Homogenization of the stationary compressible Navier-Stokes equations in domains with tiny holes

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Abstract

We consider the homogenization problem for the stationary compressible Navier-Stokes equations describing a steady flow of a compressible Newtonian fluid in a bounded three dimensional domain. We focus on the case where the domain is perforated with very tiny holes for which the diameters are much smaller than their mutual distances. We show that the homogenization process does not change the motion of the fluids: In the asymptotic limit, we obtain again the same system of equations. This coincides with similar results for the stationary Stokes and stationary *incompressible* Navier-Stokes system.

Key words:

1 Introduction

Homogenization problems in the framework of fluid mechanics have gain a lot of interest. For Stokes and stationary *incompressible* Navier-Stokes equations, Allaire in [1, 2] gave a system study

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for physical domains perforated by a family of holes of different size. More specifically, consider a system of holes of diameter $O(\varepsilon^\alpha)$, where ε is their mutual distance. In three spatial dimensions, Allaire showed that when $\alpha < 3$, the limit fluid behavior is governed by the classical Darcy's law; when $\alpha = 3$, in the limit yields Brinkman law; when $\alpha > 3$, the the equations do not change in the homogenization process and the limit problem is determined by the same system of Stokes or Navier-Stokes equations. In the case $\alpha = 1$, meaning that the size of holes is proportional to their mutual distance, the results have been extended to the incompressible evolutionary Navier-Stokes equations by Mikelič [14], and to the compressible Navier-Stokes system by Masmoudi [13], and to the complete Navier-Stokes-Fourier system in [9]. In all the aforementioned cases, the asymptotic limit gives rise to Darcy's law.

In this paper, we focus on the homogenization process for the stationary compressible Navier-Stokes equations in a domain perforated by very tiny holes, where the diameter of the holes is taken to be of size $O(\varepsilon^\alpha)$, with $\alpha > 3$, where ε denotes the mutual distance between the holes. We start by a precise description of the physical domain. We consider a bounded domain Ω in R^3 of class C^2, and a family of holes (solid obstacles) $\{T^s_{\varepsilon,k}\}$, which are simply connected smooth domains satisfying

$$T^s_{\varepsilon,k} \subset \overline{T}^s_{\varepsilon,k} \subset B_{\varepsilon,k} \subset \overline{B}_{\varepsilon,k} \subset \varepsilon C_k \tag{1.1}$$

with

$$C_k := (0,1)^3 + k, \quad k \in Z^3, \qquad B_{\varepsilon,k} := B(x_k, b_0\varepsilon^\alpha) \text{ for some } x_k \in T^s_{\varepsilon,k},\ b_0 > 0. \tag{1.2}$$

For the sake of simplicity, we suppose that all holes $T^s_{\varepsilon,k} = \varepsilon^\alpha T^s$ are similar to the same set T^s - a simply connected domain of class C^2.

The corresponding family of ε-dependent perforated domains is defined as

$$\Omega_\varepsilon := \Omega \setminus \bigcup_{k\in K_\varepsilon} \overline{T}^s_{\varepsilon,k}, \quad K_\varepsilon := \{k \mid \varepsilon\overline{C}_k \subset \Omega\}. \tag{1.3}$$

It is easy to check that the number of holes contained in Ω satisfies

$$|K_\varepsilon| = \frac{|\Omega|}{\varepsilon^3}(1 + o(1)), \quad \text{as } \varepsilon \to 0. \tag{1.4}$$

We consider the following stationary Navier-Stokes system equations in Ω_ε:

$$\mathrm{div}_x(\varrho\mathbf{u}) = 0, \tag{1.5}$$

$$\mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{u}) + \varrho\mathbf{f} + \mathbf{g}, \tag{1.6}$$

$$\mathbb{S}(\nabla_x\mathbf{u}) = \mu\left(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\right) + \eta\mathrm{div}_x\mathbf{u}\mathbb{I},\ \mu > 0,\ \eta \geq 0. \tag{1.7}$$

Here, ϱ is the fluid mass density, $\mathbf{u}$ is the velocity field, $p = p(\varrho)$ denotes the pressure, $\mathbb{S} = \mathbb{S}(\nabla_x \mathbf{u})$ stands for the Newtonian viscous stress tensor, and μ, λ are the viscosity coefficients.

The density is nonnegative and the total mass of the fluid is fixed to be

$$\int_{\Omega_\varepsilon} \varrho \, \mathrm{d}x = M > 0. \tag{1.8}$$

In addition, we impose the no-slip boundary condition

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega_\varepsilon. \tag{1.9}$$

The fluid is driven by external forces here represented by the functions $\mathbf{f}$ and $\mathbf{g}$ satisfying

$$\|\mathbf{f}\|_{L^\infty(R^3;R^3)} + \|\mathbf{g}\|_{L^\infty(R^3;R^3)} \leq c. \tag{1.10}$$

We consider the pressure law of a typical form

$$p(\varrho) = a\varrho^\gamma, \; a > 0, \tag{1.11}$$

with the adiabatic exponent $\gamma \geq 1$. The range of γ we can handle by the homogenization technique will be specified below.

For a function f defined in Ω_ε, we use the symbol $\tilde{f}$ to denote the zero-extension of f in R^3, that is

$$\tilde{f} = f \quad \text{in } \Omega_\varepsilon, \qquad \tilde{f} = 0 \quad \text{in } R^3 \setminus \Omega_\varepsilon. \tag{1.12}$$

We also use c to a generic positive constant independent of the parameter ε. However, the specific value of c could change from line to line.

1.1 Weak solutions

We start by introducing the standard concept of *weak solution* to the compressible Navier-Stokes system.

Definition 1.1 *We say that* $[\varrho, \mathbf{u}]$ *is a* finite energy weak solution *of the Navier-Stokes equations (1.5 - 1.7) supplemented with the conditions (1.8 - 1.9) in the domain* Ω_ε *if:*

$$\varrho \geq 0 \;\; a.e. \;\; in \; \Omega_\varepsilon, \; \int_{\Omega_\varepsilon} \varrho \, \mathrm{d}x = M, \; \varrho \in L^{\beta(\gamma)}(\Omega_\varepsilon), \; for \; some \; \gamma \leq \beta(\gamma), \; \mathbf{u} \in W_0^{1,2}(\Omega_\varepsilon; R^3); \tag{1.13}$$

$$\int_{\Omega_\varepsilon} \varrho \mathbf{u} \cdot \nabla_x \psi \, \mathrm{d}x = 0, \tag{1.14}$$

$$\int_{\Omega_\varepsilon} \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + p(\varrho) \mathrm{div}_x \varphi - \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \varphi + (\varrho \mathbf{f} + \mathbf{g}) \cdot \varphi \, \mathrm{d}x = 0, \tag{1.15}$$

for any test function $\psi \in C_c^\infty(\Omega_\varepsilon)$ and any test function $\varphi \in C_c^\infty(\Omega_\varepsilon; R^3)$. Moreover, there holds the energy inequality*:*

$$\int_{\Omega_\varepsilon} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \, \mathrm{d}x \leq \int_{\Omega_\varepsilon} (\varrho \mathbf{f} + \mathbf{g}) \cdot \mathbf{u} \, \mathrm{d}x. \tag{1.16}$$

Furthermore, we introduce *renormalized weak solutions.*

Definition 1.2 *We say a finite energy weak solution $[\varrho, \mathbf{u}]$ is a renormalized weak solution if its zero extension $[\tilde{\varrho}, \tilde{\mathbf{u}}]$ (see the notation (1.12)) satisfies*

$$\mathrm{div}_x(\tilde{\varrho}\tilde{\mathbf{u}}) = 0, \quad \mathrm{div}_x(b(\tilde{\varrho})\tilde{\mathbf{u}}) + (\tilde{\varrho} b'(\tilde{\varrho}) - b(\tilde{\varrho}))\mathrm{div}_x \tilde{\mathbf{u}} = 0, \quad \textit{in } \mathcal{D}'(R^3), \tag{1.17}$$

for any $b \in C^0([0,\infty)) \cap C^1((0,\infty))$ such that

$$b'(s) \leq c\, s^{-\lambda_0} \textit{ for } s \in (0,1], \quad b'(s) \leq c\, s^{\lambda_1} \textit{ for } s \in [1,\infty), \tag{1.18}$$

with

$$c > 0, \quad \lambda_0 < 1, \quad -1 < \lambda_1 \leq \frac{\beta(\gamma)}{2} - 1. \tag{1.19}$$

Remark 1.1 *The* existence *of finite energy renormalized weak solutions to the Navier-Stokes equations (1.5–1.7) for fixed $\varepsilon > 0$ is known for certain range of the adiabatic exponent γ. The first global result has been obtained by Lions [12] for the case $\gamma > 5/3$. Extensions to lower values of γ were obtained by Březina, Novotný [4], Frehse et al. [10], and Plotnikov, Sokolowski [16]. Moreover, as shown in Theorem 4.3 in the monograph of Novotný, Straškraba [15], any finite energy weak solution $[\varrho, \mathbf{u}]$ satisfies*

$$\varrho \in L^{\beta(\gamma)}, \quad \beta(\gamma) = 3(\gamma - 1) \;\; \textit{if} \;\; 3/2 < \gamma < 3, \quad \beta(\gamma) = 2\gamma \;\; \textit{if} \;\; \gamma \geq 3. \tag{1.20}$$

Remark 1.2 *In view of DiPerna-Lions's transport theory (see Section II.3 in [5] and the improvement in Lemma 3.3 in [15]), for any $r \in L^\beta(\Omega)$, $\beta \geq 2$, and any $\mathbf{v} \in W_0^{1,2}(\Omega)$, where $\Omega \subset R^3$ is a bounded C^2 domain, a couple of functions $[r, \mathbf{v}]$ satisfying*

$$\mathrm{div}_x(r\mathbf{v}) = 0 \quad \textit{in} \quad \mathcal{D}'(\Omega),$$

satisfies also the renormalized equations

$$\mathrm{div}_x(r\mathbf{v}) = 0, \quad \mathrm{div}_x(b(r)\mathbf{v}) + (r b'(r) - b(r))\mathrm{div}_x \mathbf{v} = 0, \quad \textit{in } \mathcal{D}'(R^3),$$

for any b satisfying (1.18 - 1.19) provided r and $\mathbf{v}$ have been extended to be zero outside Ω.

Hence, if $\gamma \geq 5/3$, any finite energy weak solution in the sense of Definition 1.1 is also a renormalized weak solution in the sense of Definition 1.2. The condition $\gamma \geq 5/3$ ensures that $\varrho \in L^2(\Omega_\varepsilon)$ by (1.20).

1.2 Main result

In this paper, we consider the case, where the adiabatic exponent γ in the pressure law (1.11) satisfies

$$\gamma \geq 3. \tag{1.21}$$

Note that the same restriction has been imposed by Masmoudi [13] to avoid certain sofar unsurmountable difficulties connected with the compressible Navier-Stokes system.

Our main result is the following:

Theorem 1.1 *Let*

$$M > 0, \ \gamma \geq 3$$

be given. Let $[\varrho_\varepsilon, \mathbf{u}_\varepsilon]_{0<\varepsilon<1}$ *be a family of finite energy renormalized weak solutions to the compressible Navier-Stokes system (1.5–1.9) in* Ω_ε*, where* $\mathbf{f}$, $\mathbf{g}$ *obey (1.10). Suppose that the size parameter* α *of the holes in (1.2) satisfies* $\alpha > 3$.

Then

$$\sup_{0<\varepsilon<1} \left\{ \|\varrho_\varepsilon\|_{L^{2\gamma}(\Omega_\varepsilon)} + \|\mathbf{u}\|_{W_0^{1,2}(\Omega_\varepsilon)} \right\} < c, \tag{1.22}$$

and, up to a substraction of subsequence, the extensions $[\tilde{\varrho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon]$ *satisfy*

$$\tilde{\varrho}_\varepsilon \to \varrho \text{ weakly in } L^{\beta(\gamma)}(\Omega), \quad \tilde{\mathbf{u}}_\varepsilon \to \mathbf{u} \text{ weakly in } W_0^{1,2}(\Omega). \tag{1.23}$$

where the limit $[\varrho, \mathbf{u}]$ *is a finite energy renormalized weak solution to the same problem (1.5–1.9) in the limit domain* Ω.

We find that the limit equations are the same as the original ones. This means that the homogenization process does not change the motion of the fluid when the holes (obstacles) are very small. The rest of the paper is devoted to the proof of Theorem 1.1. Although the result is formally the same as for the incompressible case studied by Allaire [1], [2], the technique for the compressible system is rather different. Moreover, in contrast with the critical case studied by Masmoudi [13], we do not perform any extra solution scaling and must establish the necessary bounds by means of the so-called Bogovskii's operator, the norm of which must be *independent* of the parameter ε. The construction of such an operator carried over in Section 2 represents the main step of the proof. The paper is finished by a short discussion in Section 3.

2 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1.

2.1 Bogovskii's operator

We introduce a suitable inverse of the divergence, commonly known as Bogovskii's operator, see Bogovskii [3], Galdi [11].

Lemma 2.1 *Let Ω_ε be defined through (1.1)–(1.3) with $\alpha \geq 3$.*

Then there exists a linear operator $\mathcal{B}_\varepsilon : L^2(\Omega_\varepsilon)/R \to W_0^{1,2}(\Omega_\varepsilon; R^3)$ such that for any $f \in L^2(\Omega_\varepsilon)$ and $\int_{\Omega_\varepsilon} f \ \mathrm{d}x = 0$, there holds

$$\mathrm{div}_x(\mathcal{B}_\varepsilon(f)) = f \quad \textit{in} \quad \Omega_\varepsilon, \quad \|\mathcal{B}_\varepsilon(f)\|_{W_0^{1,2}(\Omega_\varepsilon;R^3)} \leq c \ \|f\|_{L^2(\Omega_\varepsilon)} \tag{2.1}$$

for some c independent of ε.

The existence of such a linear operator is nowadays standard, see e.g. Novotný, Straškraba [15, Chapter 3]. The key point here is to show the uniform estimate (2.1) with the constant c independent of ε. To this end, we adapt the construction of the *restriction operator* R_ε by Allaire [1]. We start with the following result [1, Section 2.2]:

Lemma 2.2 *For Ω_ε as in (1.1)-(1.3) with $\alpha \geq 3$, there exits a linear bounded operator R_ε mapping $W_0^{1,2}(\Omega; R^3)$ to $W_0^{1,2}(\Omega_\varepsilon; R^3)$ such that*

$$\mathbf{u} \in W_0^{1,2}(\Omega_\varepsilon; R^3) \Longrightarrow R_\varepsilon(\tilde{\mathbf{u}}) = \mathbf{u} \ \textit{in} \ \Omega_\varepsilon, \tag{2.2}$$

$$\mathrm{div}_x \mathbf{u} = 0 \ \textit{in} \ \Omega \Longrightarrow \mathrm{div}_x R_\varepsilon(\mathbf{u}) = 0 \ \textit{in} \ \Omega_\varepsilon, \tag{2.3}$$

$$\|R_\varepsilon(\mathbf{u})\|_{W_0^{1,2}(\Omega_\varepsilon;R^3)} \leq c \ \|\mathbf{u}\|_{W_0^{1,2}(\Omega;R^3)}, \ \textit{c independent of } \varepsilon. \tag{2.4}$$

Inspecting the proof of Lemma 2.2 in Section 2.2 of [1], we observe that the restriction operator may be constructed in the following way:

Let $b_1 > 0$ be chosen such that

$$B(x_k, b_1\varepsilon) \subset \varepsilon C_k, \quad \overline{B}_{\varepsilon,k} = \overline{B(x_k, b_0\varepsilon^\alpha)} \subset B(x_k, b_1\varepsilon).$$

Consider the following decomposition of each cube εC_k with $k \in K_\varepsilon$:

$$\varepsilon\overline{C}_k = T^s_{\varepsilon,k} \cup \overline{E}_{\varepsilon,k} \cup \overline{F}_{\varepsilon,k} \text{ with } E_{\varepsilon,k} := B(x_k, b_1\varepsilon) \setminus T^s_{\varepsilon,k}, \ F_{\varepsilon,k} := (\varepsilon C_k) \setminus B(x_k, b_1\varepsilon).$$

For any $\mathbf{u} \in W_0^{1,2}(\Omega; R^3)$, we define R_ε through

$$R_\varepsilon(\mathbf{u}) = \mathbf{u} \text{ on } \varepsilon C_k \cap \Omega, \quad \text{for } k \notin K_\varepsilon, \tag{2.5}$$

and for $k \in K_\varepsilon$:

$$R_\varepsilon(\mathbf{u}) = \mathbf{u} \text{ on } F_{\varepsilon,k}, \quad R_\varepsilon(\mathbf{u}) = 0 \text{ on } T^s_{\varepsilon,k}, \quad R_\varepsilon(\mathbf{u}) = \mathbf{v}_{\varepsilon,k} \text{ in } E_{\varepsilon,k}, \tag{2.6}$$

where $\mathbf{v}_{\varepsilon,k} \in W^{1,2}(\varepsilon C_k; R^3)$ solves the following Stokes problem

$$\nabla p_{\varepsilon,k} - \Delta \mathbf{v}_{\varepsilon,k} = -\Delta \mathbf{u} \quad \text{in } E_{\varepsilon,k}, \tag{2.7}$$

$$\mathrm{div}_x \mathbf{v}_{\varepsilon,k} = \mathrm{div}_x \mathbf{u} + \frac{1}{|E_{\varepsilon,k}|} \int_{T^s_{\varepsilon,k}} \mathrm{div}_x \mathbf{u} \, \mathrm{d}x \quad \text{in } E_{\varepsilon,k}, \tag{2.8}$$

$$\mathbf{v}_{\varepsilon,u} = \mathbf{u} \text{ on } \partial E_{\varepsilon,k} - \partial T^s_{\varepsilon,k}, \quad \mathbf{v}_{\varepsilon,k} = 0 \text{ on } \partial T^s_{\varepsilon,k}. \tag{2.9}$$

The operator R_ε defined by (2.5–2.9) is a restriction operator satisfying (2.2–2.4).

Now we use Lemma 2.2, along with the properties of the restriction operator stated above, to prove Lemma 2.1.

Proof of Lemma 2.1. For $f \in L^2(\Omega_\varepsilon)$ with $\int_{\Omega_\varepsilon} f \, \mathrm{d}x = 0$, we consider the extension

$$\tilde{f} = f \text{ in } \Omega_\varepsilon, \quad \tilde{f} = 0 \text{ on } \Omega \setminus \Omega_\varepsilon = \bigcup_{k \in K_\varepsilon} T^s_{\varepsilon,k}. \tag{2.10}$$

Then, by employing the classical Bogovskii's operator defined on the domain Ω, we obtain $\mathbf{u} := \mathcal{B}(f) \in W_0^{1,2}(\Omega; R^3)$ such that

$$\mathrm{div}_x \mathbf{u} = \tilde{f} \text{ in } \Omega \quad \text{and } \|\mathbf{u}\|_{W_0^{1,2}(\Omega;R^3)} \le c \, \|\tilde{f}\|_{L^2(\Omega)} = c\|f\|_{L^2(\Omega_\varepsilon)} \tag{2.11}$$

for some c that only depends on Ω. Moreover, by virtue of (2.10), we have

$$\mathrm{div}_x \mathbf{u} = \tilde{f} = 0 \quad \text{on } T^s_{\varepsilon,k}.$$

Let R_ε be the restriction operator constructed through (2.5 – 2.9).

In particular, it is easy to check that equation (2.8) gives rise to

$$\mathrm{div}_x \mathbf{v}_{\varepsilon,k} = \mathrm{div}_x \mathbf{u} = f \quad \text{in } E_{\varepsilon,k}, \tag{2.12}$$

whenever $\mathbf{u}$ satifies (2.11). In addition, one has $R_\varepsilon(\mathbf{u}) = \mathbf{u}$ in $\Omega_\varepsilon \setminus (\bigcup_{k \in K_\varepsilon} E_{\varepsilon,k})$. Together with (2.6) and (2.12), we therefore conclude that

$$\mathrm{div}_x R_\varepsilon(u) = f \quad \text{in} \quad \Omega_\varepsilon.$$

Thus to prove Lemma 2.1, it is enough to define

$$\mathcal{B}_\varepsilon(f) := R_\varepsilon(\mathbf{u}) = R_\varepsilon(\mathcal{B}(\tilde{f})),$$

where $\mathcal{B}$ is the classical Bogovskii's operator on Ω. The operator norm estimate (2.1) follows from the estimate for the restriction operator in (2.4).

2.2 Uniform bounds

Our goal in this section is to show the uniform estimate (1.22) under the assumption $\gamma \geq 3$. Note that, in accordance with what we said in Remark 1.1, we already know that

$$\varrho_\varepsilon \in L^{2\gamma}(\Omega_\varepsilon), \quad \mathbf{u}_\varepsilon \in W_0^{1,2}(\Omega_\varepsilon) \tag{2.13}$$

for any *fixed* ε. However, we have to establish *uniform bounds* on the norms $\|\varrho_\varepsilon\|_{L^{2\gamma}(\Omega_\varepsilon)}$ and $\|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(\Omega_\varepsilon)}$ independent of $\varepsilon \in (0,1)$.

By the energy inequality (1.16), together with Korn's inequality and Hölder's inequality, we have

$$\mu\|\nabla_x \mathbf{u}_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \leq \|\mathbf{f}\|_{L^\infty(\Omega_\varepsilon)}\|\varrho_\varepsilon\|_{L^{\frac{6}{5}}(\Omega_\varepsilon)}\|\mathbf{u}_\varepsilon\|_{L^6(\Omega_\varepsilon)} + \|\mathbf{g}\|_{L^\infty(\Omega_\varepsilon)}\|\mathbf{u}_\varepsilon\|_{L^1(\Omega_\varepsilon)}. \tag{2.14}$$

Since $\mathbf{u}_\varepsilon \in W_0^{1,2}(\Omega_\varepsilon)$ has zero trace on the boundary, we may use Poincaré's inequality and the Sobolev embedding to obtain

$$\|\mathbf{u}_\varepsilon\|_{L^6(\Omega_\varepsilon)} \leq c\,\|\nabla_x \mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \tag{2.15}$$

for some constant c independent of the domain Ω_ε. Thus we obtain

$$\|\nabla_x \mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\mathbf{u}_\varepsilon\|_{L^6(\Omega_\varepsilon)} \leq c\,\left(\|\mathbf{f}\|_{L^\infty(\Omega_\varepsilon)}\|\varrho_\varepsilon\|_{L^{\frac{6}{5}}(\Omega_\varepsilon)} + \|\mathbf{g}\|_{L^\infty(\Omega_\varepsilon)}\right) \leq c\,\left(\|\varrho_\varepsilon\|_{L^{\frac{6}{5}}(\Omega_\varepsilon)} + 1\right). \tag{2.16}$$

Next, we introduce a test function

$$\varphi(x) := \mathcal{B}_\varepsilon\left(\varrho_\varepsilon^\gamma(x) - \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} \varrho_\varepsilon^\gamma \,\mathrm{d}x\right), \tag{2.17}$$

where $\mathcal{B}_\varepsilon$ is the Bogovskii's operator introduced in Lemma 2.1. We remark that such φ is well defined since $\varrho_\varepsilon^\gamma \in L^2(\Omega_\varepsilon)$ due to (2.13). Then by Lemma 2.1, we have

$$\mathrm{div}_x\varphi(x) = \varrho_\varepsilon^\gamma(x) - \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} \varrho_\varepsilon^\gamma \,\mathrm{d}x \quad \text{in } \Omega_\varepsilon, \tag{2.18}$$

$$\|\varphi\|_{W_0^{1,2}(\Omega_\varepsilon)} \leq c\,\left(\|\varrho_\varepsilon^\gamma\|_{L^2(\Omega_\varepsilon)} + \|\varrho_\varepsilon^\gamma\|_{L^1(\Omega_\varepsilon)}\right) \leq c\,\|\varrho_\varepsilon\|^\gamma_{L^{2\gamma}(\Omega_\varepsilon)}. \tag{2.19}$$

We plug this test function φ into (1.15) to obtain

$$\int_{\Omega_\varepsilon} p(\varrho_\varepsilon)\varrho_\varepsilon^\gamma \,\mathrm{d}x = \sum_{j=1}^4 I_j \tag{2.20}$$

with

$$I_1 := \int_{\Omega_\varepsilon} p(\varrho_\varepsilon)\,\mathrm{d}x \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} \varrho_\varepsilon^\gamma \,\mathrm{d}x, \quad I_2 := \int_{\Omega_\varepsilon} \mu\nabla_x \mathbf{u}_\varepsilon : \nabla_x\varphi \,\mathrm{d}x + \int_{\Omega_\varepsilon}\left(\frac{\mu}{3}+\eta\right)\mathrm{div}_x\mathbf{u}_\varepsilon : \mathrm{div}_x\varphi \,\mathrm{d}x, \tag{2.21}$$

$$I_3 := -\int_{\Omega_\varepsilon} \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \varphi \, \mathrm{d}x, \quad I_4 := -\int_{\Omega_\varepsilon} (\varrho_\varepsilon \mathbf{f} + \mathbf{g}) \varphi \, \mathrm{d}x.$$

Now we estimate I_j one by one. By the interpolation between L^q spaces,

$$I_1 = \frac{a}{|\Omega_\varepsilon|} \|\varrho_\varepsilon\|_{L^\gamma(\Omega_\varepsilon)}^{2\gamma} \le \frac{a}{|\Omega_\varepsilon|} \left(\|\varrho_\varepsilon\|_{L^1(\Omega_\varepsilon)}^{\theta_1} \|\varrho_\varepsilon\|_{L^{2\gamma}(\Omega_\varepsilon)}^{1-\theta_1} \right)^{2\gamma} = \frac{aM^{2\gamma\theta_1}}{|\Omega_\varepsilon|} \|\varrho_\varepsilon\|_{L^{2\gamma}(\Omega_\varepsilon)}^{2\gamma - 2\gamma\theta_1},$$

where M is the total mass and $\theta_1 \in (0,1)$ is determined by

$$\frac{1}{\gamma} = \theta_1 + \frac{1-\theta_1}{2\gamma}.$$

By (2.16) and (2.19), we have for I_2:

$$I_2 \le c \|\nabla_x \mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \|\nabla_x \varphi\|_{L^2(\Omega_\varepsilon)} \le c \, \left(\|\varrho_\varepsilon\|_{L^{\frac{6}{5}}(\Omega_\varepsilon)} + 1 \right) \|\varrho_\varepsilon\|_{L^{2\gamma}(\Omega_\varepsilon)}^\gamma \le c \, \|\varrho_\varepsilon\|_{L^{2\gamma}(\Omega_\varepsilon)}^\gamma \left(\|\varrho_\varepsilon\|_{L^{2\gamma}(\Omega_\varepsilon)} + 1 \right).$$

Again by (2.16) and (2.19), we have for I_3:

$$\begin{aligned} I_3 &\le \|\varrho_\varepsilon\|_{L^6(\Omega_\varepsilon)} \|\mathbf{u}_\varepsilon\|_{L^6(\Omega_\varepsilon)}^2 \|\nabla_x \varphi\|_{L^2(\Omega_\varepsilon)} \le c \, \|\varrho_\varepsilon\|_{L^6(\Omega_\varepsilon)} \left(\|\varrho_\varepsilon\|_{L^{\frac{6}{5}}(\Omega_\varepsilon)}^2 + 1 \right) \|\varrho_\varepsilon\|_{L^{2\gamma}(\Omega_\varepsilon)}^\gamma \\ &\le c \, \|\varrho_\varepsilon\|_{L^{2\gamma}(\Omega_\varepsilon)}^{\gamma+1} \left(\|\varrho_\varepsilon\|_{L^1(\Omega_\varepsilon)}^{2\theta_2} \|\varrho_\varepsilon\|_{L^{2\gamma}(\Omega_\varepsilon)}^{2(1-\theta_2)} + 1 \right) \\ &= c \, M^{2\theta_2} \, \|\varrho_\varepsilon\|_{L^{2\gamma}(\Omega_\varepsilon)}^{\gamma+3-2\theta_2} + c \, \|\varrho_\varepsilon\|_{L^{2\gamma}(\Omega_\varepsilon)}^{\gamma+1}, \end{aligned}$$

where $\theta_2 \in (0,1)$ is determined by

$$\frac{5}{6} = \theta_2 + \frac{1-\theta_2}{2\gamma}.$$

For I_4, we have

$$I_4 \le c \, \left(\|\varrho_\varepsilon\|_{L^2} + 1 \right) \|\varrho_\varepsilon\|_{L^{2\gamma}(\Omega_\varepsilon)}^\gamma \le c \, \|\varrho_\varepsilon\|_{L^{2\gamma}(\Omega_\varepsilon)}^\gamma + c \, \|\varrho_\varepsilon\|_{L^{2\gamma}(\Omega_\varepsilon)}^{\gamma+1}.$$

We sum up the estimates for I_j and obtain

$$a \|\varrho_\varepsilon\|_{L^{2\gamma}(\Omega_\varepsilon)}^{2\gamma} = \int_{\Omega_\varepsilon} p(\varrho_\varepsilon) \varrho_\varepsilon^\gamma \, \mathrm{d}x = \sum_{j=1}^4 I_j \le c \, \left(\|\varrho_\varepsilon\|_{L^{2\gamma}(\Omega_\varepsilon)}^{2\gamma - \beta_1} + 1 \right) \tag{2.22}$$

for some $\beta_1 > 0$. Precisely, we can choose β_1 as

$$\beta_1 = \min\{2\gamma\theta_1, \gamma - 1, \gamma - 3 + 2\theta_2\}.$$

From (2.22), we finally obtain

$$\|\varrho_\varepsilon\|_{L^{2\gamma}(\Omega_\varepsilon)} \leq c, \quad \text{for some } c \text{ independent of } \varepsilon. \tag{2.23}$$

We then go back (2.16) to derive

$$\|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(\Omega_\varepsilon)} \leq c, \quad \text{for some } c \text{ independent of } \varepsilon. \tag{2.24}$$

This completes the proof of the uniform estimate claimed in (1.22).

2.3 Equations in a fixed domain

In this section, we derive the equations for the extended functions $[\tilde{\varrho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon]$ in Ω, where $[\varrho_\varepsilon, \mathbf{u}_\varepsilon]$ is the finite energy renormalized weak solution in Theorem 1.1. In accordance with the uniform estimates (2.23 – 2.24), we have

$$\|\tilde{\varrho}_\varepsilon\|_{L^{2\gamma}(\Omega)} + \|\tilde{\mathbf{u}}_\varepsilon\|_{W_0^{1,2}(\Omega)} \leq c, \quad \text{for some } c \text{ independent of } \varepsilon. \tag{2.25}$$

2.3.1 Continuity equation

First we claim:

Proposition 2.1 *Under the assumptions in Theorem 1.1, the extended functions $\tilde{\varrho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon$ satisfy*

$$\mathrm{div}_x(\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon) = 0, \quad \mathrm{div}_x(b(\tilde{\varrho}_\varepsilon)\tilde{\mathbf{u}}_\varepsilon) + (\tilde{\varrho}_\varepsilon b'(\tilde{\varrho}_\varepsilon) - b(\tilde{\varrho}_\varepsilon))\mathrm{div}_x \tilde{\mathbf{u}}_\varepsilon = 0 \quad \textit{in } \mathcal{D}'(R^3) \tag{2.26}$$

for any $b \in C^0([0,\infty)) \cap C^1((0,\infty))$ satisfying (1.18 - 1.19).

This is a direct conclusion from the fact that $[\varrho_\varepsilon, \mathbf{u}_\varepsilon]$ is a renomalized weak solution.

2.3.2 Momentum equation

The following proposition is crucial in the proof of convergence.

Proposition 2.2 *Under the assumptions in Theorem 1.1, we have*

$$\mathrm{div}_x(\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \otimes \tilde{\mathbf{u}}_\varepsilon) + \nabla_x p(\tilde{\varrho}_\varepsilon) = \mathrm{div}_x \mathbb{S}(\nabla_x \tilde{\mathbf{u}}_\varepsilon) + \tilde{\varrho}_\varepsilon \tilde{\mathbf{f}} + \tilde{\mathbf{g}} + F_\varepsilon, \quad \textit{in } \mathcal{D}'(\Omega; R^3), \tag{2.27}$$

where F_ε is a distribution satisfying

$$|< F_\varepsilon, \varphi >_{\mathcal{D}' \times \mathcal{D}}| \leq c\ \varepsilon^\sigma \|\varphi\|_{L^r(\Omega;R^3)} + c\ \varepsilon^{\frac{3(\alpha-1)\sigma_0}{2(2+\sigma_0)}} \|\nabla_x \varphi\|_{L^{2+\sigma_0}(\Omega;R^{3\times 3}))}, \tag{2.28}$$

for any $\varphi \in C_c^\infty((0,T)\times\Omega;R^3)$ and some constants $c>0$, $\sigma>0$, $\sigma_0>0$ and $1<r<\infty$ independent of ε. In particular, we can choose

$$\sigma := \frac{\alpha-3}{4}, \quad r := \frac{12(\alpha-1)}{\alpha-3}, \quad \sigma_0 \in (0,\infty). \tag{2.29}$$

Proof of Proposition 2.2. Let $\varphi \in C_c^\infty((0,T)\times\Omega;R^3)$ be any test function. It is sufficient to show

$$\begin{aligned} I^\varepsilon &:= \int_\Omega \tilde\varrho_\varepsilon \tilde{\mathbf{u}}_\varepsilon \otimes \tilde{\mathbf{u}}_\varepsilon : \nabla_x\varphi + p(\tilde\varrho_\varepsilon)\mathrm{div}_x\varphi - \mathbb{S}(\nabla_x\tilde{\mathbf{u}}_\varepsilon) : \nabla_x\varphi + \tilde\varrho_\varepsilon\tilde{\mathbf{f}}\varphi + \tilde{\mathbf{g}}\varphi \,\mathrm{d}x \\ &\le c\,\varepsilon^\sigma \|\varphi\|_{L^r(\Omega;R^3)} + c\,\varepsilon^{\frac{3(\alpha-1)\sigma_0}{2(2+\sigma_0)}}\|\nabla_x\varphi\|_{L^{2+\sigma_0}(\Omega;R^{3\times3}))}, \end{aligned} \tag{2.30}$$

where c, σ, r and σ_0 are the constants enjoying the properties claimed in Proposition 2.2.

Using (1.1 - 1.3) we can find cut-off functions $g_\varepsilon \in C_c^\infty(\Omega)$ satisfying

$$0\le g_\varepsilon\le 1, \quad g_\varepsilon = 0 \text{ on } \bigcup_{k\in K_\varepsilon} T^s_{\varepsilon,k}, \quad g_\varepsilon = 1 \text{ in } \Omega\setminus\bigcup_{k\in K_\varepsilon} B_{\varepsilon,k}, \quad \|\nabla_x g_\varepsilon\|_{L^\infty(\Omega)} \le c\,\varepsilon^{-\alpha}. \tag{2.31}$$

Together with (1.4), we have for any $1\le q\le\infty$:

$$\|1-g_\varepsilon\|_{L^q(\Omega)} \le c\,\varepsilon^{\frac{3(\alpha-1)}{q}}, \quad \|\nabla_x g_\varepsilon\|_{L^q(\Omega)} \le c\,\varepsilon^{\frac{3(\alpha-1)}{q}-\alpha}. \tag{2.32}$$

Then direct calculation gives

$$I^\varepsilon = \int_{\Omega_\varepsilon} \varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon : \nabla_x(g_\varepsilon\varphi) + p(\varrho_\varepsilon)\mathrm{div}_x(g_\varepsilon\varphi) - \mathbb{S}(\nabla_x\mathbf{u}_\varepsilon) : \nabla_x(g_\varepsilon\varphi) + (\varrho_\varepsilon\mathbf{f}+\mathbf{g})(g_\varepsilon\varphi)\,\mathrm{d}x + \sum_{j=1}^3 I_j^\varepsilon, \tag{2.33}$$

where

$$I_1^\varepsilon := \int_\Omega \tilde\varrho_\varepsilon\tilde{\mathbf{u}}_\varepsilon\otimes\tilde{\mathbf{u}}_\varepsilon : (1-g_\varepsilon)\nabla_x\varphi - \tilde\varrho_\varepsilon\tilde{\mathbf{u}}_\varepsilon\otimes\tilde{\mathbf{u}}_\varepsilon : (\nabla_x g_\varepsilon\otimes\varphi)\,\mathrm{d}x,$$

$$I_2^\varepsilon := \int_\Omega p(\tilde\varrho_\varepsilon)(1-g_\varepsilon)\mathrm{div}_x\varphi - p(\tilde\varrho_\varepsilon)\nabla_x g_\varepsilon\cdot\varphi\,\mathrm{d}x,$$

$$I_3^\varepsilon := \int_\Omega \mathbb{S}(\nabla_x\tilde{\mathbf{u}}_\varepsilon) : (1-g_\varepsilon)\nabla_x\varphi + \mathbb{S}(\nabla_x\tilde{\mathbf{u}}_\varepsilon) : (\nabla_x g_\varepsilon\otimes\varphi)\,\mathrm{d}x.$$

Since $g_\varepsilon\varphi \in C_c^\infty(\Omega_\varepsilon)$ is a test function for the stationary Navier-Stokes equations in Ω_ε, we have

$$I^\varepsilon = \sum_{j=1}^3 I_j^\varepsilon.$$

For I_1, using hypothesis $\gamma\ge3$ and Hölder's inequality, we get

$$I_1 \le c\,\|\tilde\varrho_\varepsilon\|_{L^{2\gamma}(\Omega)}\|\tilde{\mathbf{u}}_\varepsilon\|^2_{L^6(\Omega)}(\|(1-g_\varepsilon)\nabla_x\varphi\|_{L^2(\Omega)} + \|\nabla_x g_\varepsilon\otimes\varphi\|_{L^2(\Omega)}). \tag{2.34}$$

By the uniform estimates (2.23 - 2.24 - 2.25) and Hölder's inequality, we have

$$I_1 \leq c \left(\|(1-g_\varepsilon)\|_{L^{\frac{2(2+\sigma_0)}{\sigma_0}}(\Omega)} \|\nabla_x \varphi\|_{L^{2+\sigma_0}(\Omega)} + \|\nabla_x g_\varepsilon\|_{L^{r_1}(\Omega)} \|\varphi\|_{L^{r_2}(\Omega)} \right), \tag{2.35}$$

where

$$\sigma_0 \in (0,\infty), \quad r_j \in (2,\infty), \quad \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{2}.$$

Then by (2.32), we deduce

$$I_1 \leq c\, \varepsilon^{\frac{3(\alpha-1)\sigma_0}{2(2+\sigma_0)}} \|\nabla_x \varphi\|_{L^{2+\sigma_0}(\Omega))} + c\, \varepsilon^{\frac{3(\alpha-1)}{r_1}-\alpha} \|\varphi\|_{L^{r_2}(\Omega)},$$

where the number

$$\sigma := \frac{3(\alpha-1)}{r_1} - \alpha = \frac{3(\alpha-1)}{2} - \alpha - \frac{3(\alpha-1)}{r_2} = \frac{\alpha-3}{2} - \frac{3(\alpha-1)}{r_2}$$

is strictly positive as long as

$$r_2 > \frac{6(\alpha-1)}{\alpha-3}.$$

In particular, we can choose

$$r_2 = \frac{12(\alpha-1)}{\alpha-3} \text{ such that } \sigma := \frac{3(\alpha-1)}{r_1} - \alpha = \frac{\alpha-3}{4},$$

which is exactly (2.29).

Seeing that I_2^ε and I_3^ε can be handled similarly to I_1^ε, we have completed the proof of Proposition 2.2.

2.4 Passing to the limit

By the uniform estimates (2.25), up to a substraction of subsequence, we have

$$\tilde{\varrho}_\varepsilon \to \varrho \quad \text{weakly in} \quad L^{2\gamma}(\Omega), \quad \tilde{\mathbf{u}}_\varepsilon \to \mathbf{u} \quad \text{weakly in} \quad W_0^{1,2}(\Omega; R^3). \tag{2.36}$$

It is left to show the limit $[\varrho, \mathbf{u}]$ represents a finite energy renormalized weak solution to (1.5 – 1.9) in Ω.

2.4.1 Strong convergence of velocity

By compact Sobolev embedding, we have the strong convergence

$$\tilde{\mathbf{u}}_\varepsilon \to \mathbf{u} \quad \text{strongly in} \quad L^q(\Omega; R^3) \quad \text{for any } 1 \leq q < 6. \tag{2.37}$$

Together with the weak convergence of the density, we have the weak convergence of nonlinear terms:

$$\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \to \varrho \mathbf{u} \quad \text{weakly in} \quad L^q(\Omega; R^3) \quad \text{for any } 1 < q < \tfrac{6\gamma}{3+\gamma}, \tag{2.38}$$

and

$$\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \otimes \tilde{\mathbf{u}} \to \varrho \mathbf{u} \otimes \mathbf{u} \quad \text{weakly in} \quad L^q(\Omega; R^{3\times 3}) \quad \text{for any } 1 < q < \tfrac{6\gamma}{3+2\gamma}. \tag{2.39}$$

Then we let $\varepsilon \to 0$ in the first equation of (2.26) and in equation (2.27) to deduce the following two equations in $\mathcal{D}'(\Omega)$:

$$\mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{2.40}$$

$$\mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x \overline{p(\varrho)} = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \varrho \mathbf{f} + \mathbf{g}. \tag{2.41}$$

Here $\overline{p(\varrho)}$ is the weak limit of $p(\tilde{\varrho}_\varepsilon)$ in $L^2(\Omega)$. Moreover, in accordance with Remark 1.2, the pair of functions $[\varrho, \mathbf{u}]$ satisfies the renormalized equation

$$\mathrm{div}_x(\varrho \mathbf{u}) = 0, \quad \mathrm{div}_x(b(\varrho)\mathbf{u}) + (\varrho b'(\varrho) - b(\varrho))\mathrm{div}_x \mathbf{u} = 0, \quad \text{in } \mathcal{D}'(R^3), \tag{2.42}$$

for any b satisfying (1.18 - 1.19).

Consequently, to finish the proof of Theorem 1.1, we have to show $\overline{p(\varrho)} = p(\varrho)$. This is done in the next section.

2.4.2 Convergence of the pressure

First of all, we introduce the so-called effective viscous flux which is the quantity $p(\varrho) - (\frac{4\mu}{3} + \eta)\mathrm{div}_x \mathbf{u}$. We shall show that this quantity enjoys some weak compactness property specified in the following lemma. This property, identified first by Lions [12], plays a crucial role in the existence theory in the framework of weak solutions for the compressible Navier-Stokes system.

Lemma 2.3 *For any $\psi \in C_c^\infty(\Omega)$, there holds up to a substraction of subsequence:*

$$\lim_{\varepsilon \to 0} \int_\Omega \psi \left(p(\tilde{\varrho}_\varepsilon) - (\frac{4\mu}{3} + \eta)\mathrm{div}_x \tilde{\mathbf{u}}_\varepsilon \right) \tilde{\varrho}_\varepsilon \, \mathrm{d}x = \int_\Omega \psi \left(\overline{p(\varrho)} - (\frac{4\mu}{3} + \eta)\mathrm{div}_x \mathbf{u} \right) \varrho \, \mathrm{d}x. \tag{2.43}$$

Proof of Lemma 2.3. The proof of Lemma 2.3 is quite technical but nowadays well understood. The idea is to take test functions of the form

$$\psi\nabla\Delta^{-1}(1_\Omega\tilde{\varrho}_\varepsilon), \quad \psi\nabla\Delta^{-1}(1_\Omega\varrho),$$

where $\psi \in C_c^\infty(\Omega)$ and Δ^{-1} is the Fourier multiplier on R^3 with the symbol $1/|\xi|^2$. It is straightforward to check that

$$\nabla\nabla\Delta^{-1} = (\mathcal{R}_{i,j})_{1\le i,j\le 3}$$

are the Riesz operators. In particular, using the well known properties of the singular integral operators of Carderón-Zygmund type, we have

$$\|\nabla\nabla\Delta^{-1}(r)\|_{L^q(R^3)} \le c\ \|r\|_{L^q(R^3)}.$$

for any $r \in L^q(R^3),\ 1 < q < \infty$.

We take $\psi\nabla\Delta^{-1}(1_\Omega\tilde{\varrho}_\varepsilon)$ as a test functions in the weak formulation of (2.27) and let $\varepsilon \to 0$. Then we take $\psi\nabla\Delta^{-1}(1_\Omega\varrho)$ as a test functions in the weak formulation of (2.41), and compare the results of these operations. By using the convergence results (2.36 - 2.39), compact Sobolev embedding, the fact $\gamma \ge 3$, and the property $\mathcal{R}_{i,j} = \mathcal{R}_{j,i}$, we obtain, through long but straightforward calculations, that

$$I := \lim_{\varepsilon\to 0}\int_\Omega \psi\left(p(\tilde{\varrho}_\varepsilon) - (\frac{4\mu}{3}+\eta)\mathrm{div}_x\tilde{\mathbf{u}}_\varepsilon\right)\tilde{\varrho}_\varepsilon\ \mathrm{d}x - \int_\Omega \psi\left(\overline{p(\varrho)} - (\frac{4\mu}{3}+\eta)\mathrm{div}_x\mathbf{u}\right)\varrho\ \mathrm{d}x \tag{2.44}$$

$$= \lim_{\varepsilon\to 0}\int_\Omega \tilde{\varrho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon^i\tilde{\mathbf{u}}_\varepsilon^j\psi\mathcal{R}_{i,j}(1_\Omega\tilde{\varrho}_\varepsilon)\ \mathrm{d}x - \int_\Omega \varrho\mathbf{u}^i\mathbf{u}^j\psi\mathcal{R}_{i,j}(1_\Omega\varrho)\ \mathrm{d}x.$$

On the other hand, by taking $1_\Omega\nabla\Delta^{-1}(\psi\tilde{\varrho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon)$ as a test function for the first equation of (2.26) and taking $1_\Omega\nabla\Delta^{-1}(\psi\varrho\mathbf{u})$ as a test function for (2.40), we obtain

$$\int_\Omega 1_\Omega\tilde{\varrho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon^i\mathcal{R}_{i,j}(\psi\tilde{\varrho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon)\ \mathrm{d}x = 0, \quad \int_\Omega 1_\Omega\varrho\mathbf{u}^i\mathcal{R}_{i,j}(\psi\varrho\mathbf{u})\ \mathrm{d}x = 0. \tag{2.45}$$

Plugging (2.45) into (2.44) yields

$$I = \lim_{\varepsilon\to 0}\int_\Omega \tilde{\mathbf{u}}_\varepsilon^i\left(\tilde{\varrho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon^j\psi\mathcal{R}_{i,j}(1_\Omega\tilde{\varrho}_\varepsilon) - 1_\Omega\tilde{\varrho}_\varepsilon\mathcal{R}_{i,j}(\psi\tilde{\varrho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon)\right)\ \mathrm{d}x - \int_\Omega \mathbf{u}^i\left(\varrho\mathbf{u}^j\psi\mathcal{R}_{i,j}(1_\Omega\varrho) - 1_\Omega\varrho\mathcal{R}_{i,j}(\psi\varrho\mathbf{u})\right)\ \mathrm{d}x. \tag{2.46}$$

The proof of the strong convergence of the densities is finished by means of the strong convergence of the velocity in (2.37) and the following property of commutators of the type appearing in (2.46):

Lemma 2.4 *Suppose*

$$u_\varepsilon \to u \quad \textit{weakly in} \quad L^p(R^3), \quad v_\varepsilon \to v \quad \textit{weakly in} \quad L^q(R^3), \quad \frac{1}{p}+\frac{1}{q}=\frac{1}{r}<1.$$

Then for any $1 \le i, j \le 3$,

$$u_\varepsilon \mathcal{R}_{i,j}(v_\varepsilon) - v_\varepsilon \mathcal{R}_{i,j}(u_\varepsilon) \to u\mathcal{R}_{i,j}(v) - v\mathcal{R}_{i,j}(u) \quad \textit{weakly in} \quad L^r(R^3).$$

See [8, Lemma 3.4] for the proof.

Now, we are ready to prove the following result:

Proposition 2.3 *Let* $\overline{p(\varrho)\varrho}$ *be a weak limit of* $p(\tilde{\varrho}_\varepsilon)\tilde{\varrho}_\varepsilon$ *in* $L^{\frac{2\gamma}{\gamma+1}}(\Omega)$, *we then have*

$$\overline{p(\varrho)\varrho} = \overline{p(\varrho)}\varrho \quad \textit{in} \quad L^{\frac{2\gamma}{\gamma+1}}(\Omega).$$

Proof of Proposition 2.3. Using the uniform bound for $\tilde{\varrho}_\varepsilon$ in (2.25) and the specific form of $p(\varrho)$ in (1.11), we have

$$p(\tilde{\varrho}_\varepsilon)\tilde{\varrho}_\varepsilon \to \overline{p(\varrho)\varrho} \quad \text{weakly in} \quad L^{\frac{2\gamma}{\gamma+1}}(\Omega).$$

Taking $b(s) = s\log s$ in the renormalized equations (2.26) and (2.42) we obtain

$$\mathrm{div}_x(\tilde{\varrho}_\varepsilon \log \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon) + \tilde{\varrho}_\varepsilon \mathrm{div}_x \tilde{\mathbf{u}}_\varepsilon = 0, \quad \mathrm{div}_x(\varrho \log \varrho \mathbf{u}) + \varrho \mathrm{div}_x \mathbf{u} = 0. \tag{2.47}$$

Letting $\varepsilon \to 0$ in (2.47) yields

$$\mathrm{div}_x(\overline{\varrho \log \varrho}\mathbf{u}) + \overline{\varrho_\varepsilon \mathrm{div}_x \mathbf{u}} = 0.$$

Then for any $\psi \in C_c^\infty(\Omega)$, we have

$$\int_\Omega \psi \overline{\varrho \mathrm{div}_x \mathbf{u}} \ \mathrm{d}x = \int_\Omega \nabla_x \psi \cdot (\overline{\varrho \log \varrho}\mathbf{u}) \ \mathrm{d}x, \quad \int_\Omega \psi \varrho \mathrm{div}_x \mathbf{u} \ \mathrm{d}x = \int_\Omega \nabla_x \psi \cdot (\varrho \log \varrho)\mathbf{u} \ \mathrm{d}x. \tag{2.48}$$

Passing to the limit for $\varepsilon \to 0$ in (2.43) and using (2.48) we get

$$\int_\Omega \psi \overline{p(\varrho)\varrho} - (\frac{4\mu}{3} + \eta)\nabla_x \psi \cdot \overline{(\varrho \log \varrho)}\mathbf{u} \ \mathrm{d}x = \int_\Omega \psi \overline{p(\varrho)}\varrho - (\frac{4\mu}{3} + \eta)\nabla_x \psi \cdot (\varrho \log \varrho)\mathbf{u} \ \mathrm{d}x. \tag{2.49}$$

Now we choose test functions $\psi_n \in C_c^\infty(\Omega)$ such that

$$0 \le \psi_n \le 1, \quad \psi_n(x) = 1 \text{ if } d(x, \partial\Omega) > \frac{2}{n}, \quad \psi_n(x) = 0 \text{ if } d(x, \partial\Omega) < \frac{1}{n}, \quad \|\nabla_x \psi_n\|_{L^\infty(\Omega)} \le 2n$$

and

$$\|1 - \psi_n\|_{L^q} \le c \ n^{-\frac{3}{q}}, \quad \|\nabla_x\|_{L^q} \le c \ n^{1-\frac{3}{q}}.$$

Then, letting $\psi = \psi_n$ in (2.49) and passing to the limit $n \to \infty$ we may infer that

$$\int_\Omega \overline{p(\varrho)\varrho} - \overline{p(\varrho)}\varrho \ \mathrm{d}x = 0. \tag{2.50}$$

Since the mapping $\varrho \mapsto p(\varrho)$ is strictly increasing, there holds

$$\overline{p(\varrho)\varrho} \geq \overline{p(\varrho)}\varrho, \quad \text{a.e. in} \quad \Omega,$$

which, together with (2.50), gives rise to the desired conclusion

$$\overline{p(\varrho)\varrho} = \overline{p(\varrho)}\varrho, \quad \text{a.e. in} \quad \Omega.$$

We have completed the proof of Proposition 2.3.

Our desired result $\overline{p(\varrho)} = p(\varrho)$ is a direct corollary of Proposition 2.3, due to the monotonicity and convexity of $p(\cdot)$. Accordingly, we have finished the proof of Theorem 1.1.

3 Concluding remarks

The hypothesis concerning the shape of the holes as well as their spatial distribution may be considerably relaxed. As a matter of fact, it is enough to impose the following restriction on the model hole:

There exists a constant $\omega > 0$ such that at each point $x \in \partial T^s$ there exists a closed cone C_x with vertex at x and of aperture ω such that

$$C_x \cap T^s = \{x\}.$$

Moreover, the holes need not be periodically distributed, it is enough that their mutual distance is proportional to ε, see [7] for the relevant homogenization problem in the context of *incompressible* fluids.

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