Completion of the existence theory for unsteady flows with a pressure- and shear-dependent viscosity

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Abstract

A generalization of Navier-Stokes' model is considered, where the Cauchy stress tensor depends on the pressure as well as on the shear rate in a power-law-like fashion, for values of the power-law index $r \in \left(\frac{2d}{d+2}, 2\right]$. We develop existence of generalized (weak) solutions for the resultant system of partial differential equations, including also the so far uncovered cases $r \in \left(\frac{2d}{d+2}, \frac{2d+2}{d+2}\right]$ and $r = 2$. By considering a maximal sensible range of the power-law index r, the obtained theory is in effect identical to the situation of dependence on the shear rate only.

Keywords

Existence theory, weak solution, incompressible fluid, pressure-dependent viscosity, shear-dependent viscosity, Navier boundary condition, parabolic Lipschitz truncation of Sobolev functions, flow through porous media, Neumann problem.

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1 Introduction

Let $T > 0$, $\Omega \in \mathbb{R}^d$ be an open Lipschitz domain and denote $Q = (0, T) \times \Omega$. We would like to study unsteady flows of incompressible homogeneous fluids in Ω. Setting density to be identically one for simplicity, balance of linear momentum and balance of mass for such fluids can be written down as

$$\begin{aligned} \partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div} \boldsymbol{T} &= \boldsymbol{f}, \\ \operatorname{div} \boldsymbol{v} &= 0, \end{aligned} \tag{1}$$

both holding in Q, where $\boldsymbol{f}$ represents the external forces acting on the fluid and $\boldsymbol{T}$ is the Cauchy stress tensor. When the fluid is additionally supposed to be Newtonian, the Cauchy stress is of the form

$$\boldsymbol{T} = -p\boldsymbol{I} + \nu \boldsymbol{D}\boldsymbol{v}, \tag{2}$$

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where p is the pressure (the indeterminate part of the stress),

$$\boldsymbol{D}\boldsymbol{v} = \frac{1}{2}(\nabla \boldsymbol{v} + \nabla^T \boldsymbol{v})$$

is the symmetric part of the velocity gradient and $\nu > 0$ is the shear viscosity. When $\boldsymbol{T}$ is of the form (2), the equation (1) becomes the notorious Navier-Stokes model. Unfortunately, despite all the rapt attention that this model has drawn in renown mathematicians throughout the last century and beyond, the hitherto obtained results are still far from satisfactory. Worse yet, it is well known that this model is incapable of capturing manifold features manifested by non-Newtonian fluids, such as shear-thinning or -thickening, pressure-dependent viscosity etc.

In this paper we are interested in the situation where the Cauchy stress is of the form

$$\boldsymbol{T} = -p\boldsymbol{I} + \boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v}) = -p\boldsymbol{I} + \nu(p, |\boldsymbol{D}\boldsymbol{v}|^2)\boldsymbol{D}\boldsymbol{v}, \tag{3}$$

in which the viscous stress tensor $\boldsymbol{S}$ is supposed to meet certain requirements; see Assumptions 2.1 and 2.2. This particular model goes back to two papers by Málek et al. [21, 22] and has been dealt with on multiple occasions ever since (see e.g. [5, 12, 15, 20] and the discussion below Theorem 3.1).

It has been convincingly documented in experiments that viscosity of a fluid may vary significantly with the pressure (exponentially or even more dramatically; see e.g. [1, 3] or comprehensive references in [24]). Likewise, the already mentioned shear-thinning or shear-thickening behavior can be captured through a non-constant viscosity $\nu = \nu(|\boldsymbol{D}\boldsymbol{v}|^2)$ like in the mathematically popular model of Ladyzhenskaya's. By means of the constitutive relation (3), we can capture both these dependencies in a single model. It comes at a price, sadly, for instance we are able to handle only shear-thinning, not shear-thickening, behavior (see the main result, Theorem 3.1, and the upper bound for the power exponent r).

The objective we set is to prove existence of weak solutions for the model. Therefore we have to add initial and suitable boundary conditions, for which sake let us denote $\Gamma = (0, T) \times \partial\Omega$. We consider an impermeable boundary, that is

$$\boldsymbol{v} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma,$$

where $\boldsymbol{n}$ is the unit outer normal vector of Ω. We cannot, however, resort to the no-slip boundary condition

$$\boldsymbol{v} = \boldsymbol{0} \quad \text{on } \Gamma,$$

for in that case we would be unable to construct the pressure (see the discussion below Theorem 3.1). Instead, we choose the Navier slip condition

$$\alpha \boldsymbol{v}_\tau = -(\boldsymbol{S}\boldsymbol{n})_\tau \quad \text{on } \Gamma$$

for some $\alpha \geq 0$, which is the heart of the matter here due to the dependence of $\boldsymbol{S}$ on p. For $\boldsymbol{u} : \partial\Omega \to \mathbb{R}^d$, a vector field on the boundary, we define its tangential component as

$$\boldsymbol{u}_\tau = \boldsymbol{u} - (\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{n}.$$

Note that from an instinctive point of view, the Navier slip may be regarded as a bridge between the no-slip condition ($\alpha \to \infty$) and the perfect slip condition ($\alpha = 0$).

On account of the pressure-dependent viscous stress, we have yet to add some kind of pressure anchoring, which we take in the form

$$\frac{1}{|\Omega|} \int_\Omega p(t, x)\, dx = h(t) \quad \text{in } (0, T) \tag{4}$$

for a given function h. Ideally one should like to prescribe the pressure locally (at some point) but since our pressure will be merely an integrable function, dictating its pointwise values is out of the question. A possible approximation could lie in the integral average over a given subset $\Omega_0 \subset \Omega$ but in our case, corresponding attempts led to insurmountable technical difficulties, hence (4) for simplicity.

All in all, the model to be analyzed reads

$$\left.\begin{aligned} \partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div} \boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v}) + \nabla p &= \boldsymbol{f} && \text{in } Q, \\ \operatorname{div} \boldsymbol{v} &= 0 && \text{in } Q, \\ \boldsymbol{v} \cdot \boldsymbol{n} &= 0 && \text{on } \Gamma, \\ \alpha \boldsymbol{v}_\tau &= -(\boldsymbol{S}\boldsymbol{n})_\tau && \text{on } \Gamma, \\ \boldsymbol{v}(0) &= \boldsymbol{v}_0 && \text{in } \Omega, \\ \frac{1}{|\Omega|} \int_\Omega p \, dx &= h && \text{in } (0,T). \end{aligned}\right\} \tag{5}$$

As far as the structure of this paper goes, next we are about to introduce our notation and certain assumptions, in particular those on the viscous stress $\boldsymbol{S}$, i.e. Assumptions 2.1 and 2.2. In the ensuing section, we present the result of this paper, Theorem 3.1 on existence of weak solutions to problem (5), and devote a few lines to the discussion of its relevance to past works and to the sketch of the fundamental techniques employed in the proof. In Section 4, we list various nontrivial results that are exploited in the proof of Theorem 3.1, to which the entire Section 5 and Appendix are dedicated.

2 Preliminaries

For $0 < t < T$ we write $Q_t = (0,t) \times \Omega$ and $\Gamma_t = (0,t) \times \partial\Omega$. For $r \in (1,\infty)$ we denote $r' = r/(r-1)$. For a Lebesgue measurable set Ω we denote $|\Omega|$ its Lebesgue measure. If $X(\Omega)$ is a Lebesgue or Sobolev space, we denote

$$\mathring{X}(\Omega) = \Big\{ f \in X(\Omega) \;\Big|\; \int_\Omega f(x)\, dx = 0 \Big\}.$$

For $f \in L^1(\Omega)$ we denote

$$f_\Omega = \frac{1}{|\Omega|} \int_\Omega f(x)\, dx.$$

Usually, no explicit distinction between spaces of scalar- and vector-valued functions will be made. Confusion should never come to pass as we employ small boldfaced letters to denote vectors and bold capitals for tensors. The same applies also to traces of Sobolev functions, which we denote like the original functions. Only when in need, we use Tr for a trace. Accordingly, for $r > 1$ we set

$$\begin{aligned} W^{1,r}_{\boldsymbol{n}}(\Omega) &= \big\{ \boldsymbol{f} \in W^{1,r}(\Omega) \;\big|\; \operatorname{Tr} \boldsymbol{f} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega \big\}, \\ W^{1,r}_{\boldsymbol{n},\mathrm{div}}(\Omega) &= \big\{ W^{1,r}_{\boldsymbol{n}}(\Omega) \;\big|\; \operatorname{div} \boldsymbol{f} = 0 \text{ in } \Omega \big\}, \\ W^{-1,r'}_{\boldsymbol{n}}(\Omega) &= \big(W^{1,r}_{\boldsymbol{n}}(\Omega) \big)^*, \\ X^r_{\boldsymbol{n}} &= L^r(0,T; W^{1,r}_{\boldsymbol{n}}(\Omega)) \cap L^2(0,T; L^2(\partial\Omega)), \\ X^r_{\boldsymbol{n},\mathrm{div}} &= L^r(0,T; W^{1,r}_{\boldsymbol{n},\mathrm{div}}(\Omega)) \cap L^2(0,T; L^2(\partial\Omega)), \\ \mathcal{C}^\infty_c(\Omega) &= \big\{ f \in \mathcal{C}^\infty(\Omega) \;\big|\; f \text{ is compactly supported in } \Omega \big\}. \end{aligned}$$

If $r > 0$ and $x \in \mathbb{R}^d$, let $B_r(x) = \{|y - x| < r\}$. For $f \in L^1_{loc}(\mathbb{R}^{d+1})$ and $(t, x) \in \mathbb{R}^{d+1}$, we define the parabolic maximal operator

$$\mathcal{M}^*(f)(t,x) = \sup_{0<\varrho<\infty} \frac{1}{2\varrho} \int_{t-\varrho}^{t+\varrho} \sup_{0<r<\infty} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(s,y)| \, dy \, ds.$$

When applied to functions not defined on the whole $\mathbb{R}^{d+1}$, we implicitly consider their zero extension. For more details about maximal operators see [26] or, only for the fundamental properties of $\mathcal{M}^*$ needed here, Appendix A of [13].

The symbol $\cdot$ stands for the scalar product and $\otimes$ signifies the tensor product. For open subsets A, B of $\mathbb{R}^d$, we write $A \Subset B$ if $A \subset \overline{A} \subset B$ and $\overline{A}$ is compact. We denote $(\cdot, \cdot)$ the inner product in $L^2(\Omega)$, while $(\cdot, \cdot)_S$ stands for the inner product in $L^2(S)$ for a measurable set S other than Ω. Generic constants are denoted simply by C and, when circumstances require it, we may also include quantities on which the constants depend, e.g. $C(\|\boldsymbol{v}_0\|_2)$.

The external body forces $\boldsymbol{f}$ are for the sake of convenience supposed to be of the form

$$\boldsymbol{f} = -\operatorname{div} \boldsymbol{F},$$

Consider $r \in (1, 2]$ a fixed number and $d \geq 2$. Inspired by [22], below we reproduce assumptions on the viscous stress, i.e. the smooth nonlinearity $\boldsymbol{S}$:

Assumption 2.1 *Let there be positive constants C_1 and C_2 such that for all $\boldsymbol{B}, \boldsymbol{D} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and $p \in \mathbb{R}$*

$$C_1(1 + |\boldsymbol{D}|^2)^{(r-2)/2}|\boldsymbol{B}|^2 \leq \frac{\partial \boldsymbol{S}(p, \boldsymbol{D})}{\partial \boldsymbol{D}} \cdot (\boldsymbol{B} \otimes \boldsymbol{B}) \leq C_2(1 + |\boldsymbol{D}|^2)^{(r-2)/2}|\boldsymbol{B}|^2.$$

Assumption 2.2 *Let for all $\boldsymbol{D} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and $p \in \mathbb{R}$*

$$\left|\frac{\partial \boldsymbol{S}(p, \boldsymbol{D})}{\partial p}\right| \leq \gamma_0 (1 + |\boldsymbol{D}|^2)^{(r-2)/4}, \quad \textit{with } 0 < \gamma_0 < \frac{C_1}{C_{reg}(C_1 + C_2)},$$

where C_{reg} is attributed to the solution operator of Neumann's problem on Ω; see (15) *and below.*

3 Main result

Theorem 3.1 *Let $d \geq 2$, $T > 0$, $\alpha > 0$, $2d/(d+2) < r \leq 2$ and $\Omega \in \mathcal{C}^{1,1}$ be a bounded domain in $\mathbb{R}^d$. Denote*

$$q = \frac{r(d+2)}{2d} > 1 \tag{6}$$

and consider $\boldsymbol{F} \in L^{r'}(Q)$, $h \in L^q(0, T)$ and $\boldsymbol{v}_0 \in L^2_{\boldsymbol{n},\mathrm{div}}(\Omega)$. Finally suppose that Assumptions 2.1 and 2.2 hold. Then there exists a weak solution $(\boldsymbol{v}, p)$ to the problem (5)*, that is*

$$\boldsymbol{v} \in \mathcal{C}_w([0, T]; L^2(\Omega)) \cap X^r_{\boldsymbol{n},\mathrm{div}}, \quad \partial_t \boldsymbol{v} \in L^q(0, T; W^{-1,q}_{\boldsymbol{n}}(\Omega)),$$

$$p \in L^q(0, T; L^q(\Omega)) \textit{ and } \int_\Omega p(t, x) \, dx = h(t) \textit{ for a.e. } t \in (0, T)$$

and the weak formulation is satisfied, i.e. for all $\boldsymbol{\varphi} \in W^{1,q'}_{\boldsymbol{n}}(\Omega)$ and a.e. $t \in (0, T)$ we have

$$\langle \partial_t \boldsymbol{v}(t), \boldsymbol{\varphi}\rangle - ((\boldsymbol{v} \otimes \boldsymbol{v})(t), \nabla \boldsymbol{\varphi}) + (\boldsymbol{S}(t), \boldsymbol{D}\boldsymbol{\varphi}) + \alpha(\boldsymbol{v}(t), \boldsymbol{\varphi})_{\partial\Omega} - (p(t), \operatorname{div} \boldsymbol{\varphi}) = (\boldsymbol{F}(t), \nabla \boldsymbol{\varphi}), \tag{7}$$

with $\boldsymbol{S}(t) = \boldsymbol{S}(p(t), \boldsymbol{D}\boldsymbol{v}(t))$. The initial condition is attained through $\lim_{t\to 0_+} \|\boldsymbol{v}(t) - \boldsymbol{v}_0\|_{L^2(\Omega)} = 0$.

With this result, we practically conclude the existence theory for the corresponding class of models conceived by Málek et al. in [21, 22]. More precisely, with the condition $r > 2d/(d+2)$ we have reached the same lower bound as in the case of pressure-independent viscosity $\nu = \nu(|\boldsymbol{D}\boldsymbol{v}|^2)$; see Diening et al. [13]. This bound is the best one guaranteeing compactness of the convective term $\boldsymbol{v} \otimes \boldsymbol{v}$ in $L^1(Q)$ and in this regard it may be considered optimal.

Although the range $r \in (2d/(d+2), 2)$ has already been investigated in [5], it was in the steady case and therefore the situation was considerably simpler, although the bedrock of the proof was quite similar. As for the evolutionary system like that of ours, the best result so far comes from [10], where existence for $r \in ((2d+2)/(d+2), 2)$ was proven. In [7], the problem has already been grappled with $\Omega = \mathbb{R}^3$ and $r \in (9/5, 2)$. For local results (small data, short times), see [17, 18, 25]. In [11], the model of ours is ivestigated, enriched additionally by the temperature dependence, in which case only $r \in (3d/(d+2), 2)$ can be handled, imposing a restriction $d = 2, 3$.

Apart from optimization from below, we have also finally incorporated the value $r = 2$ among amenable values of the exponent r, which has only recently been achieved for the steady-state problem in [12]. The work [10] also covers the value $r = 2$, yet under a slightly different analogue of Assumption 2.2. Similarly in [9], where the case $d = 2$ with the periodic boundary conditions is treated. Inclusion of the *critical value* $r = 2$ in our paper not only makes the theory cover the Navier-Stokes model but, more importantly, allows us to consider balance equations $(5)_1$ of the form

$$\partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \Delta \boldsymbol{v} - \operatorname{div} \boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v}) + \nabla p = \boldsymbol{f},$$

with $\boldsymbol{S}$ fulfilling Assumptions 2.1 and 2.2 with $r < 2$ if need be.

It is important to notice that we actively avoid the homogeneous Dirichlet boundary condition, corresponding informally to $\alpha = \infty$. The reason is that we need a measurable pressure for the sake of the pressure-dependent viscous stress, which in the case of zero boundary condition remains an insurmountable task. The snag lies in *incompatibility* of the Helmholtz decomposition with the Dirichlet boundary condition or, in other words, the fact that in the Neumann problem for Poisson's equation, the trace of the gradient cannot be required to be zero; only its normal component can (see (13)). This obstacle will be experienced in the flesh in (36) and below.

Even though $\alpha = \infty$ is out of the question, in Theorem 3.1 we could take $\alpha = 0$ without scruples. This situation would correspond to *the perfect-slip condition*, accounting for the fluid slipping along the boundary. From the analytical point of view, the proof would be simplified slightly as we would be completely unflapped by the trace of the velocity field. Navier's condition $(5)_4$ can be further generalized; see [8] where the so called *threshold slip* was investigated. This condition is a very natural approximation of the no-slip condition as it models a fluid adhering to the boundary until a certain *threshold stress* is experienced, after which the fluid abides by Navier's condition.

Although, as stated, the result of Theorem 3.1 is optimal in terms of the range of r, there are still opportunities for improvement. Firstly, the condition from Assumption 2.2,

$$\gamma_0 < \frac{C_1}{C_{reg}(C_1 + C_2)},$$

now depends on the set Ω through the constant C_{reg}. It is highly probable, however, that like in the steady case (see [12]), one may relax the condition to the point

$$\gamma_0 < \frac{C_1}{C_1 + C_2}. \tag{8}$$

It would require replacing the solving operator of the Neumann problem $\mathcal{N}$ (see (13)) with something *more refined*, i.e. an operator with all the properties we want from $\mathcal{N}$, enjoying additionally $C_{reg} = 1$. In [12], we were able to do so by means of the Newtonian potential. In the time-dependent case, however, this choice is no longer viable due to the loss of certain necessary compactness with respect to the time derivative.

Secondly, in $(5)_6$ it would seem more appropriate to prescribe $p_{\Omega_0}(t)$ over some (possibly small) measurable $\Omega_0 \subset \Omega$, thus to approximately fix the pressure at some point. Unluckily, not only does such a generalization lead to severe technical difficulties in the proof but, perhaps even more importantly, Assumption 2.2 was then altered to

$$\gamma_0 < \sqrt{\frac{|\Omega_0|}{|\Omega|}} \frac{C_1}{C_{reg}(C_1 + C_2)},$$

see [11]. This condition is sufficiently deterring in itself as $|\Omega_0| \to 0$ implies $\gamma_0 \to 0$. Bear in mind that this is again not the case for the steady problem, where (8) would suffice.

As far as the proof of Theorem 3.1 is concerned, we employ a two-level approximation scheme (see (128)). The inner level (limit parameter k) consists in truncation of the convective and boundary terms so that up to that point we have a sufficiently regular pressure and the velocity field is a legal test function. Getting rid of this approximation level lies virtually at the heart of this paper and the entire Section 5 is devoted to it. It is based on a pressure decomposition (see p. 12) into a lowly integrable but compact part and a highly integrable part that is at first sight only weakly convergent. Besides this decomposition, we resort to the Lipschitz truncation of functions lying in Bochner spaces (see Lemmas 4.5 and 4.6) to deal with the issue of insufficient regularity of the velocity field to make it an admissible test function in (7).

The primary objective of the outer level is to introduce the pressure. Unlike the traditional Navier-Stokes model, we cannot invoke De Rham's theorem in our situation, for the viscous stress tensor itself is pressure-dependent − the resultant pressure would be a distribution in time. Also, there would then appear two possibly distinct pressures (one in $\boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v})$ and the other generated by De Rham's theorem) and we might have to resort to some fixed-point argument to equate them. Here we construct the pressure by means of an auxiliary elliptic problem, the so called *quasicompressible approximation* (see [15]), replacing the condition on solenoidality $(5)_2$ by

$$\varepsilon p = \mathcal{N}(\operatorname{div} \boldsymbol{v}),$$

(see (13) for the definition of $\mathcal{N}$), intuitively making the velocity field only *almost* divergence-free. Since this level of approximation is comparatively simpler to lift than the truncation, we leave it for Appendix.

Moving on to the following section, we survey several nontrivial results exploited in the proof of Theorem 3.1.

4 Auxiliary tools

To begin with, we list a couple of crucial properties exhibited by the nonlinear viscous stress tensor $\boldsymbol{S}$.

Lemma 4.1 ([15], Lemmas 3.3, 3.4) *Let Assumptions 2.1 and 2.2 hold. For arbitrary* $\boldsymbol{D}^1, \boldsymbol{D}^2 \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ *and* $p^1, p^2 \in \mathbb{R}$ *we set*

$$I^{1,2} := \int_0^1 (1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} |\boldsymbol{D}^1 - \boldsymbol{D}^2|^2 \, ds,$$

with $\overline{\boldsymbol{D}}(s) = \boldsymbol{D}^2 + s(\boldsymbol{D}^1 - \boldsymbol{D}^2)$. *Then*

$$\frac{1}{2} C_1 I^{1,2} \leq (\boldsymbol{S}(p^1, \boldsymbol{D}^1) - \boldsymbol{S}(p^2, \boldsymbol{D}^2)) \cdot (\boldsymbol{D}^1 - \boldsymbol{D}^2) + \frac{\gamma_0^2}{2C_1} |p^1 - p^2|^2. \tag{9}$$

Furthermore

$$|(\boldsymbol{S}(p^1, \boldsymbol{D}^1) - \boldsymbol{S}(p^2, \boldsymbol{D}^2)| \leq \gamma_0 |p^1 - p^2| + C_2 \int_0^1 (1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} |\boldsymbol{D}^1 - \boldsymbol{D}^2| \, ds. \tag{10}$$

Finally, for all $p \in \mathbb{R}$, $r \in (1,2]$ and $\boldsymbol{D} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$

$$\boldsymbol{S}(p,\boldsymbol{D})\cdot\boldsymbol{D} \geq \frac{C_1}{2r}(|\boldsymbol{D}|^r - 1) \tag{11}$$

and

$$|\boldsymbol{S}(p,\boldsymbol{D})| \leq \frac{C_2}{r-1}(1+|\boldsymbol{D}|)^{r-1}. \tag{12}$$

The corresponding statement in [15] does not include (10). However, it is only an easy observation stemming from

$$\boldsymbol{S}(p^1,\boldsymbol{D}^1) - \boldsymbol{S}(p^2,\boldsymbol{D}^2) = \int_0^1 \frac{d}{ds}\boldsymbol{S}(p^2 + s(p^1-p^2), \boldsymbol{D}^2 + s(\boldsymbol{D}^1 - \boldsymbol{D}^2))\, ds$$

and Assumptions 2.1 and 2.2.

We also recall the Helmholtz decomposition and the L^q-regularity theory of the Neumann problem for Poisson's equation: If $q \in (1,\infty)$ and $\Omega \in \mathcal{C}^{1,1}$, let $\mathcal{N} : \mathring{L}^q(\Omega) \to \mathring{W}^{2,q}(\Omega)$ ascribe to $z \in \mathring{L}^q(\Omega)$ the unique solution v of

$$\Delta v = z \text{ in } \Omega, \qquad \nabla v\cdot\boldsymbol{n} = 0 \text{ at } \partial\Omega, \qquad v_\Omega = 0. \tag{13}$$

The Helmholtz decomposition of the space $W^{1,q}_{\boldsymbol{n}}(\Omega)^d$ allows us to resolve any $\boldsymbol{u} \in W^{1,q}_{\boldsymbol{n}}(\Omega)^d$ as a sum

$$\boldsymbol{u} = \boldsymbol{u}_{\mathrm{div}} + \nabla \boldsymbol{g}_{\boldsymbol{u}}, \tag{14}$$

where $\boldsymbol{g}_{\boldsymbol{u}} = \mathcal{N}(\operatorname{div}\boldsymbol{u})$ and $\boldsymbol{u}_{\mathrm{div}} = \boldsymbol{u} - \nabla\boldsymbol{g}_{\boldsymbol{u}}$. The L^q-continuity of $\boldsymbol{u} \mapsto \boldsymbol{u}_{\mathrm{div}}$ [16, Remark III.1.1] and the L^q-regularity for $\mathcal{N}$ with $\Omega \in \mathcal{C}^{1,1}$ [19, Proposition 2.5.2.3] imply

$$\begin{aligned} \|\mathcal{N}(z)\|_{W^{2,q}(\Omega)} &\leq C_{reg,q}\|z\|_{L^q(\Omega)}, \qquad \|\boldsymbol{u}_{\mathrm{div}}\|_{W^{1,q}(\Omega)} \leq (C_{reg,q}+1)\|\boldsymbol{u}\|_{W^{1,q}(\Omega)}, \\ \|\boldsymbol{g}_{\boldsymbol{u}}\|_{W^{1,q}(\Omega)} &\leq C(\Omega,q)\|\boldsymbol{u}\|_{L^q(\Omega)}, \qquad \|\boldsymbol{u}_{\mathrm{div}}\|_{L^q(\Omega)} \leq C(\Omega,q)\|\boldsymbol{u}\|_{L^q(\Omega)}, \end{aligned} \tag{15}$$

for any $z \in \mathring{L}^q(\Omega)$ and $\boldsymbol{u} \in W^{1,q}_{\boldsymbol{n}}(\Omega)^d$. Later on we will need especially $C_{reg} = C_{reg,2}$ which is why we utilize different notation for these constants.

Lemma 4.2 (Korn's inequality, [14], Theorem 10.15) *Let $\Omega \in \mathcal{C}^{0,1}$ and $r \in (1,\infty)$. Then there exists a positive constant $C = C(\Omega,r)$ such that for all $\boldsymbol{u} \in W^{1,r}(\Omega)$ it holds that*

$$\|\boldsymbol{u}\|_{W^{1,r}(\Omega)} \leq C\big(\|\boldsymbol{D}\boldsymbol{u}\|_{L^r(\Omega)} + \|\boldsymbol{u}\|_{L^1(\Omega)}\big). \tag{16}$$

Lemma 4.3 (Compactness of traces) *Let r and q retain their meaning from Theorem 3.1 and suppose that $\{\boldsymbol{v}^i\}_{i=1}^\infty$ is bounded in*

$$L^r(0,T;W^{1,r}_{\boldsymbol{n}}(\Omega) \cap W^{1,q}(0,T;W^{-1,q}_{\boldsymbol{n}}).$$

Then $\{\operatorname{Tr}\boldsymbol{v}^i\}_{i=1}^\infty$ is precompact in $L^r(0,T;L^r(\partial\Omega))$.

Proof. The standard Aubin-Lions lemma implies precompactness of $\{\boldsymbol{v}^i\}_{i=1}^\infty$ in $L^r(0,T;L^r(\Omega))$. Interpolation (see e.g. [23, Lemma 2.18]) then yields precompactness of $\{\boldsymbol{v}^i\}_{i=1}^\infty$ in $L^r(0,T;W^{1-\varepsilon,r}(\Omega))$ for an arbitrarily small $\varepsilon > 0$. There is also a continuous trace operator from $W^{p_1,p_2}(\Omega)$ into $W^{p_1-1/p_2,p_2}(\partial\Omega)$ for any $p_1 \in R_+$ and $p_2 \geq 1$ such that $p_1p_2 > 1$ (see [27] and the remark in [4, Lemma B.3]). Taking $\varepsilon > 0$ so small that $(1-\varepsilon)r > 1$, we have $L^r(0,T;W^{1-\varepsilon-\frac{1}{r},r}(\partial\Omega)) \hookrightarrow L^r(0,T;L^r(\partial\Omega))$ and thus also the claim. □

Lemma 4.4 (Biting lemma, [2]) *Let $S \subset \mathbb{R}^d$ have a finite Lebesgue measure and $\{f^k\}$ be a bounded sequence in $L^1(S)$. Then there exist a function $f \in L^1(S)$, a subsequence $\{f^j\}$ of $\{f^k\}$ and a non-increasing sequence of measurable sets $D^m \subset S$ with $\lim_{m\to\infty} |D^m| = 0$, such that $f^j \to f$ weakly in $L^1(S \setminus D^m)$ for every fixed m.*

Lemma 4.5 (Parabolic Lipschitz approximation I, [13], Lemma 3.11, Theorem 3.21) *Let $\Omega \subset \mathbb{R}^d$ be an open bounded set, $\boldsymbol{u} \in L^\infty(0,T;L^2(\Omega)^d) \cap L^q(0,T;W^{1,q}(\Omega)^d)$ $(1 < q < \infty)$ and $\boldsymbol{H} \in L^\sigma(0,T;L^\sigma(\Omega)^{d\times d})$ $(1 < \sigma < \infty)$ be such that*

$$-\int_Q \boldsymbol{u} \cdot \partial_t \boldsymbol{\varphi} \, dx\, dt = \int_Q \boldsymbol{H} \cdot \nabla \boldsymbol{\varphi} \, dx\, dt \tag{17}$$

for all $\boldsymbol{\varphi} \in \mathcal{C}_c^\infty(Q)$. For $\Lambda > 0$ we define

$$\mathcal{O}_\Lambda = \{\mathcal{M}^*(|\nabla \boldsymbol{u}^k|) + \mathcal{M}^*(|\boldsymbol{H}|) > \Lambda\}.$$

Let $E \subset \mathbb{R}^{d+1}$ be an open set such that $Q \cap \mathcal{O}_\Lambda \subset E \subset Q$.

Then there exists $\mathcal{L}_E \boldsymbol{u} \in L^\infty_{loc}(0,T;W^{1,\infty}_{loc}(\Omega)^d)$ such that $\mathcal{L}_E \boldsymbol{u} = \boldsymbol{u}$ in $Q \setminus E$ and[1]

$$\|\mathcal{L}_E \boldsymbol{u}\|_{L^p(Q)} \le C \|\boldsymbol{u}\|_{L^p(Q)} \qquad \text{for any } 1 \le p \le \infty. \tag{18}$$

Let $K \subset Q$ be a compact set. There is a constant $C_K > 0$ depending on K such that

$$\|\nabla \mathcal{L}_E \boldsymbol{u}\|_{L^\infty(K)} \le C\big(\Lambda + C_K \|\boldsymbol{u}\|_{L^1(E)}\big). \tag{19}$$

Furthermore, the function $(\partial_t \mathcal{L}_E \boldsymbol{u}) \cdot (\mathcal{L}_E \boldsymbol{u} - \boldsymbol{u})$ belongs to $L^1(K \cap E)$ and we have

$$\|(\partial_t \mathcal{L}_E \boldsymbol{u}) \cdot (\mathcal{L}_E \boldsymbol{u} - \boldsymbol{u})\|_{L^1(K\cap E)} \le C|E| \big(\Lambda + C_K \|\boldsymbol{u}\|_{L^1(E)}\big)^2. \tag{20}$$

Finally, for all $g \in \mathcal{C}_c^\infty(Q)$ holds the identity

$$\int_0^T \big\langle \partial_t \boldsymbol{u}(t), (\mathcal{L}_E \boldsymbol{u}(t)) g(t) \big\rangle \, dt \\ = \frac{1}{2} \int_Q \big(|\mathcal{L}_E \boldsymbol{u}|^2 - 2\boldsymbol{u} \cdot \mathcal{L}_E \boldsymbol{u}\big) \partial_t g \, dx\, dt + \int_E (\partial_t \mathcal{L}_E \boldsymbol{u}) \cdot (\mathcal{L}_E \boldsymbol{u} - \boldsymbol{u}) g \, dx\, dt. \tag{21}$$

The original version of the stated lemma contains also a certain scaling parameter[2] $\alpha > 0$. For our purposes we need the case $\alpha = 1$ only and we have adapted the statement of the lemma accordingly.

Lemma 4.6 (Parabolic Lipschitz approximation II, [6], Lemma 2.5) *Let $\Omega \subset \mathbb{R}^d$ be an open bounded set, $T > 0$ and $r \in (1, \infty)$. For any functions $\boldsymbol{H}$, $\overline{\boldsymbol{H}}$ and arbitrary sequences $\{\boldsymbol{u}^k\}$ and $\{\boldsymbol{H}^k\}$ we set*

$$a^k = |\boldsymbol{H}^k| + |\boldsymbol{H}| + |\overline{\boldsymbol{H}}| \quad \text{and} \quad b^k = |\boldsymbol{D}\boldsymbol{u}^k|$$

and suppose that for certain $C^* > 1$ and all k we have

$$\|a^k\|_{L^r(Q)} + \|b^k\|_{L^{r'}(Q)} + \sup_{t\in(0,T)} \|\boldsymbol{u}^k(t)\|_{L^2(\Omega)} \le C^*,$$

$$\boldsymbol{u}^k \to \boldsymbol{0} \quad \text{a.e. in } Q.$$

[1]The generic constants C below depend only on the dimension d.

[2]This scaling parameter α is competely unrelated to that in the boundary condition $(5)_4$.

In addition, let $\{\boldsymbol{G}^k\}$ *consist of symmetric* $\boldsymbol{G}^k$ *such that*

$$\boldsymbol{G}^k \to \boldsymbol{0} \quad \textit{strongly in } L^1(Q)^{d\times d} \tag{22}$$

and let us have the distributional identity

$$\partial_t \boldsymbol{u}^k + \operatorname{div}(\boldsymbol{H}^k - \boldsymbol{H} + \boldsymbol{G}^k) = \boldsymbol{0}.$$

Then there is $\beta > 0$ *such that for arbitrary* $\widehat{Q} \Subset Q$, $\lambda^* \in (r^{\frac{1}{(r-1)}}, \infty)$ *and* $n \in \mathbb{N}$ *there exist sequences* $\{\lambda^{k,n}\}_k \subset \mathbb{R}$, $\{B^{k,n}\}_k$ *of open sets* $B^{k,n} \subset Q$ *and* $\{\boldsymbol{u}^{k,n}\}_k$ *bounded in* $L^\infty_{loc}(0,T;W^{1,\infty}_{loc}(\Omega)^d)$ *such that*

$$\{\lambda^{k,n}\}_k \subset [\lambda^*, r^{\frac{1-r^n}{r-1}}(\lambda^*)^{r^n}], \tag{23}$$

$$\limsup_{k\to\infty} |\widehat{Q} \cap B^{k,n}| \le \frac{C(\widehat{Q})}{(\lambda^*)^r}, \tag{24}$$

$$\boldsymbol{u}^{k,n} \to \boldsymbol{0} \quad \textit{strongly in } L^s(\widehat{Q})^d \textit{ as } k \to \infty \textit{ for any } 1 \le s < \infty, \tag{25}$$

$$\boldsymbol{u}^{k,n} = \boldsymbol{u}^k \quad \textit{in } \widehat{Q} \setminus B^{k,n}, \tag{26}$$

$$\|\boldsymbol{D}\boldsymbol{u}^{k,n}\|_{L^\infty(\widehat{Q})} \le C(\widehat{Q})\lambda^{k,n}. \tag{27}$$

Morevover, for all $\tau \in \mathcal{C}^\infty_c(\widehat{Q};[0,1])$ *the following estimates hold:*

$$\limsup_{k\to\infty} \int_{\widehat{Q}\cap B^{k,n}} (|\boldsymbol{H}^k| + |\boldsymbol{H}| + |\overline{\boldsymbol{H}}|)|\boldsymbol{D}\boldsymbol{u}^{k,n}|\, dx\, dt \le C(\widehat{Q})(r(\lambda^*)^{1-r} + n^{-\beta}), \tag{28}$$

$$-\liminf_{k\to\infty} \int_0^T \langle \partial_t \boldsymbol{u}^k, \boldsymbol{u}^{k,n}\tau\rangle\, dt \le C(\widehat{Q})(r(\lambda^*)^{1-r} + n^{-1})^\beta. \tag{29}$$

Strictly speaking, the above lemma as we state it is not a precise reproduction of [6]. To avoid unnecessary generality of Orlicz spaces, our theorem pertains to a special choice of the N-function $\psi(x) = \frac{x^r}{r}$, to which we adapted all parameters of the original theorem. Dependencies of constants on fixed parameters, e.g. Ω or r, were also omitted. In (29), the estimate should also hang on $\|\tau\|_{L^\infty(\Omega)}$ but since we restrict ourselves to $\|\tau\|_{L^\infty(\Omega)} \le 1$, we may assume that the bound is really independent of the truncating function τ and that each $C(\widehat{Q})$ in (24)–(29) are the same.

5 Proof of the existence theorem

Without loss of generality we will assume $h \equiv 0$, that is to say

$$\int_\Omega p(t,x)\, dx = 0$$

for almost every time. We can think so since in the general case we would first investigate the equation for $\overline{p} = p - h$. Due to $h \in L^q(0,T)$, if $\overline{p} \in L^q(Q)$ then of course also $p \in L^q(Q)$.

There is a couple of strategies how to deal with the convective term, be it the addition of a penalty term, its mollification, or truncation (see e.g. [6, 11, 13], respectively). Here, we choose the truncation and for this purpose, let $\Phi \in \mathcal{C}^1([0,\infty))$ be a non-increasing function such that $\Phi(x) = 1$ if $x \le 1$, $\Phi(x) = 0$ if $x \ge 2$ and $\Phi(x) \in (0,1)$ otherwise, with $|\Phi'(x)| \le 2$. For $k > 0$ then define

$$\Phi_k(x) = \Phi(k^{-1}x).$$

With fixed $k > 0$, the original system (5) will be approximated by

$$\left.\begin{aligned} \partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}\Phi_k(|\boldsymbol{v}|)) - \operatorname{div} \boldsymbol{S} + \nabla p = - \operatorname{div} \boldsymbol{F} \quad & \text{in } Q, \\ \operatorname{div} \boldsymbol{v} = 0 \quad & \text{in } Q, \\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \quad & \text{on } \Gamma, \\ \alpha \boldsymbol{v}_\tau \Phi_k(|\boldsymbol{v}_\tau|) = -(\boldsymbol{S}\boldsymbol{n})_\tau \quad & \text{on } \Gamma, \\ \boldsymbol{v}(0) = \boldsymbol{v}_0 \quad & \text{in } \Omega, \\ p_\Omega = 0 \quad & \text{in } (0,T). \end{aligned}\right\} \tag{30}$$

The boundary conditions imply $\boldsymbol{v}_\tau = \boldsymbol{v}$ on Γ and therefore we will not distinguish between these two entities (see the weak formulation (31)).

Existence of weak solutions for thus truncated system can be shown by standard means (see e.g. [10, 11]) and we postpone it for Appendix. To be more precise, we suppose momentarily that the following lemma holds:

Lemma 5.1 *Under the assumptions of Theorem 3.1, for every $k > 0$ there exists a weak solution to the truncated problem* (30)*, i.e. a couple $(\boldsymbol{v}^k, p^k)$ such that*

$$\boldsymbol{v}^k \in L^r(0,T; W^{1,r}_{\boldsymbol{n},\operatorname{div}}(\Omega)), \quad \partial_t \boldsymbol{v}^k \in L^{r'}(0,T; W^{-1,r'}_{\boldsymbol{n}}(\Omega)), \quad p^k \in L^{r'}(0,T; \mathring{L}^{r'}(\Omega)),$$

satisfying[3] $\lim_{t \to 0_+} \|\boldsymbol{v}^k(t) - \boldsymbol{v}_0\|_{L^2(\Omega)} = 0$ *and*

$$\begin{aligned} \langle \partial_t \boldsymbol{v}^k(t), \boldsymbol{\varphi} \rangle - (\boldsymbol{v}^k \otimes \boldsymbol{v}^k \Phi_k(|\boldsymbol{v}^k|)(t), \nabla \boldsymbol{\varphi}) + (\boldsymbol{S}^k(t), \boldsymbol{D}\boldsymbol{\varphi}) + \alpha(\boldsymbol{v}^k \Phi_k(|\boldsymbol{v}^k|), \boldsymbol{\varphi})_{\partial\Omega} - (p^k(t), \operatorname{div} \boldsymbol{\varphi}) \\ = (\boldsymbol{F}(t), \nabla \boldsymbol{\varphi}) \end{aligned} \tag{31}$$

with $\boldsymbol{S}^k(t) = \boldsymbol{S}(p^k(t), \boldsymbol{D}\boldsymbol{v}^k(t))$, for every $\boldsymbol{\varphi} \in W^{1,r}_{\boldsymbol{n}}(\Omega)$ and a.e. $t \in (0,T)$.

5.1 Truncation removal ($k \to \infty$)

The reinstatement of the full-fledged convective term is the key limit process. Our first steps will be devoted to finding bounds independent of $k > 0$ in suitable function spaces.

Uniform estimates Taking $\boldsymbol{\varphi} = \boldsymbol{v}^k(t)$ in (31) and exploiting integration by parts and solenoidality of $\boldsymbol{v}^k$, we note that

$$(\boldsymbol{v}^k \otimes \boldsymbol{v}^k \Phi_k(|\boldsymbol{v}^k|), \nabla \boldsymbol{v}^k)_Q = \Big(\boldsymbol{v}^k, \nabla \int_0^{|\boldsymbol{v}^k|} s\, \Phi_k(s)\, ds\Big)_Q = 0,$$

owing to which (ensuing relations hold for a.e. $t \in (0,T)$)

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}^k(t)\|^2_{L^2(\Omega)} + (\boldsymbol{S}^k(t), \boldsymbol{D}\boldsymbol{v}^k(t)) + \alpha \|\Phi_k^{1/2}(|\boldsymbol{v}^k|)\boldsymbol{v}^k(t)\|^2_{L^2(\partial\Omega)} = (\boldsymbol{F}(t), \nabla \boldsymbol{v}^k(t)).$$

Due to coercivity of the stress tensor (11), the fact that $\Phi_k \le \Phi_k^{1/2}$ and Hölder's inequality,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}^k(t)\|^2_{L^2(\Omega)} + \frac{C_1}{2r}\|\boldsymbol{D}\boldsymbol{v}^k(t)\|^r_{L^r(\Omega)} + \alpha \|\Phi_k(|\boldsymbol{v}^k|)\boldsymbol{v}^k(t)\|^2_{L^2(\partial\Omega)} \\ \le \|\boldsymbol{F}(t)\|_{L^{r'}(\Omega)} \|\nabla \boldsymbol{v}^k(t)\|_{L^r(\Omega)} + \frac{C_1|\Omega|}{2r}. \end{aligned} \tag{32}$$

[3] Note that $\boldsymbol{v}^k \in \mathcal{C}([0,T]; L^2(\Omega))$.

By means of Hölder's, Young's and Korn's inequality (16), we then obtain

$$\sup_{t\in(0,T)} \|\boldsymbol{v}^k(t)\|^2_{L^2(\Omega)} + \|\boldsymbol{v}^k\|^r_{L^r(0,T;W^{1,r}(\Omega))} + \|\Phi_k(|\boldsymbol{v}^k|)\boldsymbol{v}^k\|^2_{L^2(\Gamma)} \le C\big(\|\boldsymbol{v}_0\|_{L^2(\Omega)}, \|\boldsymbol{F}\|_{L^{r'}(Q)}\big). \tag{33}$$

Using boundedness of the stress tensor (12), we get in addition

$$\|\boldsymbol{S}^k\|^{r'}_{L^{r'}(Q)} \le C\big(\|\boldsymbol{v}_0\|_{L^2(\Omega)}, \|\boldsymbol{F}\|_{L^{r'}(Q)}\big). \tag{34}$$

Combined with $L^\infty(0,T;L^2(\Omega)) \cap L^r(0,T;W^{1,r}(\Omega)) \hookrightarrow L^{2q}(Q)$ with $q > 1$ (defined in (6)), we have also

$$\|\boldsymbol{v}^n\|_{L^{2q}(Q)} \le C\big(\|\boldsymbol{v}_0\|_{L^2(\Omega)}, \|\boldsymbol{F}\|_{L^{r'}(Q)}\big). \tag{35}$$

As for a bound on the pressure p^k, due to the convective term we have to relax our requirements from the current integrability $p^k \in L^{r'}(Q)$ – we will estimate it in $L^q(Q)$. Let us consider the equation (31) with the test function

$$\boldsymbol{\varphi}^k = \nabla\mathcal{N}\big(|p^k|^{q-2}p^k - (|p^k|^{q-2}p^k)_\Omega\big), \tag{36}$$

which due to (15) satisfies

$$\begin{aligned} \|\boldsymbol{\varphi}^k\|_{L^{q'}(0,T;W^{1,q'}(\Omega))} &\le C\||p^k|^{q-1}\|_{L^{q'}(Q)} = C\|p^k\|^{q-1}_{L^q(Q)}, \\ \operatorname{div}\boldsymbol{\varphi}^k &= |p^k|^{q-2}p^k - (|p^k|^{q-2}p^k)_\Omega \quad \text{a.e. in } Q. \end{aligned}$$

Here we want to point out that had we chosen Dirichlet's boundary conditions instead of Navier's, we would now have run into serious trouble. The culprit is $\operatorname{Tr}\boldsymbol{\varphi}^k$ – in the Dirichlet setting we would be unable to justify it is actually zero, making the choice of (36) illegal for the weak formulation corresponding to Dirichlet's problem. indeed, we could choose $\boldsymbol{\varphi}^k$ differently so that $\operatorname{Tr}\boldsymbol{\varphi}^k = \boldsymbol{0}$ (e.g. by means of the Bogovskiĭ operator) but then we would face new problems stemming from the time derivative $\partial_t\boldsymbol{v}^k$ (see I_5 below and how it vanishes with our choice of $\boldsymbol{\varphi}^k$).

From (31) it holds that

$$\|p^k\|^q_{L^q(Q)} = (p^k, \operatorname{div}\boldsymbol{\varphi}^k)_Q = \sum_{i=1}^5 I_i,$$

where due to Hölder's inequality and estimates (33), (34) and (35) (note $q' \ge 2$),

$$\begin{aligned} I_1 &= -(\boldsymbol{F}, \nabla\boldsymbol{\varphi}^k)_Q \le \|\boldsymbol{F}\|_{L^{r'}(Q)}\|\nabla\boldsymbol{\varphi}^k\|_{L^r(Q)} \le C\|\boldsymbol{\varphi}^k\|_{L^{q'}(0,T;W^{1,q'}(\Omega))}, \\ I_2 &= (\boldsymbol{S}^k, \boldsymbol{D}\boldsymbol{\varphi}^k)_Q \le \|\boldsymbol{S}^k\|_{L^{r'}(Q)}\|\nabla\boldsymbol{\varphi}^k\|_{L^r(Q)} \le C\|\boldsymbol{\varphi}^k\|_{L^{q'}(0,T;W^{1,q'}(\Omega))}, \\ I_3 &= \alpha(\boldsymbol{v}^k\Phi_k(|\boldsymbol{v}^k|), \boldsymbol{\varphi}^k)_\Gamma \le \alpha\|\Phi_k(|\boldsymbol{v}^k|)\boldsymbol{v}^k\|^2_{L^2(\Gamma)}\|\boldsymbol{\varphi}^k\|_{L^2(\Gamma)} \le C\|\boldsymbol{\varphi}^k\|_{L^{q'}(0,T;W^{1,q'}(\Omega))}, \\ I_4 &= -(\boldsymbol{v}^k \otimes \boldsymbol{v}^k\Phi_k(|\boldsymbol{v}^k|), \nabla\boldsymbol{\varphi}^k)_Q \le \|\boldsymbol{v}^k\|^2_{L^{2q}(Q)}\|\boldsymbol{\varphi}^k\|_{L^{q'}(0,T;W^{1,q'}(\Omega))} \le C\|\boldsymbol{\varphi}^k\|_{L^{q'}(0,T;W^{1,q'}(\Omega))}, \\ I_5 &= \int_0^T \langle\partial_t\boldsymbol{v}^k, \boldsymbol{\varphi}^k\rangle\,dt = -\big(\partial_t\operatorname{div}\boldsymbol{v}^k, \mathcal{N}\big(|p^k|^{q-2}p^k - (|p^k|^{q-2}p^k)_\Omega\big)\big)_Q = 0. \end{aligned}$$

Thus we have the desired estimate

$$\|p^k\|_{L^q(Q)} \le C\big(\|\boldsymbol{v}_0\|_{L^2(\Omega)}, \|\boldsymbol{F}\|_{L^{r'}(Q)}\big). \tag{37}$$

Next, estimates (33), (34), (35) and (37) divulge that functionals Ψ^k defined on $L^\infty(0,T;W^{1,\infty}_{\boldsymbol{n}})$ as

$$\Psi^k(\boldsymbol{\varphi}) = (\boldsymbol{v}^k \otimes \boldsymbol{v}^k \Phi_k(|\boldsymbol{v}^k|), \nabla\boldsymbol{\varphi})_Q - (\boldsymbol{S}^k, \boldsymbol{D}\boldsymbol{\varphi})_Q - \alpha(\boldsymbol{v}^k\Phi_k(|\boldsymbol{v}^k|), \boldsymbol{\varphi})_\Gamma + (p^k, \operatorname{div}\boldsymbol{\varphi})_Q + (\boldsymbol{F}, \nabla\boldsymbol{\varphi})_Q,$$

satisfy

$$\left|\Psi^k(\boldsymbol{\varphi})\right| \le C\big(\|\boldsymbol{v}_0\|_{L^2(\Omega)}, \|\boldsymbol{F}\|_{L^{r'}(Q)}\big)\|\boldsymbol{\varphi}\|_{L^{q'}(0,T;W^{1,q'}_{\boldsymbol{n}}(\Omega))}$$

uniformly in k. In other words, from eq. (31) it follows that

$$\|\partial_t\boldsymbol{v}^k\|_{L^q(0,T;W^{-1,q}_{\boldsymbol{n}}(\Omega))} \le C\big(\|\boldsymbol{v}_0\|_{L^2(\Omega)}, \|\boldsymbol{F}\|_{L^{r'}(Q)}\big). \tag{38}$$

Limit $k \to \infty$ With help of uniform bounds (33)–(38) and the compactness lemma 4.3, we may select a subsequence $(\boldsymbol{v}^k, p^k)$ satisfying[4]

$$\begin{aligned}
\boldsymbol{v}^k &\to \boldsymbol{v} && \text{weakly in } L^r(0,T;W^{1,r}_{\boldsymbol{n},\mathrm{div}}(\Omega)), && (39)\\
\boldsymbol{v}^k &\to \boldsymbol{v} && \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)), && (40)\\
\partial_t\boldsymbol{v}^k &\to \partial_t\boldsymbol{v} && \text{weakly in } L^q(0,T;W^{-1,q}_{\boldsymbol{n}}(\Omega)), && (41)\\
\boldsymbol{v}^k &\to \boldsymbol{v} && \text{strongly in } L^s(Q) \text{ for all } s \in [1,2q), && (42)\\
\boldsymbol{v}^k &\to \boldsymbol{v} && \text{strongly in } L^r(\Gamma), && (43)\\
\Phi_k(|\boldsymbol{v}^k|)\boldsymbol{v}^k &\to \boldsymbol{v} && \text{weakly in } L^2(\Gamma), && (44)\\
\Phi_k(|\boldsymbol{v}^k|)\boldsymbol{v}^k &\to \boldsymbol{v} && \text{strongly in } L^s(\Gamma) \text{ for all } s \in [1,2), && (45)\\
\boldsymbol{v}^k &\to \boldsymbol{v} && \text{a.e. in } Q, && (46)\\
p^k &\to p && \text{weakly in } L^q(0,T;\mathring{L}^q(\Omega)), && (47)\\
\boldsymbol{S}^k &\to \overline{\boldsymbol{S}} && \text{weakly in } L^{r'}(Q). && (48)
\end{aligned}$$

We also have $\boldsymbol{v} \in \mathcal{C}_w([0,T];L^2(\Omega))$ by (40) and (41). These convergences, when applied to equation (31), produce

$$\int_0^T \langle\partial_t\boldsymbol{v}, \boldsymbol{\varphi}\rangle\,dt - (\boldsymbol{v}\otimes\boldsymbol{v}, \nabla\boldsymbol{\varphi})_Q + (\overline{\boldsymbol{S}}, \boldsymbol{D}\boldsymbol{\varphi})_Q + \alpha(\boldsymbol{v}, \boldsymbol{\varphi})_\Gamma - (p, \operatorname{div}\boldsymbol{\varphi})_Q = (\boldsymbol{F}, \nabla\boldsymbol{\varphi})_Q \tag{49}$$

for every $\boldsymbol{\varphi} \in L^{q'}(0,T;W^{1,q'}_{\boldsymbol{n}}(\Omega))$.

The next step, basically the core of this paper, consists in showing $\overline{\boldsymbol{S}} = \boldsymbol{S}$ (i.e. $\boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v})$) and this will be achieved through Vitali's theorem, since $\boldsymbol{S}(\cdot,\cdot)$ is continuous. To this end we have to show the pointwise convergence of $\boldsymbol{D}\boldsymbol{v}^k$ and p^k a.e. in Q.

Decomposition of p^k We will overcome the problem of low[5] pressure integrability by decomposing the pressure into two parts – one keeping the low q-integrability, yet converging pointwise, and the other r'-integrable, for which we then prove the pointwise convergence.

According to (31), for any $\varphi \in W^{2,q'}(\Omega)$ such that $\nabla\varphi\cdot\boldsymbol{n} = 0$ at $\partial\Omega$ and a.e. $t \in (0,T)$, we have

$$(p^k(t), \Delta\varphi) = -(\boldsymbol{v}^k \otimes \boldsymbol{v}^k\Phi_k(|\boldsymbol{v}^k|)(t), \nabla^2\varphi) + (\boldsymbol{S}^k(t), \nabla^2\varphi) + \alpha(\boldsymbol{v}^k\Phi_k(|\boldsymbol{v}^k|)(t), \nabla\varphi)_{\partial\Omega} - (\boldsymbol{F}(t), \nabla^2\varphi). \tag{50}$$

We will decompose the pressure as $p^k = p^k_1 + p^k_2$, where $p^k_2 \in L^{r'}(0,T;\mathring{L}^{r'}(\Omega))$ is the unique solution to

$$\begin{aligned}
(p^k_2(t), \Delta\varphi) &= (\boldsymbol{S}^k(t), \nabla^2\varphi) - (\boldsymbol{F}(t), \nabla^2\varphi),\\
(p^k_2(t))_\Omega &= 0
\end{aligned} \tag{51}$$

[4]We employ bars for unidentified weak limits.

[5]Relative to the ε-limit, cf. Subsection 6.1.

for any $\varphi \in W^{2,r}(\Omega)$ such that $\nabla\varphi \cdot \boldsymbol{n} = 0$ on $\partial\Omega$ and a.e. $t \in (0,T)$. For details about solvability of this equation, formally corresponding to

$$\Delta p_2^k(t) = \operatorname{div}\operatorname{div}\big(\boldsymbol{S}^k(t) - \boldsymbol{F}(t)\big),$$

see [4, (3.51)], where a procedure based on an approximation of what is here $\boldsymbol{S}^k(t) - \boldsymbol{F}(t)$ by compactly supported smooth functions is explained in more depth. Let us show $\{p_2^k\}$ is bounded in $L^{r'}(Q)$. To this end take

$$\varphi(t) = \mathcal{N}\big(|p_2^k(t)|^{r'-2}p_2^k(t) - (|p_2^k(t)|^{r'-2}p_2^k(t))_\Omega\big)$$

and recall that for $\mathcal{N}$ we have L^q-regularity (15), implying

$$\|\varphi(t)\|_{W^{2,r}(\Omega)} \le C(\Omega,r)\||p_2^k(t)|^{r'-1}\|_{L^r(\Omega)} = C(\Omega,r)\|p_2^k(t)\|_{L^{r'}(\Omega)}^{r'-1}. \tag{52}$$

Next we insert such φ into (51), obtaining

$$\begin{aligned}\|p_2^k\|_{L^{r'}(Q)}^{r'} &= (p_2^k, \Delta\varphi)_Q = (\boldsymbol{S}^k - \boldsymbol{F}, \nabla^2\varphi)_Q \le \big(\|\boldsymbol{S}^k\|_{L^{r'}(Q)} + \|\boldsymbol{F}\|_{L^{r'}(Q)}\big)\|\varphi\|_{L^r(0,T;W^{2,r}(\Omega))}\\ &\le C\|p_2^k\|_{L^{r'}(Q)}^{r'-1}\end{aligned}$$

by means of Hölder's inequality and the estimates (34) and (52). Therefore we may assume there exists $p_2 \in L^{r'}(0,T;\mathring{L}^{r'}(\Omega))$ such that

$$p_2^k \to p_2 \quad \text{weakly in } L^{r'}(Q). \tag{53}$$

By (50) and (54), the other partial pressure $p_1^k = p^k - p_2^k$ must satisfy

$$(p_1^k(t), \Delta\varphi) = -(\boldsymbol{v}^k \otimes \boldsymbol{v}^k\Phi_k(|\boldsymbol{v}^k|)(t), \nabla^2\varphi) + \alpha(\boldsymbol{v}^k\Phi_k(|\boldsymbol{v}^k|)(t), \nabla\varphi)_{\partial\Omega}, \tag{54}$$

for any $\varphi \in W^{2,q'}(\Omega)$ such that $\nabla\varphi \cdot \boldsymbol{n} = 0$ at $\partial\Omega$ and $(p_1^k(t))_\Omega = 0$ for a.e. $t \in (0,T)$. It follows from (47) and (53) that $\{p_1^k\}$ is bounded in $L^q(0,T;\mathring{L}^q(\Omega))$. We will show it also converges strongly in $L^1(0,T;L^1(\Omega))$. Let $k,l \in \mathbb{N}$ and $1 < s < q$ be arbitrary. Take

$$\varphi(t) = \mathcal{N}\big(|p_1^k - p_1^l|^{s-2}(p_1^k - p_1^l)(t) - (|p_1^k - p_1^l|^{s-2}(p_1^k - p_1^l)(t))_\Omega\big)$$

and like in (52), observe that due to L^q-regularity (15),

$$\|\varphi(t)\|_{W^{2,s'}(\Omega)} \le C(\Omega,s)\||p_1^k - p_1^l|^{s-1}(t)\|_{L^{s'}(\Omega)} = C(\Omega,s)\|(p_1^k - p_1^l)(t)\|_{L^s(\Omega)}^{s-1}. \tag{55}$$

Plugging φ into (54) yields

$$\|p_1^k - p_1^l\|_{L^s(Q)}^s = (p_1^k - p_1^l, \Delta\varphi)_Q = I_1 + I_2,$$

where, using (55),

$$\begin{aligned}I_1 &= (\boldsymbol{v}^l \otimes \boldsymbol{v}^l\Phi_l(|\boldsymbol{v}^l|)(t) - \boldsymbol{v}^k \otimes \boldsymbol{v}^k\Phi_k(|\boldsymbol{v}^k|)(t), \nabla^2\varphi)_Q,\\ &\le \|\boldsymbol{v}^l \otimes \boldsymbol{v}^l\Phi_l(|\boldsymbol{v}^l|) - \boldsymbol{v}^k \otimes \boldsymbol{v}^k\Phi_k(|\boldsymbol{v}^k|)\|_{L^s(Q)}\|\varphi\|_{L^{s'}(0,T;W^{2,s'}(\Omega))}\\ &\le C(\Omega,s)\|\boldsymbol{v}^l \otimes \boldsymbol{v}^l\Phi_l(|\boldsymbol{v}^l|) - \boldsymbol{v}^k \otimes \boldsymbol{v}^k\Phi_k(|\boldsymbol{v}^k|)\|_{L^s(Q)}\|p_1^k - p_1^l\|_{L^s(Q)}^{s-1}\end{aligned}$$

and

$$\begin{aligned}I_2 &= \alpha(\boldsymbol{v}^k\Phi_k(|\boldsymbol{v}^k|)(t) - \boldsymbol{v}^l\Phi_l(|\boldsymbol{v}^l|)(t), \nabla\varphi)_\Gamma\\ &\le \alpha\|\boldsymbol{v}^k\Phi_k(|\boldsymbol{v}^k|) - \boldsymbol{v}^l\Phi_l(|\boldsymbol{v}^l|)\|_{L^s(\Gamma)}\|\varphi\|_{L^{s'}(\Gamma)}\\ &\le C(\Omega,s)\|\boldsymbol{v}^k\Phi_k(|\boldsymbol{v}^k|) - \boldsymbol{v}^l\Phi_l(|\boldsymbol{v}^l|)\|_{L^s(\Gamma)}\|\varphi\|_{L^{s'}(0,T;W^{2,s'}(\Omega))}\\ &\le C(\Omega,s)\|\boldsymbol{v}^k\Phi_k(|\boldsymbol{v}^k|) - \boldsymbol{v}^l\Phi_l(|\boldsymbol{v}^l|)\|_{L^s(\Gamma)}\|p_1^k - p_1^l\|_{L^s(Q)}^{s-1}.\end{aligned}$$

The above computations imply that $\{p_1^k\}$ is a Cauchy sequence in $L^s(Q)$ since by the estimate (35) and the strong convergence (42), we observe

$$\begin{aligned}\lim_{k,l\to\infty} \|\boldsymbol{v}^l \otimes \boldsymbol{v}^l \Phi_l(|\boldsymbol{v}^l|) - \boldsymbol{v}^k \otimes \boldsymbol{v}^k \Phi_k(|\boldsymbol{v}^k|)\|_{L^s(Q)} &\le 2 \lim_{k\to\infty} \|\boldsymbol{v}^k \otimes \boldsymbol{v}^k \Phi_k(|\boldsymbol{v}^k|) - \boldsymbol{v} \otimes \boldsymbol{v}\|_{L^s(Q)} \\ &\le 2 \lim_{k\to\infty} \|\boldsymbol{v}^k \otimes \boldsymbol{v}^k \Phi_k(|\boldsymbol{v}^k|) - \boldsymbol{v}^k \otimes \boldsymbol{v}^k\|_{L^s(Q)} \\ &\le 4 \lim_{k\to\infty} \|\boldsymbol{v}^k\|^2_{L^{2s}(Q\cap\{|\boldsymbol{v}^k|>k\})} \\ &\le C \lim_{k\to\infty} \left|Q \cap \{|\boldsymbol{v}^k| > k\}\right|^{\frac{q-s}{qs}} \\ &= 0\end{aligned}$$

and similarly, using and the strong convergence (45),

$$\lim_{k,l\to\infty} \|\boldsymbol{v}^k \Phi_k(|\boldsymbol{v}^k|) - \boldsymbol{v}^l \Phi_l(|\boldsymbol{v}^l|)\|_{L^s(\Gamma)} = 0.$$

Hence there exists $p_1 \in L^q(0,T; \mathring{L}^q(\Omega))$ such that

$$\begin{aligned} p_1^k &\to p_1 \quad \text{weakly in } L^q(Q), \\ p_1^k &\to p_1 \quad \text{strongly in } L^1(Q). \end{aligned} \tag{56}$$

The first convergence was trivial by the already shown weak convergences (47) and (53). In particular, we may assume

$$p_1^k \to p_1 \quad \text{a.e. in } Q.$$

From (56) we also infer by the dominated convergence theorem and (12) that

$$\boldsymbol{S}(p_1^k + p_2, \boldsymbol{D}\boldsymbol{v}) \to \boldsymbol{S} \quad \text{strongly in } L^{r'}(Q). \tag{57}$$

Showing the pointwise convergence of p_2^k is all that remains. We will treat the cases $r < 2$ and $r = 2$ separately. The procedure necessitated by the former case may be accommodated to deal also with the latter (and vice versa, actually). Nonetheless, it would require to prove an improved version of Lemma 4.6, which we do not find necessary. Even though it may not be the most elegant way of tackling the issue, we have taken the path of least resistance and resolved to cover the case $r = 2$ rather by the spiritual ancestor of the aforementioned Lemma 4.6, i.e. by Lemma 4.5. This result could be in turn utilized to handle also the case $r < 2$ but it would be considerably messier than with Lemma 4.6.

5.2 Convergence for $r < 2$

Let $N \in \mathbb{N}$ be fixed. Take $Q_N \Subset \widehat{Q}_N \Subset Q$ such that

$$|Q \setminus Q_N| \le \frac{1}{N}. \tag{58}$$

Now we invoke the parabolic Lipschitz truncation lemma 4.6, set up as follows:

$$\begin{aligned} \boldsymbol{H} &= p_2 \boldsymbol{I} - \overline{\boldsymbol{S}}, \\ \boldsymbol{H}^k &= p_2^k \boldsymbol{I} - \boldsymbol{S}^k, \\ \overline{\boldsymbol{H}} &= |\boldsymbol{S}| + |\overline{\boldsymbol{S}}|, \\ \boldsymbol{u}^k &= \boldsymbol{v}^k - \boldsymbol{v}, \\ \boldsymbol{G}^k &= \boldsymbol{v}^k \otimes \boldsymbol{v}^k \Phi_k(|\boldsymbol{v}^k|) - \boldsymbol{v} \otimes \boldsymbol{v} + (p_1^k - p_1)\boldsymbol{I}. \end{aligned}$$

Next we take numbers $\lambda^* = \lambda^*(N)$ and $n = n(N)$ large enough so that the constant $C(\widehat{Q}_N)$ from (28) and (29) satisfies

$$C(\widehat{Q}_N)(r(\lambda^*)^{1-r} + n^{-\beta}) \le \frac{1}{N}, \tag{59}$$

$$C(\widehat{Q}_N)(r(\lambda^*)^{1-r} + n^{-1})^{\beta} \le \frac{1}{N}, \tag{60}$$

where the number $\beta > 0$ is produced by the said Lemma 4.6. Note that (59) also implies

$$\frac{C(\widehat{Q}_N)}{(\lambda^*)^r} \le \frac{1}{N}. \tag{61}$$

To finish the setup of Lemma 4.6, we take

$$\widehat{Q} = \widehat{Q}_N.$$

As a result, there exist sequences $\{\lambda^{k,n}\}_k \subset \mathbb{R}$, $\{B^{k,n}\}_k$ of open sets $B^{k,n} \subset Q$ and $\{\boldsymbol{u}^{k,n}\}_k$ bounded in $L^\infty_{loc}(0,T;W^{1,\infty}_{loc}(\Omega)^d)$ such that (23)–(29) hold.

Furthermore, we take $\tau^N \in \mathcal{C}^\infty_c(\widehat{Q}_N;[0,1])$ such that

$$\tau^N \equiv 1 \quad \text{in } Q_N \tag{62}$$

and

$$C(\widehat{Q}_N)\lambda^{k,n}\big|\{0 < \tau^N < 1\}\big|^{1/r} \le \frac{1}{N} \quad \text{for every } k, \tag{63}$$

which is possible by (23).

Finally, we define bad sets $E^{k,n}$ and good sets $G^{k,n}$ as

$$E^{k,n} = B^{k,n} \cup \{\tau^N < 1\}, \tag{64}$$

$$G^{k,n} = Q \setminus E^{k,n}. \tag{65}$$

Informally speaking, the bad set consists of points near the boundary or those where the Lipschitz approximation does not match the original function; see (26). From the estimate (24), bounds (58) and (61) and the property (62), it follows that

$$\limsup_{k\to\infty} |E^{k,n}| \le \frac{2}{N}. \tag{66}$$

Convergence of p_2^k Denote $\pi^k = p_2^k - p_2$. We are going to show

$$\lim_{k\to\infty} \|\pi^k\|_{L^2(Q)} = 0. \tag{67}$$

Towards this goal, we set

$$\boldsymbol{\varphi}^k = \mathcal{N}(\pi^k) \tag{68}$$

and observe that by (15) and (53), $\boldsymbol{\varphi}^k$ satisfies

$$\|\boldsymbol{\varphi}^k\|_{L^2(0,T;W^{2,2}(\Omega))} \le C_{reg}\|\pi^k\|_{L^2(Q)}, \tag{69}$$

$$\boldsymbol{\varphi}^k \to 0 \quad \text{weakly in } L^{r'}(0,T;W^{2,r'}(\Omega)). \tag{70}$$

Let $O(k^{-1})$ signify a quantity satisfying $\limsup_{k\to\infty} O(k^{-1}) \leq 0$. For quantities A^k, B^k we write $A^k \overset{k}{\lesssim} B^k$ if $A^k \leq B^k + O(k^{-1})$. With this notation[6] we develop

$$\|\pi^k\|^2_{L^2(Q)} = (\pi^k, \Delta\boldsymbol{\varphi}^k)_Q \overset{k}{\lesssim} (p_2^k, \Delta\boldsymbol{\varphi}^k)_Q \overset{k}{\lesssim} (\boldsymbol{S}^k, \nabla^2\boldsymbol{\varphi}^k)_Q$$

by (51) and the weak convergence (70). Since

$$(\boldsymbol{S}(p_1^k + p_2, \boldsymbol{D}\boldsymbol{v}), \nabla^2\boldsymbol{\varphi}^k)_Q \overset{k}{\lesssim} 0$$

by (57) and (70) (note $r' \geq 2$), we may write

$$\begin{aligned}\|\pi^k\|^2_{L^2(Q)} &\overset{k}{\lesssim} (\boldsymbol{S}^k, \nabla^2\boldsymbol{\varphi}^k)_Q \overset{k}{\lesssim} (\boldsymbol{S}^k - \boldsymbol{S}(p_1^k + p_2, \boldsymbol{D}\boldsymbol{v}), \nabla^2\boldsymbol{\varphi}^k)_Q \\ &\leq \gamma_0 \int_Q |\pi^k||\nabla^2\boldsymbol{\varphi}^k|\,dx\,dt + C_2 \int_Q \int_0^1 (1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} |\boldsymbol{D}\boldsymbol{u}^k||\nabla^2\boldsymbol{\varphi}^k|\,ds\,dx\,dt, \end{aligned} \tag{71}$$

by (10) with $\overline{\boldsymbol{D}}(s) = \boldsymbol{D}\boldsymbol{v} + s(\boldsymbol{D}\boldsymbol{v}^k - \boldsymbol{D}\boldsymbol{v})$. Denote

$$I^k = \int_0^1 (1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} |\boldsymbol{D}\boldsymbol{u}^k|^2 ds. \tag{72}$$

As $(1+|\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} \leq (1+|\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/4}$, Hölder's inequality and bound (69) applied to (71) yield

$$\|\pi^k\|^2_{L^2(Q)} \overset{k}{\lesssim} \gamma_0 C_{reg} \|\pi^k\|^2_{L^2(Q)} + C_2 C_{reg} \Big(\int_{G^{k,n}} I^k\,dx\,dt \Big)^{1/2} \|\pi^k\|_{L^2(Q)} + O(N^{-1}),$$

where we got rid of the bad set $E^{k,n}$ (see its defition (64)) by means of the bound on its measure (66), boundedness stemming from (39) and (53) and $r < 2$ as follows:

$$\begin{aligned} C_2 \int_{E^{k,n}} \int_0^1 (1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} |\boldsymbol{D}\boldsymbol{u}^k||\nabla^2\boldsymbol{\varphi}^k|\,ds\,dx\,dt &\leq C \|\boldsymbol{D}\boldsymbol{u}^k\|_{L^r(Q)} \|\pi^k\|_{L^2(Q)} |E^{k,n}|^{\frac{2-r}{2r}} \\ &\overset{k}{\lesssim} O(N^{-1}). \end{aligned} \tag{73}$$

Consequently

$$\|\pi^k\|^2_{L^2(Q)} \overset{k}{\lesssim} \left(\frac{C_2 C_{reg}}{1 - \gamma_0 C_{reg}} \right)^2 \int_{G^{k,n}} I^k\,dx\,dt + O(N^{-1}). \tag{74}$$

The integral on the right can be estimated by means of (9):

$$\int_{G^{k,n}} I^k\,dx\,dt \leq \frac{\gamma_0^2}{C_1^2} \|\pi^k\|^2_{L^2(Q)} + \frac{2}{C_1} \big(\boldsymbol{S}^k - \boldsymbol{S}(p_1^k + p_2, \boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{u}^k\big)_{G^{k,n}} \overset{k}{\lesssim} \frac{\gamma_0^2}{C_1^2} \|\pi^k\|^2_{L^2(Q)} + O(N^{-1}), \tag{75}$$

provided

$$I_1 = (\boldsymbol{S}^k, \boldsymbol{D}\boldsymbol{u}^k)_{G^{k,n}} \overset{k}{\lesssim} O(N^{-1}), \tag{76}$$

$$I_2 = -(\boldsymbol{S}(p_1^k + p_2, \boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{u}^k)_{G^{k,n}} \overset{k}{\lesssim} O(N^{-1}). \tag{77}$$

The limit inequalities (74) and (75) would then yield

$$\|\pi^k\|^2_{L^2(Q)} \overset{k}{\lesssim} \left(\frac{\gamma_0 C_2 C_{reg}}{C_1(1 - \gamma_0 C_{reg})} \right)^2 \|\pi^k\|^2_{L^2(Q)} + O(N^{-1}),$$

[6]We exploit it analogously also for other limit quantities, so for instance $O(N^{-1})$ or, later on, $O(\varepsilon)$.

implying the desired convergence (67), as long as

$$\frac{\gamma_0 C_2 C_{reg}}{C_1(1-\gamma_0 C_{reg})} < 1,$$

which does hold, however, due to Assumption 2.2, namely

$$\gamma_0 < \frac{C_1}{C_{reg}(C_1+C_2)}.$$

We must therefore justify (76) and (77). Recall that in $G^{k,n}$ we have $\boldsymbol{u}^k = \boldsymbol{u}^{k,n}$; see the definition (65). Also note that by (23), (25) and (27), for any fixed N (hence also for $n = n(N)$ and $\lambda^* = \lambda^*(N)$), we may assume

$$\nabla \boldsymbol{u}^{k,n} \to 0 \quad \text{weakly in } L^r(\widehat{Q}_N) \text{ as } k \to \infty. \tag{78}$$

We rewrite I_1 as

$$I_1 = (\boldsymbol{S}^k, \boldsymbol{D}(\tau^N \boldsymbol{u}^{k,n}))_{G^{k,n}} = (\boldsymbol{S}^k, \boldsymbol{D}(\tau^N \boldsymbol{u}^{k,n}))_Q - (\boldsymbol{S}^k, \boldsymbol{D}(\tau^N \boldsymbol{u}^{k,n}))_{\{\tau^N>0\}\setminus G^{k,n}}. \tag{79}$$

Since $\nabla \tau^N = 0$ a.e. in $G^{k,n}$ and

$$\{\tau^N > 0\} \setminus G^{k,n} = \big(\{\tau^N > 0\} \cap B^{k,n}\big) \cup \big(\{0 < \tau^N < 1\} \setminus B^{k,n}\big),$$

we recast (79) as

$$\begin{aligned} I_1 = (\boldsymbol{S}^k, \boldsymbol{D}(\tau^N \boldsymbol{u}^{k,n}))_Q - (\boldsymbol{S}^k, \boldsymbol{u}^{k,n} \otimes \nabla \tau^N)_{\{\tau^N>0\}\setminus G^{k,n}} \\ - (\boldsymbol{S}^k, \tau^N \boldsymbol{D}\boldsymbol{u}^{k,n})_{\{\tau^N>0\}\cap B^{k,n}} - (\boldsymbol{S}^k, \tau^N \boldsymbol{D}\boldsymbol{u}^{k,n})_{\{0<\tau^N<1\}\setminus B^{k,n}}. \end{aligned}$$

According to the strong convergence (25), it holds that

$$\lim_{k\to\infty} (\boldsymbol{S}^k, \boldsymbol{u}^{k,n} \otimes \nabla \tau^N)_{\{\tau^N>0\}\setminus G^{k,n}} = 0. \tag{80}$$

Additionally, by the Lipschitz bound (27), the weak convergence of $\boldsymbol{S}^k$ from (48) and then by (63),

$$\big|(\boldsymbol{S}^k, \tau^n \boldsymbol{D}\boldsymbol{u}^{k,n})_{\{0<\tau^n<1\}\setminus B^{k,n}}\big| \le C(\widehat{Q}_N)\lambda^{k,n} \big|\{0 < \tau^N < 1\}\big|^{1/r} \|\boldsymbol{S}^k\|_{L^{r'}(Q)} \le \frac{C}{N}. \tag{81}$$

As a result of (80) and (81),

$$\begin{aligned} I_1 &\overset{k}{\sim} (\boldsymbol{S}^k, \boldsymbol{D}(\tau^N \boldsymbol{u}^{k,n}))_Q - (\boldsymbol{S}^k, \tau^N \boldsymbol{D}\boldsymbol{u}^{k,n})_{\{\tau^N>0\}\cap B^{k,n}} + O(N^{-1}) \\ &\overset{k}{\sim} (\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \boldsymbol{D}(\tau^N \boldsymbol{u}^{k,n}))_Q - (\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \tau^N \boldsymbol{D}\boldsymbol{u}^{k,n})_{\{\tau^N>0\}\cap B^{k,n}} + O(N^{-1}) \end{aligned}$$

by (25) and (78) in the first term and (28) and (59) in the second one. Now we recall the weak formulations (31) and (49) and notice that $\tau^N \boldsymbol{u}^{k,n}$ is a legal test function in either of them (which mere $\boldsymbol{u}^k$ fails to meet). Substituting the term $(\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \boldsymbol{D}(\tau^N \boldsymbol{u}^{k,n}))_Q$ accordingly, we obtain

$$I_1 \overset{k}{\sim} J_1 + J_2 + J_3 + O(N^{-1}),$$

where

$$J_1 = -\int_0^T \langle \partial_t \boldsymbol{u}^k, \tau^N \boldsymbol{u}^{k,n} \rangle \, dt \overset{k}{\sim} C(\widehat{Q}_N)(r(\lambda^*)^{1-r} + n^{-1})^\beta \le \frac{1}{N}$$

by (29) and (60). Next,

$$J_2 = (\boldsymbol{G}^k, \operatorname{div}(\tau^N \boldsymbol{u}^{k,n}))_Q \overset{k}{\sim} 0$$

by (22), (23) and (27). Finally,

$$\begin{aligned} J_3 &= (\pi^k, \operatorname{div}(\tau^N \boldsymbol{u}^{k,n}))_Q - (\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \tau^N \boldsymbol{D}\boldsymbol{u}^{k,n})_{\{\tau^N>0\}\cap B^{k,n}} \\ &\overset{k}{\sim} (\pi^k, \tau^N \operatorname{div} \boldsymbol{u}^{k,n})_{\{\tau^N>0\}\cap B^{k,n}} - (\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \tau^N \boldsymbol{D}\boldsymbol{u}^{k,n})_{\{\tau^N>0\}\cap B^{k,n}} \\ &= (\boldsymbol{H}^k - \boldsymbol{H}, \tau^N \boldsymbol{D}\boldsymbol{u}^{k,n})_{\{\tau^N>0\}\cap B^{k,n}} \le \frac{1}{N}. \end{aligned}$$

by dint of (25), (28) and (59) since evidently

$$\{\tau^N > 0\} \cap B^{k,n} \subset \widehat{Q}_N.$$

Thus (76) has been shown.

As far as I_2 in (77) is concerned, we recall that $\boldsymbol{u}^k = \boldsymbol{u}^{k,n}$ in $G^{k,n}$ and notice

$$G^{k,n} = \{\tau^N \equiv 1\} \setminus \big(\{\tau^N \equiv 1\} \cap B^{k,n}\big).$$

Since $\{\tau^N \equiv 1\} \subset \widehat{Q}_N$, we recall the strong convergence (57) and the weak convergence (78) to deduce

$$\begin{aligned} I_2 &\overset{k}{\sim} -(\boldsymbol{S}, \boldsymbol{D}\boldsymbol{u}^{k,n})_{G^{k,n}} = (\boldsymbol{S}, \boldsymbol{D}\boldsymbol{u}^{k,n})_{\{\tau^N\equiv 1\}\cap B^{k,n}} - (\boldsymbol{S}, \boldsymbol{D}\boldsymbol{u}^{k,n})_{\{\tau^N\equiv 1\}} \\ &\overset{k}{\sim} (\boldsymbol{S}, \boldsymbol{D}\boldsymbol{u}^{k,n})_{\{\tau^N\equiv 1\}\cap B^{k,n}} \overset{k}{\sim} C(\widehat{Q}_N)(r(\lambda^*)^{1-r} + n^{-\beta}) \le \frac{1}{N}, \end{aligned}$$

by (28) and (59), thus showing (77) and ultimately proving also (67) for $r < 2$.

Convergence of $\boldsymbol{D}\boldsymbol{u}^k$ Recalling the definition (72), we can infer by Hölder's inequality that

$$\begin{aligned} \|\boldsymbol{D}\boldsymbol{u}^k\|^r_{L^r(S)} &\le \int_S \Big(\int_0^1 (1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} |\boldsymbol{D}\boldsymbol{u}^k|^2 (1 + |\boldsymbol{D}\boldsymbol{v}^k|^2 + |\boldsymbol{D}\boldsymbol{v}|^2)^{(2-r)/2}\, ds \Big)^{r/2} dx\, dt \\ &\le \Big(\int_S I^k \Big)^{r/2} \Big(\int_Q (1 + |\boldsymbol{D}\boldsymbol{v}^k|^2 + |\boldsymbol{D}\boldsymbol{v}|^2)^{r/2} \Big)^{(2-r)/2}, \end{aligned}$$

for any measurable $S \subset Q$, implying with help of the uniform estimate (33) in the end

$$C\|\boldsymbol{D}\boldsymbol{u}^k\|^2_{L^r(S)} \le \int_S I^k \quad \text{for any measurable } S \subset Q. \tag{82}$$

Applying Biting lemma 4.4 to

$$f^k(t,x) = |\boldsymbol{D}\boldsymbol{u}^k(t,x)|^r, \quad (t,x) \in Q,$$

there is a nonincreasing sequence of measurable sets $D^m \subset Q$ with $\lim_{m\to\infty} |D^m| = 0$, such that (without loss of generality) f^k converge weakly in $L^1(Q \setminus D^m)$ for every m. Our aim is to prove

$$\|\boldsymbol{D}\boldsymbol{u}^k\|_{L^r(Q\setminus D^m)} \overset{k}{\sim} 0 \tag{83}$$

for any $m \in \mathbb{N}$. Since $\lim_{m\to\infty} |D^m| = 0$, the pointwise convergence (for a subsequence) follows.

Let $D \in \{D^m\}$. Since f^k converge weakly in $L^1(Q \setminus D)$, they are uniformly equi-integrable in $Q \setminus D$. Let us take an arbitrary $\sigma > 0$ and $N > \sigma^{-1}$, such that

$$S \subset Q \setminus D,\ |S| < \frac{2}{N} \Rightarrow \|f^k\|_{L^1(S)} < \sigma \quad \text{for every } k. \tag{84}$$

We use this N as the starting parameter for the Lipschitz approximation scheme started at the beginning of Subsection 5.2. We may assume $|E^{k,n}| < 2N^{-1}$ for all k by (66), which combined with (82) and (84) yields

$$C\|\boldsymbol{D}\boldsymbol{u}^k\|^2_{L^r(Q\setminus D)} \le \int_{G^{k,n}\setminus D} I^k + C\|\boldsymbol{D}\boldsymbol{u}^k\|^2_{L^r(E^{k,n}\setminus D)} \le \int_{G^{k,n}} I^k + O(\sigma), \tag{85}$$

given that $I^k \ge 0$. Relations (75) and (67) then imply

$$\int_{G^{k,n}} I^k \, dx\, dt \overset{k}{\sim} \frac{\gamma_0^2}{C_1^2}\|\pi^k\|^2_{L^2(Q)} + O(N^{-1}) \overset{k}{\sim} O(N^{-1}) = O(\sigma).$$

If we plug this observation back into (85), we obtain the desired (83). Together with the compactness of the partial pressures (56) and (67), we may assume both p^k and $\boldsymbol{D}\boldsymbol{v}^k$ converge pointwise a.e. in Q, which yields ultimately $\overline{\boldsymbol{S}} = \boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v})$ for $r < 2$ by Vitali's theorem.

5.3 Convergence for $r = 2$

The above procedure, followed step by step, is rendered useless when $r = 2$ for we cannot get rid of the polluting term in (73). On the other hand, the strong convergence in $L^2(Q)$ is not essential for the pointwise convergence of a subsequence. Now we show only the strong convergence in $L^1(Q)$, arriving at the same conclusion. Although we could have skipped the case $r < 2$ entirely, given that the method applied to $r = 2$, resting on Lemma 4.5, may be presented in such a way that it conquers also the former case, we treat this situation apart for two reasons: Firstly, it is much more convenient to use Lemma 4.6 when applicable (see [13] for usage of Lemma 4.5 for a wider range of exponents). Secondly, dealing with the case $r = 2$ individually lets us balance out its slightly increased technicality with simplification of certain terms; consider e.g. I^k in (72).

Several definitions We set

$$g^k = \mathcal{M}^*\big(|\nabla \boldsymbol{u}^k|\big) + \mathcal{M}^*\big(|\boldsymbol{S}^k - \overline{\boldsymbol{S}}|\big) + \mathcal{M}^*\big(|\pi^k|\big). \tag{86}$$

By the properties of $\mathcal{M}^*$ and boundedness of the individual arguments in $L^2(Q)$ (see (39), (48) and (53)), the sequence $\{g^k\}$ is also bounded in $L^2(Q)$. Therefore[7]

$$\sum_{i=0}^{n} \int_{\left\{2^{2^{n+i}} < g^k \le 2^{2^{n+i+1}}\right\}} \big(g^k\big)^2 \, dx\, dt \le C \quad \text{for any } n \in \mathbb{N},$$

independently of k and n, which guarantees there are

$$2^{2^n} \le \lambda^{k,n} \le 2^{2^{2n}} \tag{87}$$

such that

$$\int_{\left\{\lambda^{k,n} < g^k \le (\lambda^{k,n})^2\right\}} \big(g^k\big)^2 \, dx\, dt \le \frac{C}{n} \quad \text{for any } k, n \in \mathbb{N}. \tag{88}$$

Let us define level sets related to g^k:

$$\begin{aligned} A_1^{k,n} &= \big\{g^k \le \lambda^{k,n}\big\}, \\ A_2^{k,n} &= \big\{\lambda^{k,n} < g^k \le \big(\lambda^{k,n}\big)^2\big\}, \\ A_3^{k,n} &= \big\{\big(\lambda^{k,n}\big)^2 < g^k\big\}. \end{aligned} \tag{89}$$

[7] Notice that $2^{2^{n+i+1}} = \big(2^{2^{n+i}}\big)^2$, which is the reason for our choice of such numbers.

By (88), we can bound the measure of $A_2^{k,n}$ as

$$\left|A_2^{k,n}\right| = \int_{\left\{\lambda^{k,n}<g^k\le(\lambda^{k,n})^2\right\}} 1\,dx\,dt \le \int_{\left\{\lambda^{k,n}<g^k\le(\lambda^{k,n})^2\right\}} \frac{\left(g^k\right)^2}{\left(\lambda^{k,n}\right)^2}\,dx\,dt \le \frac{C}{n\left(\lambda^{k,n}\right)^2}. \tag{90}$$

Chebyshev's inequality also implies

$$(\lambda^{k,n})^4\left|A_3^{k,n}\right| \le C. \tag{91}$$

Furthermore, we define

$$F^k = \{\mathcal{M}^*(|\boldsymbol{v}^k\otimes\boldsymbol{v}^k\Phi_k(|\boldsymbol{v}^k|) - \boldsymbol{v}\otimes\boldsymbol{v} + (p_1^k - p_1)\boldsymbol{I}|) > 1\}.$$

By means of the strong-type estimate for $\mathcal{M}^*$ and (35), (42) and (56), we obtain

$$\lim_{k\to\infty}\left|F^k\right| \le C\lim_{k\to\infty}\left\|\boldsymbol{v}^k\otimes\boldsymbol{v}^k\Phi_k(|\boldsymbol{v}^k|) - \boldsymbol{v}\otimes\boldsymbol{v} + (p_1^k - p_1)\boldsymbol{I}\right\|_{L^\sigma(Q)}^\sigma = 0 \quad \text{for some } \sigma > 1. \tag{92}$$

For fixed $n\in\mathbb{N}$ we also find $\tau^n\in\mathcal{C}_c^\infty(Q;[0,1])$ such that

$$\left|\{\tau^n < 1\}\right| \le \frac{1}{2^{2^{2n+1}}n}. \tag{93}$$

Finally we include all the adverse sets into one so that we define

$$E^{k,n} = \left(A_2^{k,n}\cup A_3^{k,n}\cup F^k\cup\{\tau^n<1\}\right)\cap Q, \tag{94}$$
$$G^{k,n} = Q\setminus E^{k,n}. \tag{95}$$

It follows easily from the definiton of $E^{k,n}$, (87), (92) and (93) that

$$\left(\lambda^{k,n}\right)^2\left|E^{k,n}\cap A_1^{k,n}\right| \overset{k}{\sim} O(n^{-1}). \tag{96}$$

We would like to engage Theorem 4.5 with $E^{k,n}$ playing the role of E. Setting

$$\boldsymbol{H}^k = \boldsymbol{v}^k\otimes\boldsymbol{v}^k\Phi_k(|\boldsymbol{v}^k|) - \boldsymbol{v}\otimes\boldsymbol{v} - \boldsymbol{S}^k + \overline{\boldsymbol{S}} + (p^k - p)\boldsymbol{I},$$

the equation (17) evidently holds with $\boldsymbol{u}^k$ and $\boldsymbol{H}^k$. The sets $E^{k,n}$ are open due to the lower semicontinuity of $\mathcal{M}^*$. Finally, subadditivity of $\mathcal{M}^*$ yields

$$\begin{aligned}\left\{g^k > \lambda^{k,n}\right\}\cup F^k &\supset \left\{\mathcal{M}^*\left(|\nabla\boldsymbol{u}^k|\right) > \lambda^{k,n}\right\}\cup\left\{\mathcal{M}^*\left(|\boldsymbol{S}^k - \overline{\boldsymbol{S}} - \pi^k\boldsymbol{I}|\right) > \lambda^{k,n}\right\}\cup F^k\\ &\supset \left\{\mathcal{M}^*\left(|\nabla\boldsymbol{u}^k|\right) > \lambda^{k,n}\right\}\cup\left\{\mathcal{M}^*\left(|\boldsymbol{H}^k|\right) > \lambda^{k,n}+1\right\}\\ &\supset \left\{\mathcal{M}^*\left(|\nabla\boldsymbol{u}^k|\right) + \mathcal{M}^*\left(|\boldsymbol{H}^k|\right) > 3\lambda^{k,n}\right\},\end{aligned}$$

implying the required property

$$\left\{\mathcal{M}^*\left(|\nabla\boldsymbol{u}^k|\right) + \mathcal{M}^*\left(|\boldsymbol{H}^k|\right) > 3\lambda^{k,n}\right\}\cap Q \subset E^{k,n}.$$

Therefore we may invoke Theorem 4.5 with $\Lambda = 3\lambda^{k,n}$. Let us denote

$$\boldsymbol{u}^{k,n} = \mathcal{L}_{E^{k,n}}\boldsymbol{u}^k.$$

Note that due to the L^p-estimate (18) and the strong convergence (42) it holds that

$$\boldsymbol{u}^{k,n}\to 0 \quad \text{strongly in } L^2(Q) \text{ as } k\to\infty \text{ for any } n\in\mathbb{N}. \tag{97}$$

Accessory calculation In this part we show a result that will be useful in a while, namely

$$\big(\boldsymbol{S}^k - \boldsymbol{S}(p_1^k + p_2, \boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{u}^k\big)_{G^{k,n}} \overset{k}{\sim} O(n^{-1}). \tag{98}$$

The individual steps to be taken will be

$$\begin{aligned}\big(\boldsymbol{S}^k - \boldsymbol{S}(p_1^k + p_2, \boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{u}^k\big)_{G^{k,n}} &\overset{k}{\sim} \big(\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \boldsymbol{D}\boldsymbol{u}^k\big)_{G^{k,n}} + O(n^{-1}) &(99)\\ &\overset{k}{\sim} \big(\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \boldsymbol{D}(\tau^n \boldsymbol{u}^{k,n})\big)_Q + O(n^{-1}) &(100)\\ &\overset{k}{\sim} O(n^{-1}). &(101)\end{aligned}$$

As for the first relation (99), in view of the strong convergence (57), it boils down to showing

$$\big(\overline{\boldsymbol{S}} - \boldsymbol{S}, \boldsymbol{D}\boldsymbol{u}^k\big)_{G^{k,n}} \overset{k}{\sim} O(n^{-1}). \tag{102}$$

By Theorem 4.5 we have $\boldsymbol{u}^k = \boldsymbol{u}^{k,n}$ in $G^{k,n}$, which set we rewrite by (94) as

$$G^{k,n} = \{\tau^n = 1\} \setminus \big((F^k \cap A_1^{k,n} \cap \{\tau^n = 1\}) \cup ((A_2^{k,n} \cup A_3^{k,n}) \cap \{\tau^n = 1\})\big). \tag{103}$$

The Lipschitz bound (19) combined with the strong convergence (97) allows us to assume that for any $n \in \mathbb{N}$,

$$\nabla \boldsymbol{u}^{k,n} \to 0 \quad \text{weakly in } L^2\big(\{\tau^n = 1\}\big) \text{ as } k \to \infty. \tag{104}$$

By (19) again and the shrinkage of F^k expressed in (92), it follows that

$$\big|\big(\overline{\boldsymbol{S}} - \boldsymbol{S}, \boldsymbol{D}\boldsymbol{u}^{k,n}\big)_{F^k \cap A_1^{k,n} \cap \{\tau^n = 1\}}\big| \le C|F^k|^{1/2}\big(\lambda^{k,n} + C_{\{\tau^n=1\}}\|\boldsymbol{u}^k\|_{L^1(Q)}\big) \overset{k}{\sim} 0. \tag{105}$$

Lastly, we easily deduce by (19), bounds on $|A_2^{k,n} \cup A_3^{k,n}|$ given in (90) and (91) that

$$\begin{aligned}\big|\big(\overline{\boldsymbol{S}} - \boldsymbol{S}, \boldsymbol{D}\boldsymbol{u}^{k,n}\big)_{(A_2^{k,n} \cup A_3^{k,n}) \cap \{\tau^n = 1\}}\big| &\le \|\overline{\boldsymbol{S}} - \boldsymbol{S}\|_{L^2(A_2^{k,n} \cup A_3^{k,n})} \|\nabla \boldsymbol{u}^{k,n}\|_{L^\infty(\{\tau^n=1\})} |A_2^{k,n} \cup A_3^{k,n}|^{1/2}\\ &\overset{k}{\sim} O(n^{-1}).\end{aligned} \tag{106}$$

Combining (103)–(106), we obtain (102) and hence also the first step of (99).

Towards showing the second step (100), we start noticing that

$$\big(\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \boldsymbol{D}\boldsymbol{u}^{k,n}\big)_{E^{k,n} \cap \{\tau^n = 1\}} \overset{k}{\sim} O(n^{-1}). \tag{107}$$

Indeed, treating the level sets (89) individually, we estimate

$$\big(\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \boldsymbol{D}\boldsymbol{u}^{k,n}\big)_{E^{k,n} \cap A_1^{k,n} \cap \{\tau^n = 1\}} = \big(\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \boldsymbol{D}\boldsymbol{u}^{k,n}\big)_{F^k \cap A_1^{k,n} \cap \{\tau^n = 1\}} \overset{k}{\sim} 0$$

as in (105) due to boundedness of $\boldsymbol{S}^k$ in $L^2(Q)$ (see (48)). Then

$$\big(\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \boldsymbol{D}\boldsymbol{u}^{k,n}\big)_{A_2^{k,n} \cap \{\tau^n = 1\}} \le Cn^{-1/2}(\lambda^{k,n})^{-1}\big(\lambda^{k,n} + C_{\{\tau^n=1\}}\|\boldsymbol{u}^k\|_{L^1(Q)}\big) \overset{k}{\sim} O(n^{-1})$$

by the bounds (19) and (90). Very similarly, using the bounds (19) and (91),

$$\big(\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \boldsymbol{D}\boldsymbol{u}^{k,n}\big)_{A_3^{k,n} \cap \{\tau^n = 1\}} \le C(\lambda^{k,n})^{-2}\big(\lambda^{k,n} + C_{\{\tau^n=1\}}\|\boldsymbol{u}^k\|_{L^1(Q)}\big) \overset{k}{\sim} O(n^{-1}).$$

Hence (107) holds and therefore also

$$\big(\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \boldsymbol{D}\boldsymbol{u}^k\big)_{G^{k,n}} = \big(\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \boldsymbol{D}\boldsymbol{u}^{k,n}\big)_{G^{k,n}} \overset{k}{\sim} \big(\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \boldsymbol{D}\boldsymbol{u}^{k,n}\big)_{\{\tau^n=1\}} + O(n^{-1}). \tag{108}$$

Next, we would like to add another negligible term, namely the Lipschitz bound (19) and properties (87) and (93) imply

$$\big(\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \tau^n \boldsymbol{D}\boldsymbol{u}^{k,n}\big)_{\{0<\tau^n<1\}} \overset{k}{\sim} C\big|\{\tau^n < 1\}\big|^{1/2}\big(\lambda^{k,n} + C_{\operatorname{spt}\tau^n}\|\boldsymbol{u}^k\|_{L^1(Q)}\big) \overset{k}{\sim} O(n^{-1}).$$

As a result, we may improve (108) into

$$\big(\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \boldsymbol{D}\boldsymbol{u}^k\big)_{G^{k,n}} \overset{k}{\sim} \big(\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \tau^n\boldsymbol{D}\boldsymbol{u}^{k,n}\big)_Q + O(n^{-1}) \overset{k}{\sim} \big(\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \boldsymbol{D}(\tau^n\boldsymbol{u}^{k,n})\big)_Q + O(n^{-1}),$$

recalling also the strong convergence of the Lipschitz approximations (97). The last inequality justifies the second step (100) and we may jubilate, for $\tau^n\boldsymbol{u}^{k,n}$ is a legal test function in both the weak formulations (31) and (49). We exploit this fact to rewrite

$$\begin{aligned}\big(\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \boldsymbol{D}(\tau^n\boldsymbol{u}^{k,n})\big)_Q &= \big(p^k - p, \operatorname{div}(\tau^n\boldsymbol{u}^{k,n})\big)_Q + \big(\boldsymbol{v}^k\otimes\boldsymbol{v}^k\Phi_k(|\boldsymbol{v}^k|) - \boldsymbol{v}\otimes\boldsymbol{v}, \nabla(\tau^n\boldsymbol{u}^{k,n})\big)_Q \\ &\quad - \int_0^T \langle\partial_t\boldsymbol{u}^k, \tau^n\boldsymbol{u}^{k,n}\rangle\,dt = I_1 + I_2 + I_3.\end{aligned} \tag{109}$$

We will demonstrate $I_i \overset{k}{\sim} O(n^{-1})$ for each $i = 1,2,3$. Beginning with I_1, the strong convergence (56) and the bound (19) yield

$$\begin{aligned} I_1 &\overset{k}{\sim} \big(p_2^k - p_2, \operatorname{div}(\tau^n\boldsymbol{u}^{k,n})\big)_Q = \big(\pi^k, \boldsymbol{u}^{k,n}\cdot\nabla\tau^n\big)_Q + \big(\pi^k, \tau^n\operatorname{div}\boldsymbol{u}^{k,n}\big)_Q \\ &\overset{k}{\sim} \big(\pi^k, \tau^n\operatorname{div}\boldsymbol{u}^{k,n}\big)_Q = \big(\pi^k, \tau^n\operatorname{div}\boldsymbol{u}^{k,n}\big)_{E^{k,n}}.\end{aligned} \tag{110}$$

We could ignore the term $(\pi^k, \boldsymbol{u}^{k,n}\cdot\nabla\tau^n)_Q$ due to the strong convergence (97) and boundedness coming from (53). Classical properties of Sobolev functions also guarantee $\operatorname{div}\boldsymbol{u}^{k,n} = \operatorname{div}\boldsymbol{u}^k = 0$ a.e. in $G^{k,n}$, which we exploited in the last equality. The rest follows the track of (107). More precisely,

$$\big(\pi^k, \tau^n\operatorname{div}\boldsymbol{u}^{k,n}\big)_{E^{k,n}\cap A_1^{k,n}} \le C\big|E^{k,n}\cap A_1^{k,n}\big|^{1/2}\big(\lambda^{k,n} + C_{\operatorname{spt}\tau^n}\|\boldsymbol{u}^k\|_{L^1(Q)}\big) \overset{k}{\sim} O(n^{-1})$$

by the observation (96). Then

$$\big(\pi^k, \tau^n\operatorname{div}\boldsymbol{u}^{k,n}\big)_{E^{k,n}\cap A_2^{k,n}} \le Cn^{-1/2}(\lambda^{k,n})^{-1}\big(\lambda^{k,n} + C_{\operatorname{spt}\tau^n}\|\boldsymbol{u}^k\|_{L^1(Q)}\big) \overset{k}{\sim} O(n^{-1})$$

by estimates stemming from (19), (53) and (90). And similarly, only switching to (91) in order to bound $|A_3^{k,n}|$,

$$\big(\pi^k, \tau^n\operatorname{div}\boldsymbol{u}^{k,n}\big)_{E^{k,n}\cap A_3^{k,n}} \le C(\lambda^{k,n})^{-2}\big(\lambda^{k,n} + C_{\operatorname{spt}\tau^n}\|\boldsymbol{u}^k\|_{L^1(Q)}\big) \overset{k}{\sim} O(n^{-1}).$$

Thus we have shown that (110) can be concluded as

$$I_1 \overset{k}{\sim} O(n^{-1}). \tag{111}$$

The term I_2 is quite effortless to tackle. Due to the strong convergences (42) and (97), we have

$$\begin{aligned} I_2 &= \big(\boldsymbol{v}^k\otimes\boldsymbol{v}^k\Phi_k(|\boldsymbol{v}^k|) - \boldsymbol{v}\otimes\boldsymbol{v}, \nabla(\tau^n\boldsymbol{u}^{k,n})\big)_Q \overset{k}{\sim} \big(\boldsymbol{v}^k\otimes\boldsymbol{v}^k\Phi_k(|\boldsymbol{v}^k|) - \boldsymbol{v}\otimes\boldsymbol{v}, \tau^n\nabla\boldsymbol{u}^{k,n}\big)_Q \\ &\le C\|\boldsymbol{v}^k\otimes\boldsymbol{v}^k\Phi_k(|\boldsymbol{v}^k|) - \boldsymbol{v}\otimes\boldsymbol{v}\|_{L^1(Q)}\big(\lambda^{k,n} + C_{\operatorname{spt}\tau^n}\|\boldsymbol{u}^k\|_{L^1(Q)}\big) \overset{k}{\sim} 0.\end{aligned} \tag{112}$$

To process the last term I_3, corresponding to the time derivative, we recall the *integration by parts* formula (21), according to which we can rewrite I_3 as

$$I_3 = \frac{1}{2}\int_Q \big(2\boldsymbol{u}\cdot\boldsymbol{u}^{k,n} - |\boldsymbol{u}^{k,n}|^2\big)\partial_t\tau^n\,dx\,dt + \int_{E^{k,n}} (\partial_t\boldsymbol{u}^{k,n})\cdot(\boldsymbol{u}-\boldsymbol{u}^{k,n})\tau^n\,dx\,dt$$
$$\overset{k}{\sim} \int_{E^{k,n}} (\partial_t\boldsymbol{u}^{k,n})\cdot(\boldsymbol{u}-\boldsymbol{u}^{k,n})\tau^n\,dx\,dt \le C\big|E^{k,n}\big|\big(\lambda^{k,n} + C_{\mathrm{spt}\,\tau^n}\|\boldsymbol{u}^k\|_{L^1(Q)}\big)^2, \tag{113}$$

first by the strong convergence (97) and then by the estimate (20). However, the sets $E^{k,n}$ by their very definition (94) satisfy trivially

$$\big|E^{k,n}\big| \le \big|A_2^{k,n}\big| + \big|A_3^{k,n}\big| + \big|F^k\big| + \big|\{\tau^n < 1\}\big|.$$

Estimates for the individual summands are contained in (90)–(93) and we plug them into (113) to infer

$$I_3 \overset{k}{\sim} C\big|E^{k,n}\big|\big(\lambda^{k,n} + C_{\mathrm{spt}\,\tau^n}\|\boldsymbol{u}^k\|_{L^1(Q)}\big)^2 \overset{k}{\sim} O(n^{-1}).$$

We insert this last result into (109) together with (111) and (112), procuring the third and final relation (101). The longed-for (98) has been hereby justified.

Pressure test function For $K > 0$ we consider the usual truncation operator $T_K : \mathbb{R} \to \mathbb{R}$

$$T_K(x) = \begin{cases} x & \text{for } |x| \le K, \\ K\,\mathrm{sgn}\,x & \text{for } |x| > K. \end{cases}$$

In contrast to the case $r < 2$ (cf. (68)), now we take

$$\boldsymbol{\varphi}^{k,n} = \mathcal{N}\big(T_{\lambda^{k,n}}\pi^k - (T_{\lambda^{k,n}}\pi^k)_\Omega\big).$$

For all $p < \infty$ we may assume due to the convergence (53) and the boundedness of $\lambda^{k,n}$ (87) that

$$T_{\lambda^{k,n}}\pi^k \to \overline{T}^n \quad \text{weakly in } L^p(Q) \text{ as } k \to \infty, \tag{114}$$
$$\overline{T}^n \to \overline{T} \quad \text{weakly in } L^2(Q) \text{ as } n \to \infty. \tag{115}$$

By the weak convergences (53) and (114) evidently

$$T_{\lambda^{k,n}}\pi^k - \pi^k \to \overline{T}^n \quad \text{weakly in } L^2(Q) \text{ as } k \to \infty.$$

Due to (53) and the bound (87), we may estimate

$$\int_Q |T_{\lambda^{k,n}}\pi^k - \pi^k| \le 2\int_{\{|\pi^k|>\lambda^{k,n}\}} |\pi^k| \le 2\int_{\{|\pi^k|>\lambda^{k,n}\}} \frac{|\pi^k|^2}{\lambda^{k,n}} = O(n^{-1}),$$

specifying the weak convergence (115) more closely as

$$\overline{T}^n \to 0 \quad \text{weakly in } L^2(Q) \text{ as } n \to \infty.$$

By the same token (up to a subsequence)

$$T_{\lambda^{k,n}}\pi^k - (T_{\lambda^{k,n}}\pi^k)_\Omega \to \overline{T}_0^n \quad \text{weakly in } L^p(Q) \text{ as } k \to \infty,$$
$$\overline{T}_0^n \to 0 \quad \text{weakly in } L^2(Q) \text{ as } n \to \infty. \tag{116}$$

Back to $\boldsymbol{\varphi}^{k,n}$, the property (15) entails for any $1 < p < \infty$ that

$$\|\boldsymbol{\varphi}^{k,n}\|_{L^p(0,T;W^{2,p}(\Omega))} \le C_{reg,p}\|T_{\lambda^{k,n}}\pi^k - (T_{\lambda^{k,n}}\pi^k)_\Omega\|_{L^p(Q)} \tag{117}$$
$$\le C_{reg,p}\,\lambda^{k,n}. \tag{118}$$

As a result, and also owing to (116), we may assume that for all $p < \infty$

$$\boldsymbol{\varphi}^{k,n} \to \overline{\boldsymbol{\varphi}}^n \quad \text{weakly in } L^p(0,T;W^{2,p}(\Omega)) \text{ as } k \to \infty, \tag{119}$$
$$\overline{\boldsymbol{\varphi}}^n \to 0 \quad \text{weakly in } L^2(0,T;W^{2,2}(\Omega)) \text{ as } n \to \infty. \tag{120}$$

Convergence of p_2^k Let $n \in \mathbb{N}$. We are going to show

$$(\pi^k, T_{\lambda^{k,n}}\pi^k)_Q \overset{k}{\sim} O(n^{-1}), \tag{121}$$

implying $\pi^k \to 0$ strongly in $L^1(Q)$, hence $\pi^k \to 0$ a.e. in Q for a subsequence. We write

$$(\pi^k, T_{\lambda^{k,n}}\pi^k)_Q = (\pi^k, T_{\lambda^{k,n}}\pi^k - (T_{\lambda^{k,n}}\pi^k)_\Omega)_Q = (\pi^k, \Delta\boldsymbol{\varphi}^{k,n})_Q \overset{k}{\sim} (\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \nabla^2\boldsymbol{\varphi}^{k,n})_Q$$

by the weak formulation for p_2^k (51), strong convergence (43) and weak convergences (53) and (119). We carry on by means of the strong convergence (57):

$$\begin{aligned}(\pi^k, T_{\lambda^{k,n}}\pi^k)_Q &\overset{k}{\sim} (\boldsymbol{S}^k - \overline{\boldsymbol{S}}, \nabla^2\boldsymbol{\varphi}^{k,n})_Q \overset{k}{\sim} (\boldsymbol{S}^k - \boldsymbol{S}(p_1^k + p_2, \boldsymbol{Dv}), \nabla^2\boldsymbol{\varphi}^{k,n})_Q - (\boldsymbol{S} - \overline{\boldsymbol{S}}, \nabla^2\overline{\boldsymbol{\varphi}}^n)_Q \\ &\le \gamma_0 \int_Q |\pi^k||\nabla^2\boldsymbol{\varphi}^{k,n}|\,dx\,dt + C_2\int_Q |\boldsymbol{Du}^k||\nabla^2\boldsymbol{\varphi}^{k,n}|\,dx\,dt + O(n^{-1}),\end{aligned} \tag{122}$$

by (10) and (120). We will concentrate on the second integral, decomposing Q into four subdomains (see (89), (94) and (95) for definitions):

$$Q = (E^{k,n} \cap A_1^{k,n}) \cup (E^{k,n} \cap A_2^{k,n}) \cup (E^{k,n} \cap A_3^{k,n}) \cup G^{k,n}.$$

Accordingly

$$\int_Q |\boldsymbol{Du}^k||\nabla^2\boldsymbol{\varphi}^{k,n}|\,dx\,dt = I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned}I_1 &= \int_{E^{k,n}\cap A_1^{k,n}} |\boldsymbol{Du}^k||\nabla^2\boldsymbol{\varphi}^{k,n}|\,dx\,dt \le \|\boldsymbol{Du}^k\|_{L^\infty(A_1^{k,n})}\|\nabla^2\boldsymbol{\varphi}^{k,n}\|_{L^2(Q)}|E^{k,n}\cap A_1^{k,n}|^{1/2} \\ &\le C\lambda^{k,n}|E^{k,n}\cap A_1^{k,n}|^{1/2} \overset{k}{\sim} O(n^{-1}),\end{aligned}$$

by the observation (96), $|\boldsymbol{Du}^k| \le \lambda^{k,n}$ a.e. in $A_1^{k,n}$ and the estimate of $\boldsymbol{\varphi}^{k,n}$ (117). Next

$$I_2 = \int_{E^{k,n}\cap A_2^{k,n}} |\boldsymbol{Du}^k||\nabla^2\boldsymbol{\varphi}^{k,n}|\,dx\,dt \le \|\boldsymbol{Du}^k\|_{L^2(A_2^{k,n})}\|\nabla^2\boldsymbol{\varphi}^{k,n}\|_{L^2(Q)} \le C\|g^k\|_{L^2(A_2^{k,n})} = O(n^{-1}),$$

by the key property of $A_2^{k,n}$ (88) and the estimate (117), and

$$\begin{aligned}I_3 &= \int_{E^{k,n}\cap A_3^{k,n}} |\boldsymbol{Du}^k||\nabla^2\boldsymbol{\varphi}^{k,n}|\,dx\,dt \le \|\boldsymbol{Du}^k\|_{L^2(Q)}\|\nabla^2\boldsymbol{\varphi}^{k,n}\|_{L^p(Q)}|A_3^{k,n}|^{\frac{p-2}{2p}} \le C(\lambda^{k,n})^{\frac{4-p}{p}} \\ &\overset{k}{\sim} O(n^{-1})\end{aligned}$$

for any $p > 4$ by the bound on $|A_3^{k,n}|$ (91) and (118) for a fixed $p > 4$. Finally,

$$\begin{aligned}I_4 &= \int_{G^{k,n}} |\boldsymbol{Du}^k||\nabla^2\boldsymbol{\varphi}^{k,n}|\,dx\,dt \\ &\le \Big(\frac{\gamma_0^2}{C_1^2}\|\pi^k\|^2_{L^2(A_1^{k,n})} + \frac{2}{C_1}\big(\boldsymbol{S}^k - \boldsymbol{S}(p_1^k + p_2, \boldsymbol{Dv}), \boldsymbol{Du}^k\big)_{G^{k,n}}\Big)^{1/2}\|\nabla^2\boldsymbol{\varphi}^{k,n}\|_{L^2(Q)} \\ &\overset{k}{\sim} \frac{\gamma_0}{C_1}\|\pi^k\|_{L^2(A_1^{k,n})}\|\nabla^2\boldsymbol{\varphi}^{k,n}\|_{L^2(Q)} + O(n^{-1}),\end{aligned}$$

by (9), $G^{k,n} \subset A_1^{k,n}$, the accessory calculation (98) and the estimate (117). We see that only the last term I_4 adds a palpable contribution to (122), which hence simplifies into

$$
\begin{aligned}
(\pi^k, T_{\lambda^{k,n}}\pi^k)_Q &\overset{k}{\sim} \gamma_0 \left(1 + \frac{C_2}{C_1}\right) \|\pi^k\|_{L^2(A_1^{k,n})} \|\nabla^2 \boldsymbol{\varphi}^{k,n}\|_{L^2(Q)} + O(n^{-1}) \\
&\leq \gamma_0 C_{reg} \left(\frac{C_1 + C_2}{C_1}\right) \|\pi^k\|_{L^2(A_1^{k,n})} \|T_{\lambda^{k,n}}\pi^k - (T_{\lambda^{k,n}}\pi^k)_\Omega\|_{L^2(Q)} + O(n^{-1}) \\
&\leq \gamma_0 C_{reg} \left(\frac{C_1 + C_2}{C_1}\right) \|\pi^k\|_{L^2(A_1^{k,n})} \|T_{\lambda^{k,n}}\pi^k\|_{L^2(Q)} + O(n^{-1}),
\end{aligned} \tag{123}
$$

by (117) and an elementary manipulation

$$
\|T_{\lambda^{k,n}}\pi^k - (T_{\lambda^{k,n}}\pi^k)_\Omega\|_{L^2(Q)}^2 = \|T_{\lambda^{k,n}}\pi^k\|_{L^2(Q)}^2 - |\Omega|(T_{\lambda^{k,n}}\pi^k)_\Omega^2 \leq \|T_{\lambda^{k,n}}\pi^k\|_{L^2(Q)}^2.
$$

What remains is to relate $(\pi^k, T_{\lambda^{k,n}}\pi^k)_Q$ to the right-hand side in a better way: Recalling the definition of g^k (86), we have trivially

$$
|T_{\lambda^{k,n}}\pi^k| \leq |\pi^k| \leq g^k \quad \text{a.e. in } Q.
$$

Therefore, and by the estimates (88) and (91), we observe

$$
\|T_{\lambda^{k,n}}\pi^k\|_{L^2(Q)}^2 \leq \|\pi^k\|_{L^2(A_1^{k,n})}^2 + \|g^k\|_{L^2(A_2^{k,n})}^2 + \|\lambda^{k,n}\|_{L^2(A_3^{k,n})}^2 \leq \|\pi^k\|_{L^2(A_1^{k,n})}^2 + O(n^{-1}).
$$

Then we add an obvious inequality

$$
\|\pi^k\|_{L^2(A_1^{k,n})}^2 \leq (\pi^k, T_{\lambda^{k,n}}\pi^k)_Q
$$

and (123) combined with $0 < \gamma_0 < \dfrac{C_1}{C_{reg}(C_1 + C_2)}$ from Assumption 2.2 becomes the desired (121) and we may hence assume (bearing in mind the already proved result for p_1^k (56))

$$
p^k \to p \quad \text{a.e. in } Q. \tag{124}
$$

Convergence of $\boldsymbol{D}\boldsymbol{u}^k$ This time the Biting lemma will be engaged on

$$
f^k(t,x) = |\pi^k(t,x)|^2 + |\boldsymbol{D}\boldsymbol{u}^k(t,x)|^2, \quad (t,x) \in Q,
$$

with our sight set on

$$
\|\boldsymbol{D}\boldsymbol{u}^k\|_{L^r(Q \setminus D^m)} \overset{k}{\sim} 0
$$

for any $m \in \mathbb{N}$, where D^m are the sets provided by the Biting lemma, like in (83). Assuming without loss of generality that f^k are themselves weakly convergent in $L^1(Q \setminus D_m)$, in particular they are equi-integrable in $Q \setminus D^m$, for any $m \in \mathbb{N}$, Vitali's theorem and the pointwise convergence (124) imply

$$
\pi^k \to 0 \quad \text{strongly in } L^2(Q \setminus D^m) \text{ for every } m \in \mathbb{N}. \tag{125}
$$

Let $m_0 \in \mathbb{N}$ be fixed. Equi-integrability of f^k and the definition of $E^{k,n}$ (94) imply

$$
\|\boldsymbol{D}\boldsymbol{u}^k\|_{L^2(Q \setminus D^{m_0})}^2 \overset{k}{\sim} \|\boldsymbol{D}\boldsymbol{u}^k\|_{L^2(G^{k,n} \setminus D^{m_0})}^2 + O(n^{-1}) \leq \|\boldsymbol{D}\boldsymbol{u}^k\|_{L^2(G^{k,n} \setminus D^m)}^2 + O(n^{-1})
$$

for any $m \geq m_0$. Take $m(n) \geq m_0$ fulfilling

$$
\left|D^{m(n)}\right| \leq \frac{1}{2^{2^{2n+1}} n}. \tag{126}
$$

Applying the estimate (9) and convergence (125), we obtain

$$C\|\boldsymbol{D}\boldsymbol{u}^k\|^2_{L^2(G^{k,n}\setminus D^{m(n)})} \overset{k}{\sim} (\boldsymbol{S}^k - \boldsymbol{S}(p_1^k + p_2, \boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{u}^k)_{G^{k,n}\setminus D^{m(n)}}$$
$$\overset{k}{\sim} -(\boldsymbol{S}^k - \boldsymbol{S}(p_1^k + p_2, \boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{u}^{k,n})_{G^{k,n}\cap D^{m(n)}} + O(n^{-1}),$$

where we recalled the accessory calculation (98), i.e.

$$\big(\boldsymbol{S}^k - \boldsymbol{S}(p_1^k + p_2, \boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{u}^k\big)_{G^{k,n}} \overset{k}{\sim} O(n^{-1}),$$

for the second relation. The rest is assured by the Lipschitz bound (19) and (126):

$$(\boldsymbol{S}^k - \boldsymbol{S}(p_1^k + p_2, \boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{u}^{k,n})_{G^{k,n}\cap D^{m(n)}} \le C|D^{m(n)}|^{1/2}\big(\lambda^{k,n} + C_{\{\tau^n=1\}}\|\boldsymbol{u}^k\|_{L^1(Q)}\big) \overset{k}{\sim} O(n^{-1}),$$

yielding

$$\|\boldsymbol{D}\boldsymbol{u}^k\|^2_{L^2(Q\setminus D^{m_0})} \overset{k}{\sim} O(n^{-1}),$$

hence also the pointwise convergence (for a subsequence) of $\boldsymbol{D}\boldsymbol{u}^k$. Together with the compactness of the pressure (124) we obtain also $\overline{\boldsymbol{S}} = \boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v})$ for $r = 2$.

5.4 Initial condition

Proceeding exactly like in the Galerkin approximation (see Appendix), we could justify

$$(\boldsymbol{v}_0 - \boldsymbol{v}(0), \boldsymbol{w}) = 0 \quad \text{for all } \boldsymbol{w} \in W^{1,q'}_{\boldsymbol{n}}(\Omega),$$

i.e. $\boldsymbol{v}(0) = \boldsymbol{v}_0$. Next we will show

$$\boldsymbol{v}^k(t) \to \boldsymbol{v}(t) \quad \text{weakly in } L^2(\Omega) \text{ for all } t \in (0,T). \tag{127}$$

Let $t \in (0,T)$, then $\{\boldsymbol{v}^k(t)\}_k$ is bounded in $L^2(\Omega)$ and we may assume that for a subsequence

$$\boldsymbol{v}^{k_m}(t) \to \overline{\boldsymbol{v}} \quad \text{weakly in } L^2(\Omega).$$

Recall (31) and take $\boldsymbol{\varphi} = \boldsymbol{w}\chi_{(0,t)}$ for an arbitrary $\boldsymbol{w} \in W^{1,q'}_{\boldsymbol{n}}(\Omega)$. Then

$$(\boldsymbol{v}^{k_m}(t), \boldsymbol{w}) - (\boldsymbol{v}_0, \boldsymbol{w}) = (\boldsymbol{v}^{k_m} \otimes \boldsymbol{v}^{k_m}\Phi_{k_m}(|\boldsymbol{v}^{k_m}|), \nabla\boldsymbol{w})_{Q_t} - (\boldsymbol{S}^{k_m}, \boldsymbol{D}\boldsymbol{w})_{Q_t}$$
$$- \alpha(\boldsymbol{v}^{k_m}\Phi_{k_m}(|\boldsymbol{v}^{k_m}|), \boldsymbol{w})_{\Gamma_t} + (p^{k_m}, \operatorname{div}\boldsymbol{w})_{Q_t} + (\boldsymbol{F}, \nabla\boldsymbol{w})_{Q_t},$$

which tends for $m \to \infty$ to

$$(\overline{\boldsymbol{v}}, \boldsymbol{w}) - (\boldsymbol{v}_0, \boldsymbol{w}) = (\boldsymbol{v} \otimes \boldsymbol{v}, \nabla\boldsymbol{w})_{Q_t} - (\boldsymbol{S}, \boldsymbol{D}\boldsymbol{w})_{Q_t} - \alpha(\boldsymbol{v}, \boldsymbol{w})_{\Gamma_t} + (p, \operatorname{div}\boldsymbol{w})_{Q_t} + (\boldsymbol{F}, \nabla\boldsymbol{w})_{Q_t}$$
$$= (\boldsymbol{v}(t), \boldsymbol{w}) - (\boldsymbol{v}_0, \boldsymbol{w}),$$

by the already proved weak formulation (7). Therefore $\overline{\boldsymbol{v}} = \boldsymbol{v}(t)$ and we may extend the result beyond a mere subsequence, in other words (127) holds.

Regarding the strong convergence to the initial value in $L^2(\Omega)$, in the weak formulation (31) we can take $\boldsymbol{\varphi} = \boldsymbol{v}^k\chi_{(0,t)}$ for any $t \in (0,T)$, obtaining

$$\|\boldsymbol{v}^k(t)\|^2_{L^2(\Omega)} - \|\boldsymbol{v}_0\|^2_{L^2(\Omega)} = (\boldsymbol{F}, \nabla\boldsymbol{v}^k)_{Q_t} - (\boldsymbol{S}^k, \boldsymbol{D}\boldsymbol{v}^k)_{Q_t} - \alpha(\boldsymbol{v}^k\Phi_k(|\boldsymbol{v}^k|), \boldsymbol{v}^k)_{\Gamma_t} \le (\boldsymbol{F}, \nabla\boldsymbol{v}^k)_{Q_t} + Ct,$$

by means of the property (11) and non-negativity of the boundary term.

Adding (127) and the lower semicontinuity of the norm then yields

$$\lim_{t\to 0_+} \|\boldsymbol{v}(t) - \boldsymbol{v}_0\|^2_{L^2(\Omega)} = \lim_{t\to 0_+} \|\boldsymbol{v}(t)\|^2_{L^2(\Omega)} - \|\boldsymbol{v}_0\|^2_{L^2(\Omega)}$$
$$\le \lim_{t\to 0_+} \liminf_{k\to\infty} \|\boldsymbol{v}^k(t)\|^2_{L^2(\Omega)} - \|\boldsymbol{v}_0\|^2_{L^2(\Omega)}$$
$$\le \lim_{t\to 0_+} \big((\boldsymbol{F}, \nabla\boldsymbol{v})_{Q_t} + Ct\big) = 0.$$

With this last fragment we have established the claim of Theorem 3.1. □

6 Appendix

In this ancillary part we prove Lemma 5.1. Towards that aim, with fixed $\varepsilon, k > 0$, the original problem (5) will be further approximated by the following *quasicompressible* system:

$$\left.\begin{aligned} \partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}\Phi_k(|\boldsymbol{v}|)) - \operatorname{div} \boldsymbol{S} + \nabla p &= -\operatorname{div} \boldsymbol{F} && \text{in } Q, \\ \operatorname{div} \boldsymbol{v} &= \varepsilon \Delta p && \text{in } Q, \\ \nabla p \cdot \boldsymbol{n} &= 0 && \text{on } \Gamma, \\ \boldsymbol{v} \cdot \boldsymbol{n} &= 0 && \text{on } \Gamma, \\ \alpha \boldsymbol{v}_\tau \Phi_k(|\boldsymbol{v}_\tau|) &= -(\boldsymbol{S}\boldsymbol{n})_\tau && \text{on } \Gamma, \\ \boldsymbol{v}(0) &= \boldsymbol{v}_0 && \text{in } \Omega, \\ p_\Omega &= 0 && \text{in } (0,T). \end{aligned}\right\} \tag{128}$$

Like in the case of the system with only the convective term truncated, we are insterested in existence of weak solutions. In the following lemma we both particularize this concept and affirm the existential question.

Lemma 6.1 *Under the assumptions of Theorem 3.1, for every $\varepsilon, k > 0$ there exists a weak solution to the approximate problem* (128)*, i.e. a couple $(\boldsymbol{v}^{\varepsilon,k}, p^{\varepsilon,k})$ satisfying*

$$\begin{aligned} \boldsymbol{v}^{\varepsilon,k} &\in L^r(0,T; W^{1,r}_{\boldsymbol{n}}(\Omega)), \\ \partial_t \boldsymbol{v}^{\varepsilon,k} &\in L^{r'}(0,T; W^{-1,r'}_{\boldsymbol{n}}(\Omega)), \\ p^{\varepsilon,k} &\in L^2(0,T; \mathring{W}^{1,2}(\Omega)) \cap L^{r'}(Q) \end{aligned}$$

and for all $\boldsymbol{\varphi} \in W^{1,r}_{\boldsymbol{n}}(\Omega)$ and a.e. $t \in (0,T)$, it holds that

$$\begin{aligned} \langle \partial_t \boldsymbol{v}^{\varepsilon,k}(t), \boldsymbol{\varphi}\rangle - (\boldsymbol{v}^{\varepsilon,k} \otimes \boldsymbol{v}^{\varepsilon,k}\Phi_k(|\boldsymbol{v}^{\varepsilon,k}|)(t), \nabla\boldsymbol{\varphi}) + (\boldsymbol{S}(p^{\varepsilon,k}(t), \boldsymbol{D}\boldsymbol{v}^{\varepsilon,k}(t)), \boldsymbol{D}\boldsymbol{\varphi}) \\ + \alpha(\boldsymbol{v}^{\varepsilon,k}\Phi_k(|\boldsymbol{v}^{\varepsilon,k}|)(t), \boldsymbol{\varphi})_{\partial\Omega} - (p^{\varepsilon,k}(t), \operatorname{div}\boldsymbol{\varphi}) = (\boldsymbol{F}(t), \nabla\boldsymbol{\varphi}), \end{aligned} \tag{129}$$

as well as for every $\psi \in W^{1,2}(\Omega)$ and a.e. $t \in (0,T)$ the identity

$$\varepsilon\, (\nabla p^{\varepsilon,k}(t), \nabla\psi) = -(\operatorname{div} \boldsymbol{v}^{\varepsilon,k}(t), \psi). \tag{130}$$

The initial condition is being attained in the form $\lim_{t\to 0_+} \|\boldsymbol{v}^{\varepsilon,k}(t) - \boldsymbol{v}_0\|_{L^2(\Omega)} = 0.$

Proof. Let $\{\boldsymbol{w}_i\}_{i\in\mathbb{N}} \subset W^{1,2}_{\boldsymbol{n}}(\Omega)$ be an orthogonal basis in $W^{1,2}_{\boldsymbol{n}}(\Omega)$ and an orthonormal basis in $L^2(\Omega)$. We also standardly require of the basis that L^2-projections

$$P^n \boldsymbol{u} = \sum_{i=1}^n (\boldsymbol{u}, \boldsymbol{w}_i)\boldsymbol{w}_i, \qquad \boldsymbol{u} \in L^2(\Omega),\ n \in \mathbb{N},$$

be orthogonal in $W^{1,2}_{\boldsymbol{n}}(\Omega)$. Note that $P^n\boldsymbol{v}_0$ converges to $\boldsymbol{v}_0$ in $L^2(\Omega)$ for $n \to \infty$.

Galerkin approximation Dropping the ε, k-indices (both parameters stay fixed), for $n \in \mathbb{N}$ we construct Faedo-Galerkin approximations

$$\begin{aligned} \boldsymbol{v}^n(t,x) &= \sum_{i=1}^n c_i^n(t)\boldsymbol{w}_i(x), \\ p^n(t,x) &= \mathcal{N}\left(\frac{\operatorname{div} \boldsymbol{v}^n}{\varepsilon}\right)(t,x) = \frac{1}{\varepsilon}\sum_{i=1}^n c_i^n(t)\mathcal{N}(\operatorname{div} \boldsymbol{w}_i)(x). \end{aligned} \tag{131}$$

Recall (13) for the definition of $\mathcal{N}$. What is to be found are absolutely continuous functions $\{c_i^n\}_{i=1}^n$, extensible to the whole $[0,T]$ and satisfying

$$(\partial_t \boldsymbol{v}^n(t), \boldsymbol{w}_i) - (\boldsymbol{v}^n \otimes \boldsymbol{v}^n \Phi_k(|\boldsymbol{v}^n|)(t), \nabla \boldsymbol{w}_i) + (\boldsymbol{S}^n(t), \boldsymbol{D}\boldsymbol{w}_i) + \alpha(\boldsymbol{v}^n \Phi_k(|\boldsymbol{v}^n|)(t), \boldsymbol{w}_i)_{\partial\Omega} - (p^n(t), \operatorname{div} \boldsymbol{w}_i)$$
$$= (\boldsymbol{F}(t), \nabla \boldsymbol{w}_i) \qquad \text{for all } i = 1, \dots, n, \quad (132)$$

where $\boldsymbol{S}^n(t) = \boldsymbol{S}(p^n(t), \boldsymbol{D}\boldsymbol{v}^n(t))$. We also set $\boldsymbol{v}^n(0) = P^n \boldsymbol{v}_0$.

The functions $\{c_i^n\}_{i=1}^n$ would be found standardly with help of the Carathéodory theory, at least for a short time interval. The extensibility onto the whole of $[0,T]$ will follow from the uniform estimates derived presently.

Uniform estimates Multiplying eq. (132) by $c_i^n(t)$ and summing the n equalities yields

$$\frac{1}{2}\frac{d}{dt} \|\boldsymbol{v}^n(t)\|_{L^2(\Omega)}^2 - (\boldsymbol{v}^n \otimes \boldsymbol{v}^n \Phi_k(|\boldsymbol{v}^n|)(t), \nabla \boldsymbol{v}^n(t)) + (\boldsymbol{S}^n(t), \boldsymbol{D}\boldsymbol{v}^n(t)) + \alpha \|\Phi_k^{1/2}(|\boldsymbol{v}^n|)\boldsymbol{v}^n(t)\|_{L^2(\partial\Omega)}^2$$
$$- (p^n(t), \operatorname{div} \boldsymbol{v}^n(t)) = (\boldsymbol{F}(t), \nabla \boldsymbol{v}^n(t)).$$

Due to eq. (131), boundedness of the truncated convective term and (11),

$$\frac{1}{2}\frac{d}{dt} \|\boldsymbol{v}^n(t)\|_{L^2(\Omega)}^2 + \frac{C_1}{2r} \|\boldsymbol{D}\boldsymbol{v}^n(t)\|_{L^r(\Omega)}^r + \varepsilon \|\nabla p^n(t)\|_{L^2(\Omega)}^2$$
$$\le \big(\|\boldsymbol{F}(t)\|_{L^{r'}(\Omega)} + C(k)\big) \|\nabla \boldsymbol{v}^n(t)\|_{L^r(\Omega)} + \frac{C_1|\Omega|}{2r}. \quad (133)$$

Hölder's inequality now implies

$$\sup_{t\in(0,T)} \|\boldsymbol{v}^n(t)\|_{L^2(\Omega)}^2 \le 2\big(\|\boldsymbol{F}\|_{L^{r'}(Q)} + C(k)\big) \|\nabla \boldsymbol{v}^n\|_{L^r(Q)} + \|\boldsymbol{v}_0\|_{L^2(\Omega)}^2 + \frac{TC_1|\Omega|}{r},$$

which we apply in (133), getting

$$\sup_{t\in(0,T)} \|\boldsymbol{v}^n(t)\|_{L^2(\Omega)}^2 + \frac{C_1}{r} \|\boldsymbol{D}\boldsymbol{v}^n\|_{L^r(Q)}^r + 2\varepsilon \|\nabla p^n\|_{L^2(Q)}^2$$
$$\le 2\big(\|\boldsymbol{F}\|_{L^{r'}(Q)} + C(k)\big) \|\nabla \boldsymbol{v}^n\|_{L^r(Q)} + \|\boldsymbol{v}_0\|_{L^2(\Omega)}^2 + \frac{TC_1|\Omega|}{r}.$$

Now we recall Korn's inequality (16) and then utilize Young's inequality to deduce

$$\sup_{t\in(0,T)} \|\boldsymbol{v}^n(t)\|_{L^2(\Omega)}^2 + \|\boldsymbol{v}^n\|_{L^r(0,T;W^{1,r}(\Omega))}^r + \varepsilon \|\nabla p^n\|_{L^2(Q)}^2 \le C\big(k, \|\boldsymbol{v}_0\|_{L^2(\Omega)}, \|\boldsymbol{F}\|_{L^{r'}(Q)}\big),$$

finally implying, using (12) for the stress tensor $\boldsymbol{S}$ and Poincaré's inequality for the pressure,

$$\sup_{t\in(0,T)} \|\boldsymbol{v}^n(t)\|_{L^2(\Omega)}^2 + \|\boldsymbol{v}^n\|_{L^r(0,T;W^{1,r}(\Omega))}^r + \|\boldsymbol{S}^n\|_{L^{r'}(Q)}^{r'} + \varepsilon \|p^n\|_{L^2(0,T;W^{1,2}(\Omega))}^2$$
$$\le C\big(k, \|\boldsymbol{v}_0\|_{L^2(\Omega)}, \|\boldsymbol{F}\|_{L^{r'}(Q)}\big). \quad (134)$$

The time derivative $\partial_t \boldsymbol{v}^n$ will be momentarily estimated in $L^2(0,T;W_{\boldsymbol{n}}^{-1,2}(\Omega))$. Noting that $W_{\boldsymbol{n}}^{1,2}(\Omega)$ is densely and continuously embedded in $L^2(\Omega)$, for $\boldsymbol{\varphi} \in W_{\boldsymbol{n}}^{1,2}(\Omega)$ we may write

$$\begin{aligned}
\langle \partial_t \boldsymbol{v}^n(t), \boldsymbol{\varphi}\rangle &= (\partial_t \boldsymbol{v}^n(t), P^n \boldsymbol{\varphi}) \\
&\le 4k^2 \big\|\nabla P^n \boldsymbol{\varphi}\big\|_{L^1(\Omega)} + \big\|\boldsymbol{S}^n(t)\big\|_{L^{r'}(\Omega)} \big\|\boldsymbol{D} P^n \boldsymbol{\varphi}\big\|_{L^r(\Omega)} + 2\alpha k \big\|P^n \boldsymbol{\varphi}\big\|_{L^2(\partial\Omega)} \\
&\quad + \big\|p^n(t)\big\|_{L^2(\Omega)} \big\|\nabla P^n \boldsymbol{\varphi}\big\|_{L^2(\Omega)} + \big\|\boldsymbol{F}(t)\big\|_{L^{r'}(\Omega)} \big\|\nabla P^n \boldsymbol{\varphi}\big\|_{L^r(\Omega)} \\
&\le C \|\nabla \boldsymbol{\varphi}\|_{L^2(\Omega)} \big(4k^2 + \|\boldsymbol{S}^n(t)\|_{L^{r'}(\Omega)} + 2\alpha k + \|p^n(t)\|_{L^2(\Omega)} + \|\boldsymbol{F}(t)\|_{L^{r'}(\Omega)}\big).
\end{aligned}$$

The first inequality follows from the eq. (132), while the latter step made use of orthogonality of P^n on $W^{1,2}_{\boldsymbol{n}}(\Omega)$, as well as Hölder's inequality ($r \leq 2$) and the trace theorem for Sobolev functions. Combining the last inequality with (134) yields the desired

$$\int_0^T \|\partial_t \boldsymbol{v}^n(t)\|^2_{W^{-1,2}_{\boldsymbol{n}}(\Omega)} \, dt \leq C\big(k, \|\boldsymbol{v}_0\|_{L^2(\Omega)}, \|\boldsymbol{F}\|_{L^{r'}(Q)}\big). \tag{135}$$

Limit $n \to \infty$ With bounds (134)–(135), we may invoke the traditional compactness arguments like reflexivity, the Banach-Alaoglu theorem, the Aubin-Lions lemma with $W^{1,r}_{\boldsymbol{n}} \hookrightarrow\hookrightarrow L^2 \hookrightarrow W^{-1,2}_{\boldsymbol{n}}$ and Lemma 4.3, to select a subsequence (labelled again $(p^n, \boldsymbol{v}^n)$) such that for $n \to \infty$

$$\begin{aligned}
\boldsymbol{v}^n &\to \boldsymbol{v} && \text{weakly in } L^r(0,T; W^{1,r}_{\boldsymbol{n}}(\Omega)), && (136)\\
\boldsymbol{v}^n &\to \boldsymbol{v} && \text{weakly}^* \text{ in } L^\infty(0,T; L^2(\Omega)), && (137)\\
\partial_t \boldsymbol{v}^n &\to \partial_t \boldsymbol{v} && \text{weakly in } L^2(0,T; W^{-1,2}_{\boldsymbol{n}}(\Omega)), && (138)\\
\boldsymbol{v}^n &\to \boldsymbol{v} && \text{strongly in } L^2(Q), && (139)\\
\boldsymbol{v}^n &\to \boldsymbol{v} && \text{strongly in } L^r(\Gamma), && (140)\\
\|\boldsymbol{v}^n(t)\|_2 &\to \|\boldsymbol{v}(t)\|_2 && \text{a.e. in } (0,T), && (141)\\
\boldsymbol{v}^n &\to \boldsymbol{v} && \text{a.e. in } Q, && (142)\\
p^n &\to p && \text{strongly in } L^2(0,T; \mathring{W}^{1,2}(\Omega)), && (143)\\
p^n &\to p && \text{a.e. in } Q, && (144)\\
\boldsymbol{S}^n &\to \overline{\boldsymbol{S}} && \text{weakly in } L^{r'}(Q). && (145)
\end{aligned}$$

We were able to deduce the strong convergence of p^n from (15) and (139).

Considering the continuity of $\mathcal{N}$ and properties of $\{\boldsymbol{w}_i\}_{i\in\mathbb{N}}$, we apply the convergence results (136)–(145) to the equations (131) (132) to acquire

$$\varepsilon p = \mathcal{N}(\operatorname{div} \boldsymbol{v}) \tag{146}$$

and

$$\int_0^T \langle \partial_t \boldsymbol{v}, \boldsymbol{\varphi}\rangle \, dt = (\boldsymbol{v} \otimes \boldsymbol{v}\Phi_k(|\boldsymbol{v}|), \nabla\boldsymbol{\varphi})_Q - (\overline{\boldsymbol{S}}, \boldsymbol{D}\boldsymbol{\varphi})_Q - \alpha(\boldsymbol{v}\Phi_k(|\boldsymbol{v}|), \boldsymbol{\varphi})_\Gamma + (p, \operatorname{div}\boldsymbol{\varphi})_Q + (\boldsymbol{F}, \nabla\boldsymbol{\varphi})_Q \tag{147}$$

for every $\boldsymbol{\varphi} \in L^2(0,T; W^{1,2}_{\boldsymbol{n}}(\Omega))$.

Improved pressure integrability The bound (134) is insufficient to infer $p \in L^{r'}(Q)$ but we are able to deduce it all the same, even uniformly in ε. The first thing we notice is that

$$p \in L^2(0,T; L^{r'}(\Omega))$$

since $r' < 2d/(d-2)$. This observation carries over to the equation (147), where it allows us to infer $\partial_t \boldsymbol{v} \in L^2(0,T; W^{-1,r'}_{\boldsymbol{n}}(\Omega))$ and we may take $\boldsymbol{\varphi} \in L^2(0,T; W^{1,r}_{\boldsymbol{n}}(\Omega))$.

For $L > 0$ denote χ_L the indicator function of the set $\{\|p(t)\|_{L^{r'}(\Omega)} < L\}$. We will consider

$$\boldsymbol{\varphi} = \chi_L \nabla\mathcal{N}\big(|p|^{r'-2}p - (|p|^{r'-2}p)_\Omega\big).$$

Notice from (15) that

$$\begin{aligned}
\|\boldsymbol{\varphi}(t)\|_{W^{1,r}(\Omega))} &\le C(\Omega,r)\chi_L(t)\||p(t)|^{r'-1}\|_{L^r(\Omega)} = C(\Omega,r)\|\chi_L(t)p(t)\|^{r'-1}_{L^{r'}(\Omega)},\\
\|\boldsymbol{\varphi}\|_{L^r(0,T;W^{1,r}(\Omega))} &\le C(\Omega,r)\|\chi_L p\|^{r'-1}_{L^{r'}(Q)},\\
\|\boldsymbol{\varphi}\|_{L^\infty(0,T;W^{1,r}(\Omega))} &\le C(\Omega,r)L^{r'-1},\\
\operatorname{div}\boldsymbol{\varphi} &= \big(|p|^{r'-2}p - (|p|^{r'-2}p)_\Omega\big)\chi_L \quad \text{a.e. in } Q.
\end{aligned} \tag{148}$$

In particular, we can make use of $\boldsymbol{\varphi}$ in the equation (147), implying

$$\|p\chi_L\|^{r'}_{L^{r'}(Q)} = (p, \operatorname{div}\boldsymbol{\varphi})_Q = \sum_{i=1}^{5} I_i, \tag{149}$$

where, by (147) and Hölder's inequality,

$$\begin{aligned}
I_1 &= -(\boldsymbol{F},\nabla\boldsymbol{\varphi})_Q \le \|\boldsymbol{F}\|_{L^{r'}(Q)}\|\nabla\boldsymbol{\varphi}\|_{L^r(Q)} \le C\,\|\boldsymbol{\varphi}\|_{L^r(0,T;W^{1,r}(\Omega))},\\
I_2 &= (\overline{\boldsymbol{S}},\boldsymbol{D}\boldsymbol{\varphi})_Q \le \|\overline{\boldsymbol{S}}\|_{L^{r'}(Q)}\|\nabla\boldsymbol{\varphi}\|_{L^r(Q)} \le C\,\|\boldsymbol{\varphi}\|_{L^r(0,T;W^{1,r}(\Omega))},\\
I_3 &= -(\boldsymbol{v}\otimes\boldsymbol{v}\Phi_k(|\boldsymbol{v}|),\nabla\boldsymbol{\varphi})_Q \le C(k)\,\|\boldsymbol{\varphi}\|_{L^r(0,T;W^{1,r}(\Omega))},\\
I_4 &= \alpha(\boldsymbol{v}\Phi_k(|\boldsymbol{v}|),\boldsymbol{\varphi})_\Gamma \le C(k)\|\boldsymbol{\varphi}\|_{L^r(\Gamma)} \le C(k)\,\|\boldsymbol{\varphi}\|_{L^r(0,T;W^{1,r}(\Omega))},\\
I_5 &= \int_0^T \langle\partial_t\boldsymbol{v},\boldsymbol{\varphi}\rangle\,dt = \int_0^T \langle\partial_t\nabla\mathcal{N}(\operatorname{div}\boldsymbol{v}),\boldsymbol{\varphi}\rangle\,dt = \varepsilon\int_0^T\langle\partial_t\nabla p,\boldsymbol{\varphi}\rangle\,dt,
\end{aligned} \tag{150}$$

by the Helmholtz decomposition (14) and the relation (146). If p were smooth, then

$$I_5 = -\varepsilon\int_0^T (\partial_t p, |p|^{r'-2}p)\chi_L\,dt = -\frac{\varepsilon}{r'}\,\|p(T)\|^{r'}_{L^{r'}(\Omega)}\,\chi_L(T) \le 0.$$

In the general case we could use an approximation by smooth functions to conclude $I_5 \le 0$. All in all, from (148), (149) and the estimates on I_1–I_5 we have

$$\|p\chi_L\|^{r'}_{L^{r'}(Q)} \le C(k).$$

independently of $L > 0$, which entails $L^{r'}$-integrability of the pressure

$$\|p\|_{L^{r'}(Q)} \le C(k). \tag{151}$$

Therefore the right-hand side of the equation (147) is well-defined for any $\boldsymbol{\varphi} \in L^r(0,T;W^{1,r}_{\boldsymbol{n}}(\Omega))$ and we conclude $\partial_t\boldsymbol{v} \in L^{r'}(0,T;W^{-1,r'}_{\boldsymbol{n}}(\Omega))$.

Initial condition Attainment of the initial condition is almost trivial: Let $\zeta \in C^1_c([0,T))$, such that $\zeta(0) = -1$. Multiply the eq. (132) with ζ, integrate over $(0,T)$ and perform the limit $n \to \infty$. Then

$$\begin{aligned}
(\boldsymbol{v}_0,\boldsymbol{w}_i) = \lim_{n\to\infty}(\boldsymbol{v}^n(0),\boldsymbol{w}_i) &= (\boldsymbol{v},\zeta'\boldsymbol{w}_i)_Q + (\boldsymbol{v}\otimes\boldsymbol{v}\Phi_k(|\boldsymbol{v}|),\nabla(\zeta\boldsymbol{w}_i))_Q - (\overline{\boldsymbol{S}},\boldsymbol{D}(\zeta\boldsymbol{w}_i))_Q\\
&- \alpha(\boldsymbol{v}\Phi_k(|\boldsymbol{v}|),(\zeta\boldsymbol{w}_i))_\Gamma + (p,\operatorname{div}(\zeta\boldsymbol{w}_i))_Q + (\boldsymbol{F},\nabla(\zeta\boldsymbol{w}_i))_Q \qquad \text{for all } i\in\mathbb{N}.
\end{aligned} \tag{152}$$

If we in (147) take $\boldsymbol{\varphi} = \zeta\boldsymbol{w}_i$ and compare the equation with (152), we obtain

$$(\boldsymbol{v}_0 - \boldsymbol{v}(0),\boldsymbol{w}_i) = 0 \quad \text{for all } i\in\mathbb{N},$$

so that $\boldsymbol{v}(0) = \boldsymbol{v}_0$. Since $\boldsymbol{v} \in \mathcal{C}([0,T];L^2(\Omega))$, we are finished.

Identification of $\overline{\boldsymbol{S}}$ What remains is to show $\overline{\boldsymbol{S}} = \boldsymbol{S}$ (i.e. $\boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v})$). Since $\boldsymbol{S}(\cdot,\cdot)$ is continuous and we already have (144), it suffices to verify the pointwise convergence of $\boldsymbol{D}\boldsymbol{v}^n$ a.e. in Q. Then $\overline{\boldsymbol{S}} = \boldsymbol{S}$ by Vitali's theorem.

Observe that we may without loss of generality assume in (141) that $\|\boldsymbol{v}^n(T)\|_{L^2(\Omega)} \to \|\boldsymbol{v}(T)\|_{L^2(\Omega)}$ for $n \to \infty$. Indeed so; if it were otherwise, we would solve our equation from the beginning on a larger time interval, say $(0, T+1)$. Then we could assume there is $T \le \tau \le T+1$ such that $\|\boldsymbol{v}^n(\tau)\|_{L^2(\Omega)} \to \|\boldsymbol{v}(\tau)\|_{L^2(\Omega)}$ for $n \to \infty$, and we would prove all convergencies on $(0, \tau)$, only to restrict ourselves to $(0, T)$ in the end.

Define

$$I^n = \int_0^1 (1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} |\boldsymbol{D}\boldsymbol{v}^n - \boldsymbol{D}\boldsymbol{v}|^2 \, ds, \qquad \overline{\boldsymbol{D}}(s) = \boldsymbol{D}\boldsymbol{v} + s(\boldsymbol{D}\boldsymbol{v}^n - \boldsymbol{D}\boldsymbol{v}).$$

With the strong convergence (143), the relation (9) implies

$$\begin{aligned} 0 \le C \limsup_{n\to\infty} \int_Q I^n &\le \limsup_{n\to\infty} (\boldsymbol{S}^n - \boldsymbol{S}, \boldsymbol{D}(\boldsymbol{v}^n - \boldsymbol{v}))_Q = \limsup_{n\to\infty} (\boldsymbol{S}^n, \boldsymbol{D}\boldsymbol{v}^n)_Q - (\overline{\boldsymbol{S}}, \boldsymbol{D}\boldsymbol{v})_Q \\ &\le \sum_{i=1}^5 \limsup_{n\to\infty} I_i, \end{aligned} \tag{153}$$

where, by (132) and (147), the terms I_i are are handled by convergencies (136)–(143) as follows:[8]

$$\begin{aligned} I_1 &= (\boldsymbol{F}, \nabla(\boldsymbol{v}^n - \boldsymbol{v}))_Q \overset{n}{\sim} 0 \\ I_2 &= (p^n, \operatorname{div} \boldsymbol{v}^n)_Q - (p, \operatorname{div} \boldsymbol{v})_Q = \varepsilon \|\nabla p\|^2_{L^2(Q)} - \varepsilon \|\nabla p^n\|^2_{L^2(Q)} \overset{n}{\sim} 0, \\ I_3 &= (\boldsymbol{v}^n \otimes \boldsymbol{v}^n \Phi_k(|\boldsymbol{v}^n|), \nabla \boldsymbol{v}^n)_Q - (\boldsymbol{v} \otimes \boldsymbol{v} \Phi_k(|\boldsymbol{v}|), \nabla \boldsymbol{v})_Q \overset{n}{\sim} 0, \\ I_4 &= \frac{1}{2} \int_0^T \frac{d}{dt} \big(\|\boldsymbol{v}\|^2_{L^2(\Omega)} - \|\boldsymbol{v}^n\|^2_{L^2(\Omega)} \big) \, dt \\ &= \frac{1}{2} \big(\|\boldsymbol{v}(T)\|^2_{L^2(\Omega)} - \|\boldsymbol{v}^n(T)\|^2_{L^2(\Omega)} + \|\boldsymbol{v}^n(0)\|^2_{L^2(\Omega)} - \|\boldsymbol{v}(0)\|^2_{L^2(\Omega)} \big) \overset{n}{\sim} 0, \\ I_5 &= \alpha(\boldsymbol{v}\Phi_k(|\boldsymbol{v}|), \boldsymbol{v})_\Gamma - \alpha(\boldsymbol{v}^n \Phi_k(|\boldsymbol{v}^n|), \boldsymbol{v}^n)_\Gamma \overset{n}{\sim} 0. \end{aligned}$$

Therefore (153) entails

$$\lim_{n\to\infty} \int_Q I^n = 0 \tag{154}$$

and now we are practically finished, for Hölder's inequality yields

$$\begin{aligned} \big\| \boldsymbol{D}(\boldsymbol{v}^n - \boldsymbol{v}) \big\|^r_{L^r(Q)} &\le \int_Q \Big(\int_0^1 (1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} |\boldsymbol{D}(\boldsymbol{v}^n - \boldsymbol{v})|^2 \, ds \Big)^{r/2} (1 + |\boldsymbol{D}\boldsymbol{v}^n|^2 + |\boldsymbol{D}\boldsymbol{v}|^2)^{r(2-r)/4} \, dx \, dt \\ &\le \Big(\int_Q I^n \Big)^{r/2} \Big(\int_Q (1 + |\boldsymbol{D}\boldsymbol{v}^n|^2 + |\boldsymbol{D}\boldsymbol{v}|^2)^{r/2} \Big)^{(2-r)/2}, \end{aligned} \tag{155}$$

which tends to zero with $n \to \infty$ by (154). □

6.1 Vanishing artificial compressibility ($\varepsilon \to 0_+$)

Now we justify the limit $\varepsilon \to 0_+$ for solutions yielded by Lemma 6.1, proving thus Lemma 5.1. Let us again drop the index k and denote the solutions at hand simply $(\boldsymbol{v}^\varepsilon, p^\varepsilon)$.

[8] The symbol $\overset{n}{\sim}$ has an analogical meaning to $\overset{k}{\sim}$ introduced under (166).

Uniform estimates Taking $\varphi = p^\varepsilon$ in (130), $\boldsymbol{\varphi} = \boldsymbol{v}^\varepsilon$ in (129) and summing up the resultant identities, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}^\varepsilon(t)\|_{L^2(\Omega)}^2 - (\boldsymbol{v}^\varepsilon \otimes \boldsymbol{v}^\varepsilon \Phi_k(|\boldsymbol{v}^\varepsilon|)(t), \nabla \boldsymbol{v}^\varepsilon(t)) + (\boldsymbol{S}^\varepsilon(t), \boldsymbol{D}\boldsymbol{v}^\varepsilon(t)) + \alpha\|\Phi_k^{1/2}(|\boldsymbol{v}^\varepsilon|)\boldsymbol{v}^\varepsilon(t)\|_{L^2(\partial\Omega)}^2 + \varepsilon\|\nabla p^\varepsilon(t)\|_{L^2(\Omega)}^2 = (\boldsymbol{F}(t), \nabla \boldsymbol{v}^\varepsilon(t)),$$

where $\boldsymbol{S}^\varepsilon(t) = \boldsymbol{S}(p^\varepsilon(t), \boldsymbol{D}\boldsymbol{v}^\varepsilon(t))$. Following the same steps as in the proof of Lemma 6.1, we could show

$$\sup_{t\in(0,T)}\|\boldsymbol{v}^\varepsilon(t)\|_{L^2(\Omega)}^2 + \|\boldsymbol{v}^\varepsilon\|_{L^r(0,T;W^{1,r}(\Omega))}^r + \|\boldsymbol{S}^\varepsilon\|_{L^{r'}(\Omega)}^{r'} + \varepsilon\|p^\varepsilon\|_{L^2(0,T;W^{1,2}(\Omega))}^2 \leq C\big(k, \|\boldsymbol{v}_0\|_{L^2(\Omega)}, \|\boldsymbol{F}\|_{L^{r'}(Q)}\big), \quad (156)$$

which can be combined with the weak formulation for the pressure (130) to obtain

$$\int_0^T \|\operatorname{div} \boldsymbol{v}^\varepsilon\|_{W_{\boldsymbol{n}}^{-1,2}(\Omega)}^2 \leq \sqrt{\varepsilon}\, C\big(k, \|\boldsymbol{v}_0\|_{L^2(\Omega)}, \|\boldsymbol{F}\|_{L^{r'}(Q)}\big).$$

As far as an ε-uniform estimate of p^ε is concerned, we still have (151). Combining (156) with (151) and the starting equation (129) also yields the last estimate

$$\|\partial_t \boldsymbol{v}^\varepsilon\|_{L^{r'}(0,T;W_{\boldsymbol{n}}^{-1,r'}(\Omega))} \leq C\big(k, \|\boldsymbol{v}_0\|_{L^2(\Omega)}, \|\boldsymbol{F}\|_{L^{r'}(Q)}\big).$$

Limit $\varepsilon \to 0_+$ The uniform bounds hitherto deduced allow us to pick a subsequence $(\boldsymbol{v}^\varepsilon, p^\varepsilon)$ satisfying

$$\begin{aligned}
\boldsymbol{v}^\varepsilon &\rightharpoonup \boldsymbol{v} && \text{weakly in } L^r(0,T;W_{\boldsymbol{n},\mathrm{div}}^{1,r}(\Omega)), && (157)\\
\boldsymbol{v}^\varepsilon &\rightharpoonup \boldsymbol{v} && \text{weakly* in } L^\infty(0,T;L^2(\Omega)), && (158)\\
\partial_t \boldsymbol{v}^\varepsilon &\rightharpoonup \partial_t \boldsymbol{v} && \text{weakly in } L^{r'}(0,T;W_{\boldsymbol{n}}^{-1,r'}(\Omega)), && (159)\\
\boldsymbol{v}^\varepsilon &\to \boldsymbol{v} && \text{strongly in } L^2(Q), && (160)\\
\boldsymbol{v}^\varepsilon &\to \boldsymbol{v} && \text{strongly in } L^r(\Gamma), && (161)\\
\boldsymbol{v}^\varepsilon &\to \boldsymbol{v} && \text{a.e. in } Q, && (162)\\
p^\varepsilon &\rightharpoonup p && \text{weakly in } L^{r'}(0,T;\mathring{L}^{r'}(\Omega)), && (163)\\
\boldsymbol{S}^\varepsilon &\rightharpoonup \overline{\boldsymbol{S}} && \text{weakly in } L^{r'}(Q). && (164)
\end{aligned}$$

Applying (157)–(164) to eq. (129), we get

$$\int_0^T \langle \partial_t \boldsymbol{v}, \boldsymbol{\varphi}\rangle\, dt - (\boldsymbol{v}\otimes\boldsymbol{v}\Phi_k(|\boldsymbol{v}|), \nabla\boldsymbol{\varphi})_Q + (\overline{\boldsymbol{S}}, \boldsymbol{D}\boldsymbol{\varphi})_Q + \alpha(\boldsymbol{v}\Phi_k(|\boldsymbol{v}|), \boldsymbol{\varphi})_\Gamma - (p, \operatorname{div}\boldsymbol{\varphi})_Q = (\boldsymbol{F}, \nabla\boldsymbol{\varphi})_Q$$

for every $\boldsymbol{\varphi} \in L^r(0,T;W_{\boldsymbol{n}}^{1,r}(\Omega))$. As far as attainment of the initial condition is concerned, we could proceed identically like in the Galerkin approximation (notice $\boldsymbol{v}^\varepsilon(0) = \boldsymbol{v}_0$ for all $\varepsilon > 0$) and hence we skip it.

Identification of the weak limit $\overline{\boldsymbol{S}}$ is thus the only remaining issue of the ε-limit. Yearning to invoke Vitali's theorem again, we are in a slightly more problematic situation at this moment as we have lost compactness of the pressure. The equality $\overline{\boldsymbol{S}} = \boldsymbol{S}$ (i.e. $\boldsymbol{S}(p, \boldsymbol{D}\boldsymbol{v})$) now therefore demands showing not only the pointwise convergence of $\boldsymbol{D}\boldsymbol{v}^\varepsilon$ but also of p^ε a.e. in Q.

Convergence of p^ε We will deduce

$$p^\varepsilon \to p \quad \text{strongly in } L^2(Q).$$

Define $\boldsymbol{\varphi}^\varepsilon = \mathcal{N}(p^\varepsilon - p)$ and observe that by (15) and (163)

$$\|\boldsymbol{\varphi}^\varepsilon\|_{L^2(0,T;W^{2,2}(\Omega))} \leq C_{reg}\,\|p^\varepsilon - p\|_{L^2(Q)}\,, \tag{165}$$

$$\boldsymbol{\varphi}^\varepsilon \to 0 \quad \text{weakly in } L^{r'}(0,T;W^{2,r'}(\Omega)). \tag{166}$$

Let $O(\varepsilon)$ signify a quantity satisfying $\limsup_{\varepsilon\to 0_+} O(\varepsilon) \leq 0$. For quantities A^ε, B^ε we write $A^\varepsilon \overset{\varepsilon}{\lesssim} B^\varepsilon$ if $A^\varepsilon \leq B^\varepsilon + O(\varepsilon)$. Then

$$\|p^\varepsilon - p\|^2_{L^2(Q)} = (p^\varepsilon - p, \Delta\boldsymbol{\varphi}^\varepsilon)_Q \overset{\varepsilon}{\lesssim} (p^\varepsilon, \Delta\boldsymbol{\varphi}^\varepsilon)_Q = (\boldsymbol{S}^\varepsilon, \nabla^2\boldsymbol{\varphi}^\varepsilon)_Q + \sum_{i=1}^{5} I_i, \tag{167}$$

where by the equation (129), convergences (157)–(162) and (166), the individual summands are dealt with as

$$\begin{aligned}
I_1 &= -(\boldsymbol{F}, \nabla^2\boldsymbol{\varphi}^\varepsilon)_Q \overset{\varepsilon}{\lesssim} 0,\\
I_2 &= \alpha(\boldsymbol{v}^\varepsilon\Phi_k(|\boldsymbol{v}^\varepsilon|), \nabla\boldsymbol{\varphi}^\varepsilon)_\Gamma \overset{\varepsilon}{\lesssim} 0,\\
I_3 &= -(\boldsymbol{v}^\varepsilon \otimes \boldsymbol{v}^\varepsilon\Phi_k(|\boldsymbol{v}^\varepsilon|), \nabla^2\boldsymbol{\varphi}^\varepsilon)_Q \overset{\varepsilon}{\lesssim} 0,\\
I_4 &= -\int_0^T \langle \partial_t\boldsymbol{v}^\varepsilon, \nabla\mathcal{N}(p)\rangle\,dt \overset{\varepsilon}{\lesssim} -\int_0^T \langle \partial_t\boldsymbol{v}, \nabla\mathcal{N}(p)\rangle\,dt = 0,\\
I_5 &= \int_0^T \langle \partial_t\boldsymbol{v}^\varepsilon, \nabla\mathcal{N}(p^\varepsilon)\rangle\,dt \overset{\varepsilon}{\lesssim} 0,
\end{aligned}$$

being a clone of I_5 in (150) with r' changed to 2. Hence the sum in (167) can be ignored and

$$\begin{aligned}
\|p^\varepsilon - p\|^2_{L^2(Q)} &\overset{\varepsilon}{\lesssim} (\boldsymbol{S}^\varepsilon, \nabla^2\boldsymbol{\varphi}^\varepsilon)_Q \overset{\varepsilon}{\lesssim} (\boldsymbol{S}^\varepsilon - \boldsymbol{S}, \nabla^2\boldsymbol{\varphi}^\varepsilon)_Q\\
&\leq \gamma_0\int_Q |p^\varepsilon - p||\nabla^2\boldsymbol{\varphi}^\varepsilon|\,dx\,dt + C_2\int_Q\int_0^1 (1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2}|\boldsymbol{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})||\nabla^2\boldsymbol{\varphi}^\varepsilon|\,ds\,dx\,dt,
\end{aligned} \tag{168}$$

by the property (10) with $\overline{\boldsymbol{D}}(s) = \boldsymbol{D}\boldsymbol{v} + s(\boldsymbol{D}\boldsymbol{v}^\varepsilon - \boldsymbol{D}\boldsymbol{v})$. Denote

$$I^\varepsilon = \int_0^1 (1 + |\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2}|\boldsymbol{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})|^2 ds.$$

Since $(1+|\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/2} \leq (1+|\overline{\boldsymbol{D}}(s)|^2)^{(r-2)/4}$, Hölder's inequality and bound (165) applied to (168) yield

$$\|p^\varepsilon - p\|^2_{L^2(Q)} \overset{\varepsilon}{\lesssim} \gamma_0 C_{reg}\,\|p^\varepsilon - p\|^2_{L^2(Q)} + C_2 C_{reg}\Big(\int_Q I^\varepsilon\,dx\,dt\Big)^{1/2}\|p^\varepsilon - p\|_{L^2(Q)}$$

entailing (note $1 - \gamma_0 C_{reg} > 0$ by Assumption 2.2)

$$\|p^\varepsilon - p\|^2_{L^2(Q)} \overset{\varepsilon}{\lesssim} \left(\frac{C_2 C_{reg}}{1 - \gamma_0 C_{reg}}\right)^2 \int_Q I^\varepsilon\,dx\,dt. \tag{169}$$

Using (9), we can estimate the integral on the right as

$$\int_Q I^\varepsilon \,dx\,dt \le \frac{2}{C_1}\big(\boldsymbol{S}^\varepsilon - \boldsymbol{S}, \boldsymbol{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})\big)_Q + \frac{\gamma_0^2}{C_1^2}\,\|p^\varepsilon - p\|^2_{L^2(Q)}$$
$$\overset{\varepsilon}{\sim} \frac{2}{C_1}\big(\boldsymbol{S}^\varepsilon, \boldsymbol{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})\big)_Q + \frac{\gamma_0^2}{C_1^2}\,\|p^\varepsilon - p\|^2_{L^2(Q)} \overset{\varepsilon}{\sim} \frac{\gamma_0^2}{C_1^2}\,\|p^\varepsilon - p\|^2_{L^2(Q)}\,, \tag{170}$$

as long as

$$(\boldsymbol{S}^\varepsilon, \boldsymbol{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v}))_Q \overset{\varepsilon}{\sim} 0. \tag{171}$$

Notice that (169) and (170) would then imply

$$\lim_{\varepsilon\to 0_+} \|p^\varepsilon - p\|_{L^2(Q)} = 0 \tag{172}$$

provided also

$$\frac{\gamma_0 C_2 C_{reg}}{C_1(1-\gamma_0 C_{reg})} < 1,$$

which does hold, however, due to Assumption 2.2, namely

$$\gamma_0 < \frac{C_1}{C_{reg}(C_1 + C_2)}.$$

We must therefore justify (171). Set $\boldsymbol{\varphi}^\varepsilon = \boldsymbol{v}^\varepsilon - \boldsymbol{v}$ in the weak formulation (129), whence

$$\big(\boldsymbol{S}^\varepsilon, \boldsymbol{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})\big)_Q = \sum_{i=1}^{5} I_i,$$

where, exploiting convergences (157)–(162),

$$I_1 = (\boldsymbol{F}, \nabla\boldsymbol{\varphi}^\varepsilon)_Q \overset{\varepsilon}{\sim} 0,$$
$$I_2 = -\alpha(\boldsymbol{v}^\varepsilon \Phi_k(|\boldsymbol{v}^\varepsilon|), \boldsymbol{\varphi}^\varepsilon)_\Gamma \overset{\varepsilon}{\sim} 0,$$
$$I_3 = (\boldsymbol{v}^\varepsilon \otimes \boldsymbol{v}^\varepsilon \Phi_k(|\boldsymbol{v}^\varepsilon|), \nabla\boldsymbol{\varphi}^\varepsilon)_Q \overset{\varepsilon}{\sim} 0,$$
$$I_4 = (p^\varepsilon, \operatorname{div}\boldsymbol{\varphi}^\varepsilon)_Q = -\varepsilon(\nabla p^\varepsilon, \nabla p^\varepsilon)_Q \overset{\varepsilon}{\sim} 0,$$
$$I_5 = -\int_0^T \langle \partial_t \boldsymbol{v}^\varepsilon, \boldsymbol{\varphi}^\varepsilon\rangle\,dt = -\frac{1}{2}\int_0^T \frac{d}{dt}\,\|\boldsymbol{v}^\varepsilon - \boldsymbol{v}\|^2_{L^2(\Omega)}\,dt - \int_0^T \langle \partial_t \boldsymbol{v}, \boldsymbol{\varphi}^\varepsilon\rangle\,dt \overset{\varepsilon}{\sim} 0,$$

thus proving (171) and justifying (172).

Convergence of $\boldsymbol{D}\boldsymbol{v}^\varepsilon$ The inequality (155) in the current situation takes form

$$\big\|\boldsymbol{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})\big\|^r_{L^r(Q)} \le \Big(\int_Q I^\varepsilon\Big)^{r/2}\Big(\int_Q (1 + |\boldsymbol{D}\boldsymbol{v}^\varepsilon|^2 + |\boldsymbol{D}\boldsymbol{v}|^2)^{r/2}\Big)^{(2-r)/2} \overset{\varepsilon}{\sim} 0,$$

by (170) and (172). Consequently, we may assume the pointwise convergence of both p^ε and $\boldsymbol{D}\boldsymbol{v}^\varepsilon$ a.e. in Q, which proves $\overline{\boldsymbol{S}} = \boldsymbol{S}$ and thus concludes the entire ε-limit. □

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