeat.

The equations which comprise all terms of order $O(\varepsilon^0)$ are

$$\partial_t U_{0,p} - A(\partial_x) U_{1,p} + (ip + A_0) U_{2,p} = F(U_0)_p, \quad \text{for all } p. \tag{5.6}$$

Here $F(U_0)_p$ is the p-th coefficient of the Fourier series of $F(U_0)$ in θ. Precisely,

$$F(U_0)_p = (0_d^T, -\tilde{f}_p, 0)^T,\ \tilde{f}_p := f(u_0)_p = \frac{1}{2\pi}\int_0^{2\pi} e^{-ip\theta} \check{f}(\theta) d\theta, \tag{5.7}$$

where

$$\check{f}(\theta) := f(u_0)(\theta) := f(e^{-i\theta}\bar{g}_0 + e^{i\theta} g_0). \tag{5.8}$$

Here we used the notation for the corresponding components:

$$U_n =: \begin{pmatrix} w_n \\ v_n \\ u_n \end{pmatrix}, \quad \text{for any } n \in \mathbb{Z},\ n \geq 0. \tag{5.9}$$

Lemma 5.4. *For $\tilde{f}_p$ defined by (5.7) and (5.8), we have the estimates for any $p \in \mathbb{Z}$:*

$$(5.10)\qquad \begin{aligned} &\|\tilde{f}_p(t)\|_{H^\sigma} \le C(f, \|g_0(t)\|_{L^\infty})\|g_0(t)\|_{H^\sigma}, \quad \text{for any } 0 \le \sigma < m, \\ &\|\tilde{f}_p(t)\|_{H^\sigma} \le \frac{C(f, \|g_0(t)\|_{L^\infty})}{1+|p|}\|g_0(t)\|_{H^\sigma}, \quad \text{for any } 0 \le \sigma < m-1, \end{aligned}$$

where the dependency of the constant C is as follows

$$C(f, \|g_0(t)\|_{L^\infty}) = C\left(\sum_{|\alpha|\le m} \|f^{(\alpha)}(g_0(t))\|_{L^\infty}, \|g_0(t)\|_{L^\infty}\right).$$

Proof of Lemma 5.4. Since $f \in C^m,\ m > s > d/2+4$, then we have

$$(5.11)\qquad \|f(u)\|_{H^\sigma} \le C\left(\sum_{|\alpha|\le m} \|f^{(\alpha)}(u)\|_{L^\infty}, \|u\|_{L^\infty}\right)\|u\|_{H^\sigma}, \quad \text{for any } 0 \le \sigma < m.$$

For a proof of this fact (5.11), we refer to Theorem 5.2.6 in [16]. Then it is direct to deduce $(5.10)_1$. We can obtain $(5.10)_2$ by observing for any $p \neq 0$:

$$\tilde{f}_p := \frac{1}{2\pi}\int_0^{2\pi} (ip)^{-1}e^{-ip\theta}f'(e^{-i\theta}\bar{g}_0 + e^{i\theta}g_0)(-ie^{-i\theta}\bar{g}_0 + ie^{i\theta}g_0)d\theta.$$

□

When $p = 0$, equation (5.6) is equivalent to

$$(5.12)\qquad U_{2,0} = (h_2^T, 0, \mathrm{div}h_1 - \tilde{f}_0)^T \quad \text{for some vector valued function } h_2 \in \mathbb{R}^d.$$

When $p = 1$, equation (5.6) is equivalent to

$$(5.13)\qquad \begin{cases} 2i\partial_t g_0 - \Delta g_0 + \tilde{f}_1 = 0, \\ U_{2,1} = g_2 e_+ + (\nabla^T g_1, \partial_t g_0, 0)^T, \quad \text{for some scalar function } g_2. \end{cases}$$

Here the equation $(5.13)_1$ is exactly the nonlinear Schrödinger equation (1.15). The way to determine the initial data of g_0 is the same as in (3.14) in Section 3.1. Since The initial data $g(0) \in H^s,\ s > d/2+4$, then by Lemma 5.4 and the classical theory for the local well-posedness of Schröginder equations (see for instance Chapter 8 of [16]), the Cauchy problem $(5.13)_1$-(3.14) admits a unique solution $g_0 \in C([0,T_0^*); H^s) \cap C^1([0,T_0^*); H^{s-2})$ where T_0^* is the maximal existence time. Moreover, there holds the estimate for any $T < T_0^*$:

$$(5.14)\qquad \|\partial_t g_0\|_{L^\infty(0,T;H^{s-2})} \le C\|g_0\|_{L^\infty(0,T;H^s)} \le C\|(\phi_0, \psi_0)\|_{H^s}.$$

When $|p| \ge 2$, equation (5.6) is equivalent to

$$(5.15)\qquad (ip + A_0)U_{2,p} = \tilde{f}_p \Longleftrightarrow U_{2,p} = (ip + A_0)^{-1}\tilde{f}_p.$$

5.2.2 First order approximation

In this subsection, we are working under the assumptions in Theorem 1.3.

If we stop the WKB expansion by taking

$$g_1 = g_2 = h_1 = h_2 = 0$$

in (3.7), (5.12) and $(5.13)_2$, we construct an approximate solution

$$U_a^{(1)} := U_0 + \varepsilon U_1 + \varepsilon^2 U_2, \tag{5.16}$$

where

$$\begin{aligned} &U_0 := \varepsilon^{it/\varepsilon^2} g_0 e_+ + c.c., \qquad U_1 := \varepsilon^{it/\varepsilon^2} \begin{pmatrix} \nabla g_0 \\ 0 \\ 0 \end{pmatrix} + c.c., \\ &U_2 := \begin{pmatrix} 0_d \\ 0 \\ -\tilde{f}_0 \end{pmatrix} + \left(\varepsilon^{it/\varepsilon^2} \begin{pmatrix} 0_d \\ \partial_t g_0 \\ 0 \end{pmatrix} + c.c. \right) + \sum_{|p|\geq 2} e^{ipt/\varepsilon^2} U_{2,p}. \end{aligned} \tag{5.17}$$

Here *c.c.* means complex conjugate and $z + c.c. = 2\Re z$ is two times of the real part of z. Then we have:

Proposition 5.5. *Under the assumptions in Theorem 1.3, the approximate solution $U_a^{(1)}$ constructed by (5.16) and (5.17) satisfies $U_a^{(1)} \in C([0, T_0^*); H^{s-2})$ and solves*

$$\begin{cases} \partial_t U_a^{(1)} - \dfrac{1}{\varepsilon} A(\partial_x) U_a^{(1)} + \dfrac{1}{\varepsilon^2} A_0 U_a^{(1)} = F(U_a^{(1)}) - \varepsilon R_\varepsilon^{(1)}, \\ U_a^{(1)}(0) = U(0) - \varepsilon \Psi_\varepsilon^{(1)}. \end{cases} \tag{5.18}$$

Moreover, for any $T < T_0^*$, there holds the estimate

$$\sup_{0<\varepsilon<1} \left(\|R_\varepsilon^{(1)}\|_{L^\infty(0,T;H^{s-4})} + \|\Psi_\varepsilon^{(1)}\|_{H^{s-4}} \right) < +\infty. \tag{5.19}$$

Proof of Proposition 5.5. By $(5.10)_2$ in Lemma 5.4 and (5.15), we have

$$\|U_{2,p}(t)\|_{H^{s-1}} \leq \frac{C(f, \|g_0(t)\|_{L^\infty})}{(1+|p|)^2} \|g_0(t)\|_{H^{s-1}}, \quad |p| \geq 2.$$

By (5.14) and (5.17), we obtain for any $t < T_0^*$:

$$\|U_2(t)\|_{H^{s-2}} \leq \sum_{p\in\mathbb{Z}} \|U_{2,p}(t)\|_{H^{s-2}} \leq C \|g_0(t)\|_{H^s}. \tag{5.20}$$

Together with (5.14) and (5.17), we conclude $U_a^{(1)} \in C([0,T_0^*); H^{s-2})$ solving (5.18) with the remainders

(5.21)

$$R_\varepsilon^{(1)} := \frac{F(U_a^{(1)}) - F(U_0)}{\varepsilon} + \left(\varepsilon^{it/\varepsilon^2} \begin{pmatrix} \nabla \partial_t g_0 \\ 0 \\ 0 \end{pmatrix} + c.c. \right) + A(\partial_x) U_2 - \varepsilon \sum_{p\in\mathbb{Z}} e^{ipt/\varepsilon^2} \partial_t U_{2,p},$$

$$\Psi_\varepsilon^{(1)} := \left(\varepsilon \nabla^T \varphi_\varepsilon, \psi_\varepsilon, \varphi_\varepsilon\right)^T - \varepsilon U_2(0).$$

By (1.3), (1.10), (2.5) and (5.20), we have the uniform estimate (5.19) for $\Psi_\varepsilon^{(1)}$.

It is left to show the uniform estimate (5.19) for $R_\varepsilon^{(1)}$. Direct calculation gives

$$\frac{F(U_a^{(1)}) - F(U_0)}{\varepsilon} = \int_0^1 F'(U_0 + \varepsilon\tau(U_1 + \varepsilon U_2)) \cdot (U_1 + \varepsilon U_2) d\tau.$$

Since $f \in C^m,\ m > s$, so des F. Then

$$\frac{1}{\varepsilon} \|F(U_a^{(1)}) - F(U_0)(t)\|_{H^\sigma} \leq C(f, \sum_j \|U_j\|_{L^\infty}) \sum_j \|U_j\|_{H^\sigma}, \quad \text{for any } 0 \leq \sigma < m-1.$$

Again by (5.17) and (5.14), we have for any $t < T_0^*$:

$$\frac{1}{\varepsilon} \|F(U_a^{(1)}) - F(U_0)(t)\|_{H^{s-4}} \leq C.$$

The proof of the uniform estimates for other terms in $R_\varepsilon^{(1)}$ is similar and rather direct by using the estimate (5.11). We omit the details.

□

This approximate solution $U_a^{(1)}$ in Proposition 5.5 will be used to prove Theorem 5.1 in Section 5.3.

5.2.3 Second order approximation

First of all, we point out that in this subsection, we are working under the assumptions in Theorem 1.4.

We can continue the WKB process from the end of Section 5.2.1 where we achieved the equations of order $O(\varepsilon^0)$.

The equations in the terms of order $O(\varepsilon)$ are

$$\partial_t U_{1,p} - A(\partial_x) U_{2,p} + (ip + A_0) U_{3,p} = -(0_d^T, \tilde{\tilde{f}}_p, 0)^T, \quad \text{for all } p, \tag{5.22}$$

where

$$\tilde{\tilde{f}}_p := (f'(u_0)u_1)_p = \frac{1}{2\pi} \int_0^{2\pi} e^{-ip\theta} f'(e^{-i\theta}\bar{g}_0 + e^{i\theta} g_0)(e^{-i\theta}\bar{g}_1 + e^{i\theta} g_1) d\theta. \tag{5.23}$$

Here we used the notations in (5.9).

A similar proof as that of Lemma 5.4, we can obtain:

Lemma 5.6. *There holds the estimates for any $p \in \mathbb{Z}$:*

$$\|\tilde{\tilde{f}}_p(t)\|_{H^\sigma} \leq C(f, \|(g_0, g_1)(t)\|_{L^\infty})\|(g_0, g_1)(t)\|_{H^\sigma}, \quad \text{for any } 0 \leq \sigma < m-1,$$
$$\|\tilde{\tilde{f}}_p(t)\|_{H^\sigma} \leq \frac{C(f, \|(g_0, g_1)(t)\|_{L^\infty}}{1+|p|}\|(g_0, g_1)(t)\|_{H^\sigma}, \quad \text{for any } 0 \leq \sigma < m-2,$$

where the dependency of the constant C is as follows

$$C(f, \|(g_0, g_1)(t)\|_{L^\infty}) = C\left(\sum_{|\alpha|\leq m} \|f^{(\alpha)}(g_0(t))\|_{L^\infty}, \|(g_0, g_1)(t)\|_{L^\infty}\right).$$

When $p = 0$, equation (5.22) becomes

$$\partial_t U_{1,0} - A(\partial_x)U_{2,0} + A_0 U_{3,0} = -(0_d^T, \tilde{\tilde{f}}_0, 0)^T$$

which is equivalent to

$$\partial_t h_1 = 0, \quad U_{3,0} = (h_3^T, 0, \mathrm{div} h_2 - \tilde{\tilde{f}}_0)^T \tag{5.24}$$

for some vector valued function $h_3 \in \mathbb{R}^d$.

When $p = 1$, equation (5.22) becomes

$$\partial_t U_{1,1} - A(\partial_x)U_{2,1} + (i + A_0)U_{3,1} = -(0_d^T, \tilde{\tilde{f}}_1, 0)^T,$$

which is equivalent to

$$\begin{cases} 2i\partial_t g_1 - \Delta g_1 + \tilde{\tilde{f}}_1 = 0, \\ U_{3,1} = g_3 e_+ + (\nabla^T g_2, \partial_t g_1, 0)^T, \quad \text{for some scalar function } g_3. \end{cases} \tag{5.25}$$

We find that g_1 satisfies a Schrödinger equation where the source term $\tilde{\tilde{f}}_1$ is actually linear in g_1 (see (5.23)). The initial data $g_1(0)$ is determined such that $U_1(0) = (\nabla^T\varphi_0, \psi_1, \varphi_1)^T$ which is the first order ($O(\varepsilon)$) perturbation of $U(0)$ (see (2.5) and (1.8)). This imposes

$$g_1(0) + \bar{g}_1(0) = \varphi_1, \quad ig_1(0) - i\bar{g}_1(0) = \psi_1,$$

which is equivalent to

$$g_1 = \frac{\varphi_1 - i\psi_1}{2} \in H^s. \tag{5.26}$$

This is exactly the initial condition in (1.22). Since $m > s+1$, $s > d/2+4$, by Lemma 5.6 and the classical theory, the Cauchy problem $(5.25)_1$-(5.26) admits a unique solution in Sobolev space $C([0, T_1^*), H^s)$. Here we have the maximal existence time

$T_1^* = T_0^*$ where T_0^* is the maximal existence time for the solution $g_0 \in C([0, T_0^*), H^s)$ to $(5.13)_1$, because $\tilde{\tilde{f}}_1$ is linear in g_1. Moreover, there holds for any $T < T_0^*$:

$$\|\partial_t g_1\|_{L^\infty(0,T;H^{s-2})} \leq C\|g_1\|_{L^\infty(0,T;H^s)} \leq C\|(\phi_1, \psi_1)\|_{H^s}. \tag{5.27}$$

When $|p| \geq 2$, equation (5.22) becomes

$$-A(\partial_x)U_{2,p} + (ip + A_0)U_{3,p} = -(0_d^T, \tilde{\tilde{f}}_p, 0)^T,$$

which is equivalent to

$$U_{3,p} = (ip + A_0)^{-1}\left(A(\partial_x)U_{2,p} - (0_d^T, \tilde{\tilde{f}}_p, 0)^T\right). \tag{5.28}$$

We stop the WKB expansion and take

$$g_2 = g_3 = h_1 = h_2 = h_3 = 0$$

in (3.7), (5.12), (5.24) and $(5.25)_2$. Then we construct another approximate solution

$$U_a^{(2)} := U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \varepsilon^3 U_3, \tag{5.29}$$

where

$$U_0 := \varepsilon^{it/\varepsilon^2} g_0 e_+ + c.c., \qquad U_1 := \varepsilon^{it/\varepsilon^2}\begin{pmatrix}\nabla g_0\\ ig_1\\ g_1\end{pmatrix} + c.c.,$$

$$\begin{aligned} U_2 &:= \begin{pmatrix}0_d\\ 0\\ -\tilde{f}_0\end{pmatrix} + \left(\varepsilon^{it/\varepsilon^2}\begin{pmatrix}\nabla g_1\\ \partial_t g_0\\ 0\end{pmatrix} + c.c.\right) + \sum_{|p|\geq 2} e^{ipt/\varepsilon^2}U_{2,p},\\ U_3 &:= \begin{pmatrix}0_d\\ 0\\ -\tilde{\tilde{f}}_0\end{pmatrix} + \left(\varepsilon^{it/\varepsilon^2}\begin{pmatrix}0_d\\ \partial_t g_1\\ 0\end{pmatrix} + c.c.\right) + \sum_{|p|\geq 2} e^{ipt/\varepsilon^2}U_{3,p}. \end{aligned} \tag{5.30}$$

Then we have:

Proposition 5.7. *Under the assumptions in Theorem 1.4, the approximate solution* $U_a^{(2)}$ *constructed by* (5.29) *and* (5.30) *satisfies* $U_a^{(2)} \in C([0, T_0^*); H^{s-3})$ *and solves*

$$\begin{cases} \partial_t U_a^{(2)} - \dfrac{1}{\varepsilon}A(\partial_x)U_a^{(2)} + \dfrac{1}{\varepsilon^2}A_0 U_a^{(2)} = F(U_a^{(2)}) - \varepsilon^2 R_\varepsilon^{(2)},\\ U_a^{(2)}(0) = U(0) - \varepsilon^2\Psi_\varepsilon^{(2)}. \end{cases}$$

Moreover, for any $T < T_0^*$*, there holds the estimates*

$$\sup_{0<\varepsilon<1}\left(\|R_\varepsilon^{(2)}\|_{L^\infty(0,T;H^{s-4})} + \|\Psi_\varepsilon^{(2)}\|_{H^{s-4}}\right) < +\infty. \tag{5.31}$$

The proof of Proposition 5.7 is the same as that of Proposition 5.5. So we omit the details here.

The approximate solution $U_a^{(2)}$ in Proposition 5.7 will be used to prove Theorem 5.2 in Section 5.4.

5.3 Proof of Theorem 5.1

Associate with the approximate solution $U_a^{(1)}$ in Proposition 5.5, we define the error

$$\dot{U} := \frac{U - U_a^{(1)}}{\varepsilon}, \tag{5.32}$$

where $U \in C([0, T_\varepsilon^*); H^{s-4})$ is the exact solution to (2.1)-(2.5). Then at least over the time interval $[0, \min\{T_\varepsilon^*, T_0^*\})$, $\dot{U}$ solves

$$\begin{cases} \partial_t \dot{U} - \frac{1}{\varepsilon} A(\partial_x)\dot{U} + \frac{1}{\varepsilon^2} A_0 \dot{U} = \frac{1}{\varepsilon}\left(F(U_a^{(1)} + \varepsilon \dot{U}) - F(U_a^{(1)})\right) + R_\varepsilon^{(1)}, \\ \dot{U}(0) = \Psi_\varepsilon^{(1)}, \end{cases} \tag{5.33}$$

where $R_\varepsilon^{(1)}$ and $\Psi_\varepsilon^{(1)}$ satisfy the uniform estimate in (5.19).

Concerning the well-posedness of Cauchy problem (5.33), we have the following proposition.

Proposition 5.8. *Under the assumptions in Theorem 1.3, the Cauchy problem (5.33) admits a unique solution $\dot{U} \in C([0, \widetilde{T}_\varepsilon^*); H^{s-4})$ where $\widetilde{T}_\varepsilon^*$ is the maximal existence time. Moreover, there holds*

$$\liminf_{\varepsilon \to 0} \widetilde{T}_\varepsilon^* \geq T_0^*, \tag{5.34}$$

and for any $T < \min\{T_0^*, \widetilde{T}_\varepsilon^*\}$:

$$\sup_{0<\varepsilon<1} \|\dot{U}\|_{L^\infty(0,T;H^{s-2})} \leq C(T). \tag{5.35}$$

Proof of Proposition 5.8. We calculate

$$\frac{1}{\varepsilon}\left(F(U_a^{(1)} + \varepsilon \dot{U}) - F(U_a^{(1)})\right) = F'(U_a)\dot{U} + \varepsilon \int_0^1 F''(U_a + \varepsilon\tau\dot{U})\dot{U}^2 \frac{(1-\tau)^2}{2} d\tau.$$

Since $U_a^{(1)} \in C([0, T_0^*); H^{s-2})$ and $F \in C^m,\ m > s$ with $s > d/2 + 4$, then for any $t < T_0^*$, there holds

$$\begin{aligned} &\left\| \frac{1}{\varepsilon}\left(F(U_a^{(1)} + \varepsilon \dot{U}) - F(U_a^{(1)})\right)(t) \right\|_{H^{s-4}} \leq \\ &\left(C(F, \|\dot{U}_a(t)\|_{H^{s-4}}) + \varepsilon\, C(F, \|\dot{U}_a(t)\|_{H^{s-4}}, \|\dot{U}(t)\|_{H^{s-4}}) \right) \|\dot{U}(t)\|_{H^{s-4}}. \end{aligned} \tag{5.36}$$

The system in $\dot{U}$ is semi-linear symmetric hyperbolic and the initial datum is uniformly bounded in H^{s-4}. By (5.36), the local-in-time well-posedness of Cauchy problem (5.33) in Sobolev space H^{s-4} is classical (see for instance Chapter 7 of [16]).

Moreover, if we denote $\widetilde{T}_\varepsilon^*$ to be the maximal existence time, the classical solution is in $C([0,\widetilde{T}_\varepsilon^*);H^{s-4})$ and there holds the estimate

$$\sup_{0<\varepsilon<1}\|\dot{U}\|_{L^\infty(0,T;H^{s-2})}\leq C(T),\quad \text{for any } T<\min\{\widetilde{T}_\varepsilon^*,T_0^*\}$$

and the criterion of the life-span

$$\widetilde{T}_\varepsilon^*<\infty\Longrightarrow\lim_{t\to\widetilde{T}_\varepsilon^*}\|\dot{U}\|_{L^\infty}=\infty. \tag{5.37}$$

It is left to prove (5.34) to finish the proof. Let $T<T_0^*$ be a arbitrary number. It is sufficient to show there exists $\varepsilon_0>0$ such that $\widetilde{T}_\varepsilon^*>T$ for any $0<\varepsilon<\varepsilon_0$. By classical energy estimates in Sobolev spaces for semi-linear symmetric hyperbolic system, we have for any $t<\min\{T,\widetilde{T}_\varepsilon^*\}$:

$$\frac{d}{dt}\|\dot{U}(t)\|_{H^{s-4}}\leq C(T)\left(1+\varepsilon\, C(\|\dot{U}(t)\|_{H^{s-4}})\right)\|\dot{U}(t)\|_{H^{s-4}}+\|R_\varepsilon^{(1)}\|_{L^\infty(0,T;H^s)}.$$

Here $C(\|\dot{U}(t)\|_{H^{s-4}})$ is continuous and increasing in $\|\dot{U}(t)\|_{H^{s-4}}$. Then Gronwall's inequality implies

$$\begin{aligned}\|\dot{U}(t)\|_{H^{s-4}}\leq &\exp\left(\int_0^t\left(C(T)(1+\varepsilon\, C(\|\dot{U}(\tau)\|_{H^{s-4}}))\right)d\tau\right)\|\dot{U}(0)\|_{H^{s-4}}\\ &+T\|R_\varepsilon^{(1)}\|_{L^\infty(0,T;H^{s-4})}.\end{aligned}\tag{5.38}$$

Let

$$M(T):=\exp\left(2C(T)T\right)\|\dot{U}(0)\|_{H^{s-4}}+T\|R_\varepsilon^{(1)}\|_{L^\infty(0,T;H^{s-4})}.$$

We then define

$$\mathbf{T}:=\sup\left\{t:\|\dot{U}\|_{L^\infty(0,t;H^{s-4})}\leq M(T)\right\}.$$

If $\mathbf{T}\leq\min\{T,\widetilde{T}_\varepsilon^*\}$, then for any $t<\mathbf{T}$, the inequality (5.38) implies

$$\|\dot{U}(t)\|_{H^{s-4}}\leq\exp\left\{TC(T)[1+\varepsilon\, C(M(T))]\right\}\|\dot{U}(0)\|_{H^{s-4}}+T\|R_\varepsilon^{(1)}\|_{L^\infty(0,T;H^{s-4})}.$$

Let

$$\varepsilon_0:=\left\{2C(M(T))\right\}^{-1}.$$

Then for any $0<\varepsilon<\varepsilon_0$, there holds

$$\|\dot{U}(t)\|_{H^{s-4}}\leq\exp\left(\frac{3}{2}C(T)T\right)\|\dot{U}(0)\|_{H^{s-4}}+T\|R_\varepsilon^{(1)}\|_{L^\infty(0,T;H^{s-4})}.$$

The classical continuation argument implies that

$$\mathbf{T}>\min\{T,\widetilde{T}_\varepsilon^*\},\quad\text{for any } 0<\varepsilon<\varepsilon_0. \tag{5.39}$$

By (5.37), we have $\widetilde{T}_\varepsilon^*\geq\mathbf{T}$. Together with (5.39), we deduce $\widetilde{T}_\varepsilon^*>T$. Since $T<T_0^*$ is a arbitrary number, we obtain (5.34) and complete the proof. $\square$

Now we are ready to prove Theorem 2.1. Given $U_a^{(1)}$ as in Proposition 5.5 and $\dot{U}$ the solution of (5.33), we can reconstruct U through (5.32):

$$U = U_a^{(1)} + \varepsilon \dot{U}$$

which solves (2.1)-(2.5). This implies that the maximal existence time T_ε^* of the solution $U \in C([0, T_\varepsilon^*); H^{s-4})$ satisfies

$$T_\varepsilon^* \geq \min\{T_0^*, \widetilde{T}_\varepsilon^*\}.$$

By (5.34) in Proposition 5.8, we obtain (5.1) in Theorem 5.1.

Finally, by (5.14), (5.17) and (5.35), we deduce (5.2) and we complete the proof of Theorem 5.1.

5.4 Proof of Theorem 5.2

Associate with the approximate solution $U_a^{(2)}$ in Proposition 5.7, we define the error

$$\dot{V} := \frac{U - U_a^{(2)}}{\varepsilon^2}. \tag{5.40}$$

Then the equation and initial datum for $\dot{V}$ are

$$\begin{cases} \partial_t \dot{V} - \dfrac{1}{\varepsilon} A(\partial_x) \dot{V} + \dfrac{1}{\varepsilon^2} A_0 \dot{V} = \dfrac{1}{\varepsilon^2} \Big(F(U_a^{(1)} + \varepsilon^2 \dot{V}) - F(U_a^{(1)}) \Big) + R_\varepsilon^{(2)}, \\ \dot{V}(0) = \Psi_\varepsilon^{(2)}, \end{cases} \tag{5.41}$$

where $R_\varepsilon^{(2)}$ and $\Psi_\varepsilon^{(2)}$ satisfy the uniform estimate in (5.31).

Similar to Proposition 5.8, we have

Proposition 5.9. *Under the assumptions in Theorem 5.2, the Cauchy problem (5.41) admits a unique solution $\dot{V} \in C([0, \hat{T}_\varepsilon^*); H^{s-4})$ where $\hat{T}_\varepsilon^*$ is the maximal existence time. Moreover, there holds*

$$\liminf_{\varepsilon \to 0} \hat{T}_\varepsilon^* \geq T_0^* \tag{5.42}$$

and for any $T < \min\{T_0^*, \hat{T}_\varepsilon^*\}$:

$$\sup_{0<\varepsilon<1} \|\dot{V}\|_{L^\infty(0,T;H^{s-4})} \leq C(T). \tag{5.43}$$

The proof is the same as the proof of Proposition 5.8. Theorem 5.2 follows from Proposition 5.9 through a similar argument as in the end of Section 5.3.

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