Non-Darcy Flows in the porous media for compressible fluids and application

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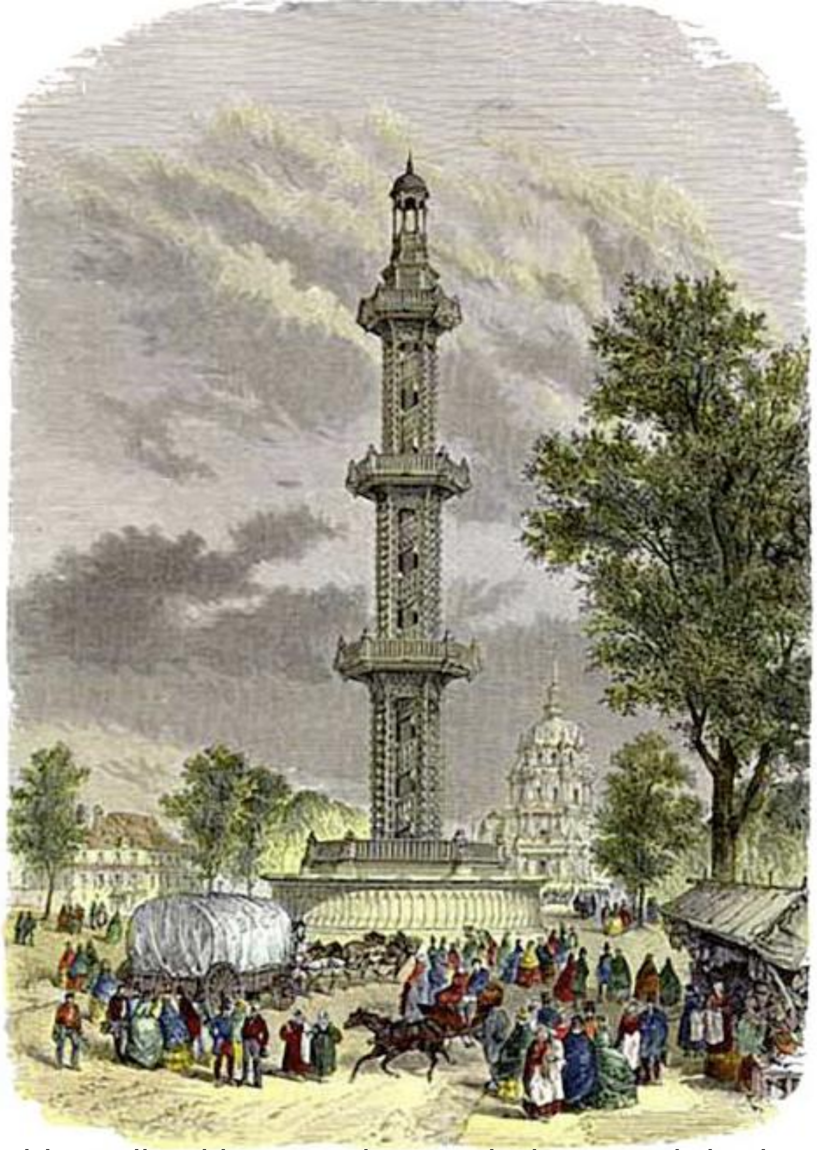
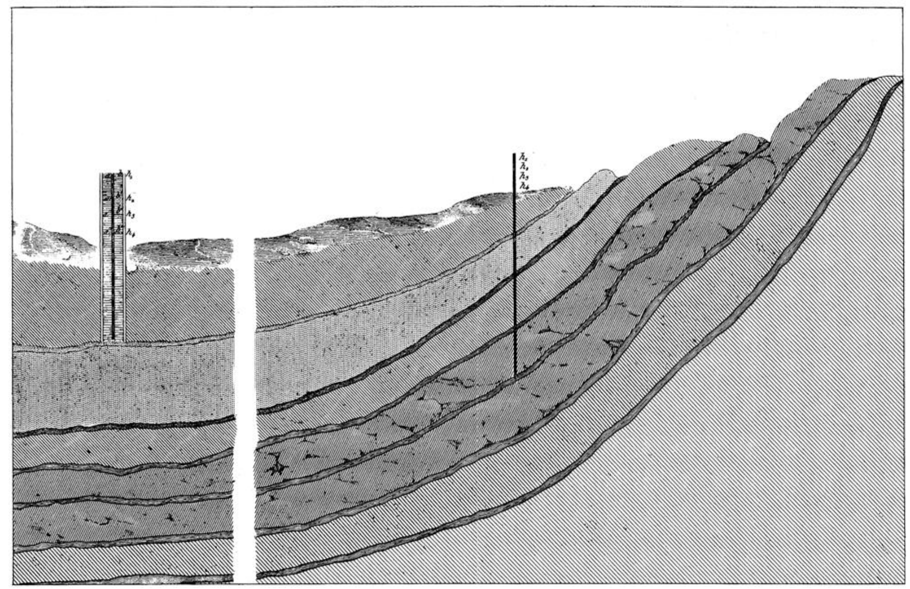
History of the underground fluid flow starts by Darcy with clean water supply



This well, with upgrades made by Dupuit in the early 1850s, was one of the major municipal supplies of water for the SW side of Paris, along with the Perrier pumping station at Chaillot, which supplied water from the Seine

Stratigraphy and structural geology of the Paris basin [from Darcy, 1856]



Comprehensive principles ….established by Darcy and Dupuit, by R.W. Ritzi, and P. Bobeck, Water resources research,,v44,2008.

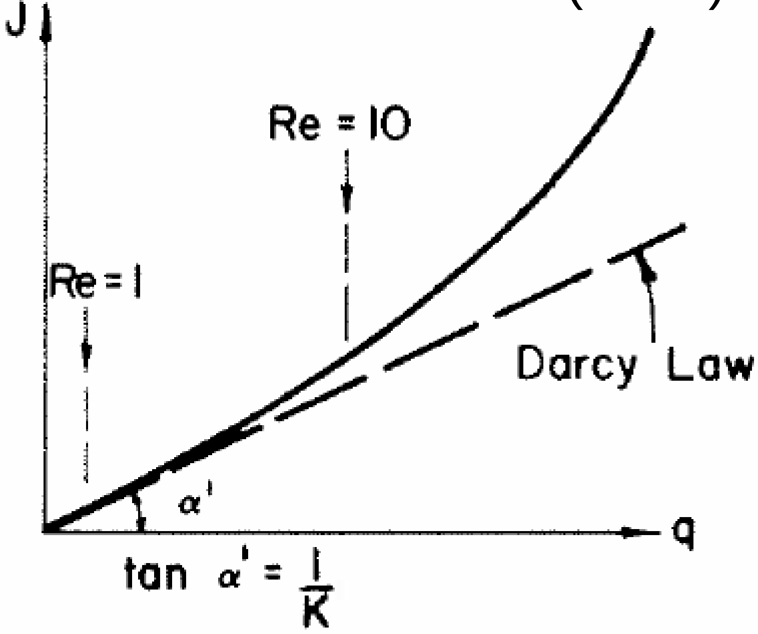
Experimental Observation

Darcy approximation $\alpha q = \Delta P$

Forchheimer Two terms approximation $\alpha q + \beta q^2 = \Delta P$

Forchheimer Power approximation $\alpha q + c(n) q^n = \Delta P, \quad 1 \le n < 2$

Forchheimer Three terms (cubic) approximation $Aq + Bq^2 + Cq^3 = \Delta P$

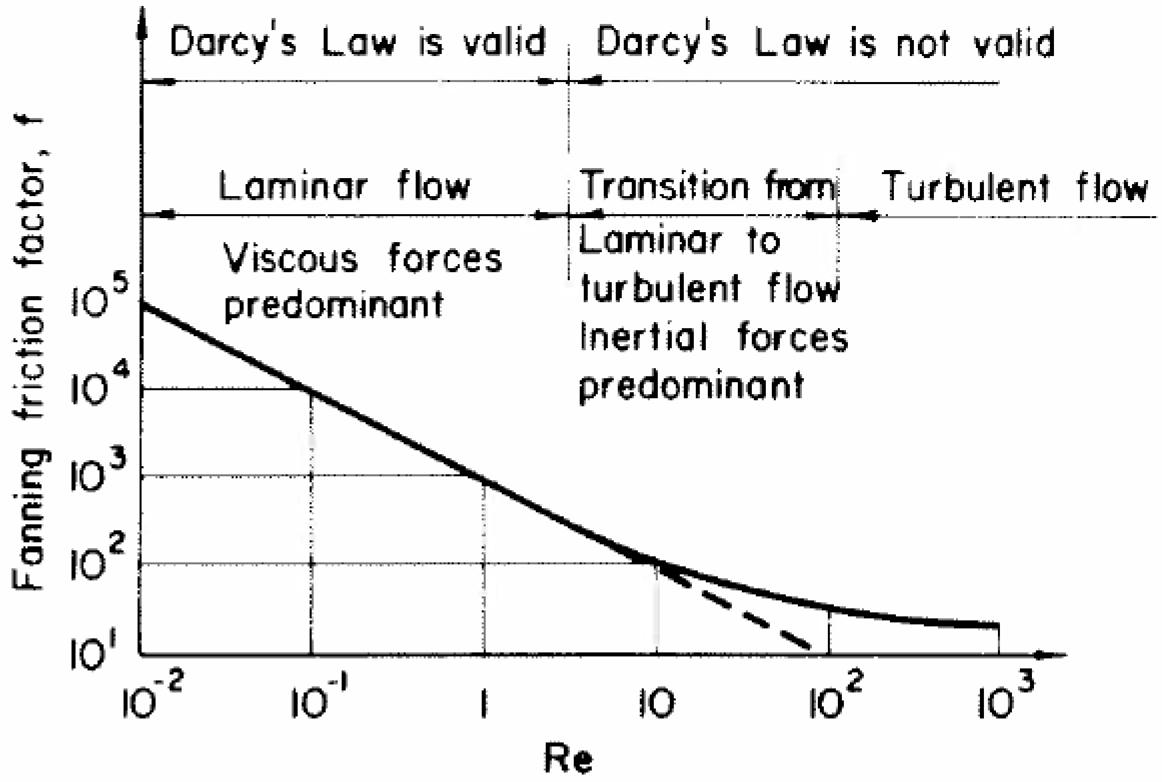


$$\mu / k = \alpha \ , \quad \Delta P = P_1 - P_2$$

$$J = \Delta P$$

Muskat, M., The Flow of Homogeneous Fluids through Porous Media, 1937, latest edition, Springer 1982

Cause of Deviation is not Understood Fully



Bear, J., Dynamics of Fluids in Porous Media ,Dover Publications, Inc., New York, 1972

Constitutive Generalized Forchheimer Equations

$$\vec{v} g(|\vec{v}|) = -\mathrm{K}\nabla p$$

$$g(s) = a_0 + a_1 s^{\alpha_1} + \ldots + a_n s^{\alpha_n}$$

Algebraic polynomial with positive coefficients

Two term Forchheimer Law

(1) $$\alpha\vec{v} + \beta\sqrt{(\mathrm{K}'\vec{v},\vec{v})}\,\vec{v} = -\mathrm{K}\nabla p$$

$K -$ **Dimensionless permeability tensor**

Power Forchheimer Law

$$\mathrm{K}' = \mathrm{K}^{-1}$$

(2) $$\alpha\vec{v} + c^n\left[\sqrt{(\mathrm{K}'\vec{v},\vec{v})}\right]^{n-1}\vec{v} = -\mathrm{K}\nabla p$$

$$\nabla p = \left(p_{x_1}, \ldots, p_{x_n}\right)$$

Three term cubic Forchheimer Law

(3) $$\alpha\vec{v} + B\sqrt{(\mathrm{K}'\vec{v},\vec{v})}\,\vec{v} + C(\mathrm{K}'\vec{v},\vec{v})\vec{v} = -\mathrm{K}\nabla p$$

*E. Aulisa, L. Bloshanskaya, L. Hoang, **A. I.** , Analyses of Gen. Forchheimer Flows of Compressible fluid …, J. Math. Phys. 50, (2009).*

Generalized Forchheimer Equation Nonlinear Darcy Equation

$$\alpha\vec{v} = -K\nabla p$$

Darcy vector field

$$\vec{v} = -K(\eta(\nabla p))\mathrm{K}\nabla p$$

Generalized Nonlinear Darcy Vector Field

$\vec{v}(x,t) -$ velocity

$p(x,t) -$ pressure

$\eta(\nabla p) = \left(\mathrm{K}\nabla p, \nabla p\right) -$ Bilinear form

Proposition. **For any algebraic polynomial *g(s)* exists *G* positive function such that , generalized nonlinear Darcy vector field (5) , solves generalized Forchheimer equation.**

E. Aulisa, L. Bloshanskaya, L. Hoang, A. I. , Analysis of Gen. Forchheimer Flows of Compressible Fluid in the porous media…, J. Math. Phys. 50, (2009).

In case of two-terms Forchheimer

$$K(\eta) = \frac{2}{\alpha + \sqrt{\alpha^2 + \beta\sqrt{\eta}}}$$

E. Aulisa, A. I. P. P. Valk´o, J. R. Walton, Mathematical Frame-Work For Productivity Index of The Well for Fast Forchheimer (non-Darcy) Flow in Porous Media. Mathematical Models and Methods in Applied Sciences, 19(8), 1241-1275., 2009

Governing equations

Momentum equation –Generalized Nonlinear Darcy Vector Field

(5) $$\vec{v} = -K(\eta(\nabla p))\mathrm{K}\nabla p$$

Continuity equation

(6) $$\rho' p_t = -\rho\nabla\vec{v} - \rho'\vec{v}\nabla p$$

Equation of state for slightly compressible fluids

(7) $$\rho' = \gamma^{-1}\rho$$

Then (6) is reduced to

$$p_t = -\gamma\nabla\vec{v} - \vec{v}\nabla p$$

Initial Boundary Value Problem for Non-linear parabolic equation

$$\frac{\partial p}{\partial t} = \gamma \nabla \left(K(\eta(\nabla p))K\nabla p \right) \text{ in the domain } U$$

$$Q(t) = \int_{\Gamma_w} \left(\vec{v}(x,t), \vec{n} \right) ds_x$$

$$\left(\vec{v}(x,t), \vec{n} \right) = 0 \text{ on } \Gamma_e$$

$$p(x,0) = f(x)$$

Structural Properties of degenerate degenerate permeability *K*

$$\frac{C_2}{(1+\xi)^a} \le K(\xi) \le \frac{C_1}{(1+\xi)^a}, \quad a = \frac{\alpha_N}{1+\alpha_N}$$

$$H(\xi) = \int_0^\xi K(s^{1/2})\,ds \sim K(\xi)\xi^2$$

$$\Phi(\vec{u}_1,\vec{u}_2) = \left(K(|\vec{u}_1|)\vec{u}_1 - K(|\vec{u}_2|)\vec{u}_2, \vec{u}_1 - \vec{u}_2\right)$$

Weighted monotonicity: :

$$\int_U \Phi(\nabla p_2,\nabla p_1)\,dx \ge C_0\left(1+|\nabla p_2|_{L_{2-a}} \wedge |\nabla p_1|_{L_{2-a}}\right)^{-a} \|\nabla p_2 - \nabla p_1\|^2$$

E. Aulisa, L. Bloshanskaya, L. Hoang, A. I. , Analysis of Gen. Forchheimer Flows of Compressible Fluid in the porous media…, J. Math. Phys. 50, (2009).

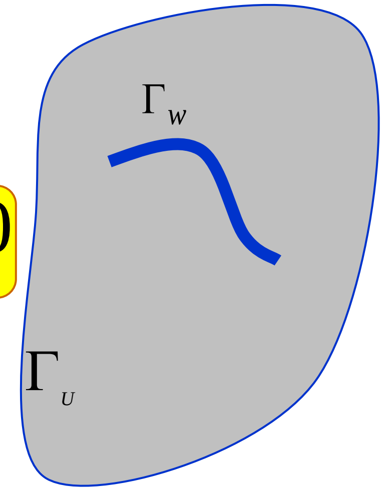
Time Invariant Solution. Auxiliary Problem

(1) $$(p_s)_t = \gamma \nabla \left(K(\eta(\nabla p_s)) K \nabla p_s \right)$$

(2) $$\int_{\Gamma_w} \left(\vec{v}(x,t), \vec{n} \right) ds_x = Q = const$$

(4) $$(\vec{v}, \vec{n})\Big|_{\Gamma_U} = 0$$

(3) $$p_s\Big|_{\Gamma_w} = -At + \phi_0(x)$$



(1s-2s) $$-A = Q/|U| = \nabla \left(K\left(\eta(\nabla w) \right) \nabla w \right)$$

(3s) $$w(x) = \phi_0(x) \quad \text{on } \Gamma_w$$

(4s) $$\frac{\partial}{\partial \vec{v}} w(x) = 0 \quad \text{on } \Gamma_e$$

__Theorem .__ Solution of IBVP exist and unique iff

$$p_s(x,0) = w(x)$$

*E. Aulisa, **A. I.** P. P. Valko, J. R. Walton, “Math. Frame-Work For PI of The Well for Fast Forchheimer (non-Darcy) Flow in porous media”, Mathematical Models & Methods in Applied Science, v.. 19, (2009),*
*E. Aulisa, L. Bloshanskaya, **A. I.** Long-term dynamics for well productivity index for nonlinear flows , J. Math. Phys. 52 (2011)*

Productivity Index - Basic Engineering Parameter

$$Q(t) = \int_{\Gamma_w} \left(\vec{v}(x,t), \vec{n}\right) ds_x$$

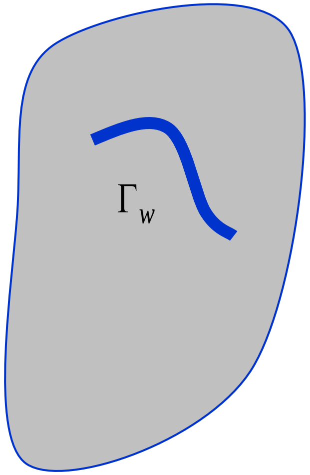
Total Flux on the well

$$PDD(t) = \overline{p}_{\Gamma_w}(t) - \overline{p}_U(t)$$

Pressure drawdown

$$\overline{p}_S(t) = |S|^{-1} \int_S p(x,t)\,dx$$

Average Pressure



<u>*Productivity Index/ Diffusive Capacity*</u>

$$J_K(p,\vec{v},t) = J(t) = \frac{Q(t)}{PDD(t)}$$

Definition : Pseudo-Steady State (PSS) regime

If $Q(t) = Q = const$ and $PDD(t) = const$

Consequently $$J_G(p,\vec{v},t) = const$$

A. I., *Khalmanova, D., Valk´o, P. P., and Walton, J. R., "On a Mathematical Model of the Productivity Index of a Well from Reservoir Engineering," SIAM J. Appl. Math. 65, 1952 (2005).*

Pseudo-steady State Regime Exists

Definition :

$p_s(x,t)$ *Is called pseudo-steady state solution*

Theorem . *PSS regime exists and is unique*

$$J_K(p_s, \vec{v}_s, t) = \frac{Q}{\overline{\phi}_0 - \overline{w}} = const,$$

here $\overline{f}$ *is integral average.*

Proof.

Indeed, pressure function $p_s(x,t) = w(x) + At$ *solves transient problem (1-3)*

And corresponding pressure drawdown is equal to

$$PDD(t) = \overline{\phi}_0 - \overline{w} = const$$

*E. Aulisa, **A. I.** P. P. Valko, J. R. Walton, "Math. Frame-Work For PI of The Well for Fast Forchheimer (non-Darcy) Flow in porous media", Mathematical Models & Methods in Applied Science, v.. 19, (2009),*

*E. Aulisa, L. Bloshanskaya, **A. I.** Long-term dynamics for well productivity index for nonlinear flows , J. Math. Phys. 52 (2011),*

Variational Formulation for Auxiliary Problem, Variational Interpretation for Diffusive Capacitance

<u>*Proposition 4*</u> *Equation*

$$-Q/|U| = div\left(K\left(|\nabla w|\right)\nabla w\right)$$

Serves as Euler-Lagrange equation for functional

$$I_K(u) = \int\left(\frac{1}{2}F(|\nabla u|)|\nabla u|^2 - Au\right)dx$$

$$\frac{1}{2}F'(\eta)\eta + F(\eta) = K(\eta)$$

<u>*Theorem 5.*</u> *Variational interpretation of diffusive capacitance*

a) in linear Darcy case

$$I_{Darcy}(W) = 1/J_{Darcy}$$

b) in case of Forchheimer

Power Law,

$$g(s) = a_0 + a_1 s^{\alpha_1}, \alpha_1 > 1$$

$$I_{Power} \geq 1/J_{Power}$$

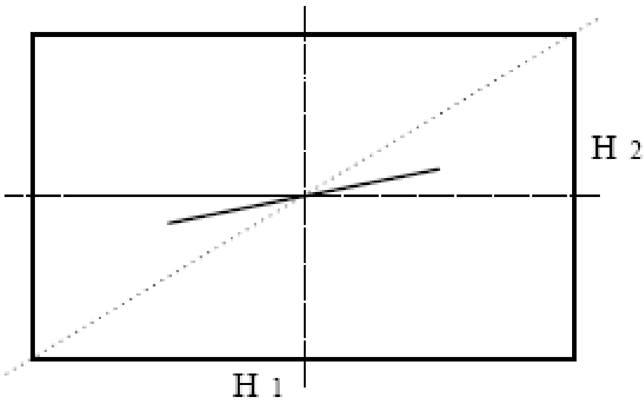
Two terms Law

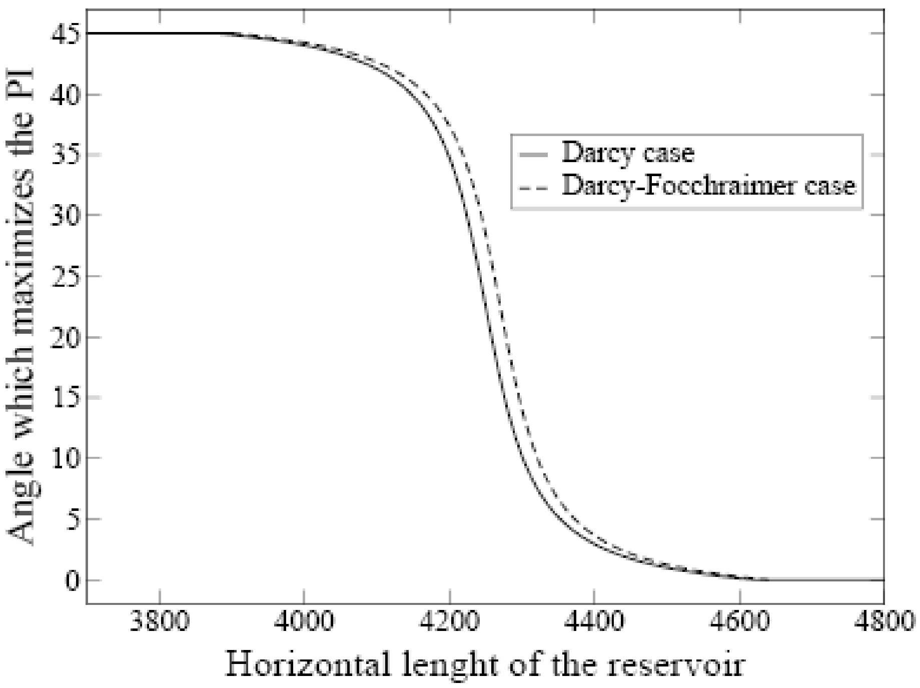
$$g(s) = a_0 + a_1 s$$

$$I_{Two\_Terms} \leq 1/J_{Two\_Terms}$$

E. Aulisa, A. Cakmak, ***A. I.*** *A. Solynin, "Variational principle and steady state invariant for non-linear hydrodynamic interactions in porous media Advances in Dynamical . Systems 14 (S2) (2007).*

Gap Between Variational Interpretation in Linear and Non-linear Cases





Geometric Interpretation

Theorem. Let

$$G(\nabla w) \quad \nabla\left(\frac{2}{\alpha+\sqrt{\alpha^2+\beta|\nabla w|}}\nabla w\right) = -\frac{Q}{|U|}$$

PSS Two terms Forchheimer Equation

$$\nabla\left(\frac{1}{\sqrt{1+|\nabla u|^2}}\nabla u\right) = -\frac{Q}{|U|}$$

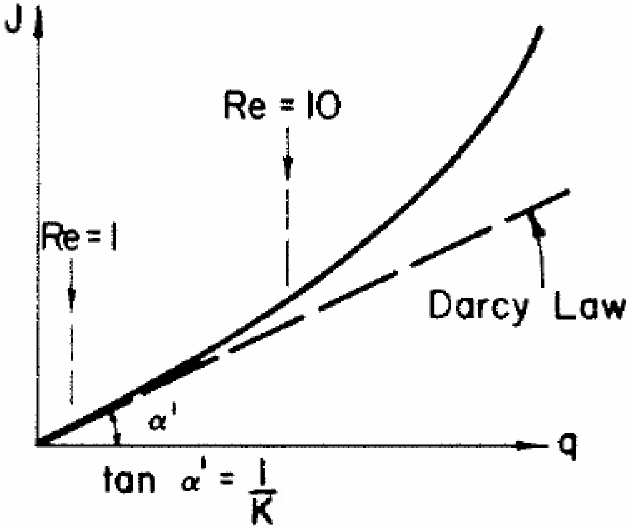
Constant Mean Curvature Equation

Then function $\eta(|\nabla u|)$ *exists, such that*

$$\nabla w = \sqrt{\eta(|\nabla u|/|\nabla u|^2}\nabla u$$

*M. Toda, M, **A.I.,** A, Aulisa, E, "Geometric Frame-Work for Modeling Non-Linear Flows …", J. of Non-linear Anal. v.. 11, 3, 2010.*
*E, Aulisa, **A.I.** , Toda, M. "Geom. Meth. in the Anal. of Non-linear…" Proc. of the AMS, Spectral Theory and Geometric Analysis,2011*

Impact of the non-linearity on the deviation $J_{g,PSS}$ from J_{Darcy}



$$J_{Darcy} = \frac{1}{\overline{w}_D}; \qquad \Delta w_D = -1/|U|$$

$$w_D\big|_{\Gamma_w} = 0 \ , \partial w_D / \partial \vec{n}\big|_{\Gamma_U} = 0$$

$$J_g(Q) = \frac{Q}{\overline{w}_F}; \qquad \nabla\left(K(\nabla w_F)\nabla w_F\right) = -Q/|U|$$

$$w_F\big|_{\Gamma_w} = 0 \ , \partial w_F / \partial \vec{n}\big|_{\Gamma_U} = 0$$

Comparison Theorem

$$\left|1 - \frac{J_{g,PSS}(Q)}{J_{Darcy}} - \right| \le \max\left|\nabla w_{Darcy}\right| QR$$

, *here* $R = \max\limits_{0 \le \xi < \infty} \sum_{j=1}^{k} a_j \frac{\xi^{\alpha_i - 1}}{g(\xi)}$

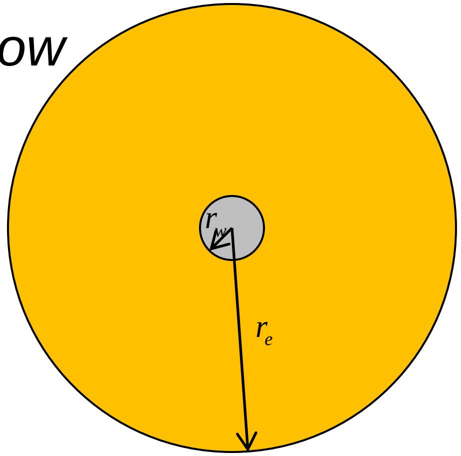
Two terms (quadratic)Forchheimer	*Three terms (cubic) Forchheimer*
$R = a_1 / a_0$	$R = a_1 / a_0 + \dfrac{a_2 / a_0}{2\sqrt{a_2 / a_0 + a_1 / a_0}}$

E. Aulisa, L. Bloshanskaya, ***A. I****. Long-term dynamics for well productivity index for nonlinear flows , J. Math. Phys. 52 (2011)*

Engineering Application - Skin Factor

$$J_{Darcy} = \frac{2\pi / a_0}{\ln \frac{r_e}{r_w} - 3/4}, \quad \text{for} \quad r_e >> r_w$$

Radial flow



Routine empirical approach, skin factor obtained by matching

$$J_{Forch} = \frac{2\pi / a_0}{\ln \frac{r_e}{r_w} - 3/4 + skin}, \quad \text{for} \quad r_e >> r_w$$

Application of the mathematical framework for general Forchheimer polynomial

$$J_{g,PSS} = \frac{1}{J_{Darcy}^{-1} + S_F^{-1}} \approx \frac{2\pi / a_0}{\ln \frac{r_e}{r_w} - 3/4 + 2\pi / a_0 S^{-1}}, \quad \text{for} \quad r_e >> r_w$$

$$S^{-1} = \sum_{j=1}^{k} \frac{a_j Q^{\alpha_j}}{(2\pi)^{\alpha_j + 1} r_e^{2(\alpha_j + 2)}} \int_0^{r_e} \frac{\left(r_e^2 - r^2\right)^{\alpha_j + 2}}{r^{\alpha_j + 1}} dr$$

*E. Aulisa, **A. I.,** P. P. Valko, J. R. Walton, "A new method for evaluation the productivity index…", SPE, v.14, n., 2009 .*
*E. Aulisa, L. Bloshanskaya, **A. I**. "Long-term dynamics for well productivity index for nonlinear flows ", J. Math. Phys. 52 (2011)*

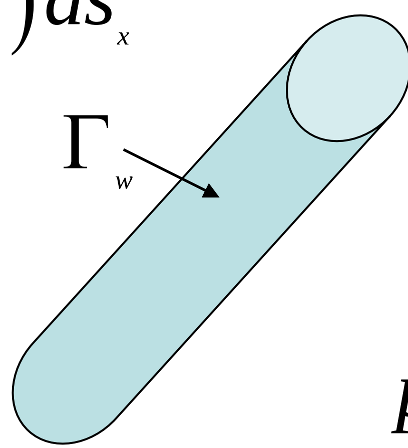
Important generalization of the idealized pseudo-steady state regime

Total well production

$$\int_{\Gamma_w} \left(K(\left| \nabla p_q \right| \partial p_q / \partial n \right) ds_x$$

Is specified

Γ_w



Bottom hole Pressure

$$p_A(x,t)\Big|_{\Gamma_w} -$$ Is specified

Pseudo-steady State Regime - Attractor

$$\left(p_s\right)_t = \gamma \nabla\left(K(\nabla p_s)\nabla p_s\right), \text{ and initial Data } \quad p_s(x,0) = w(x)$$

Satisfies simultaneously two conditions on the boundary

total flux *and* *Dirichlet (given pressure)*

$$\int_{\Gamma_w}\left(K(\left|\nabla p_s\right| \partial p_s / \partial n\right) ds_x = Q(t)$$

$$p_s(x,t) = -At + \phi_0(x) \quad \text{on } \Gamma_w$$

Total Flux $\int_{\Gamma_w}\left(K(\left|\nabla p_q\right| \partial p_q / \partial n\right) ds_x = Q(t) \to Q$

Dirichlet $p_A\big|_{\Gamma_w} = -At + \phi(x,t)$

$$\phi(x,t) \underset{H}{\Rightarrow} \phi_0(x)$$

Goal

$$J_K(p_q, \vec{v}_q, t) \Rightarrow J_K(p_s, \vec{v}_s, t) = const$$

$$J_K(p_A, \vec{v}_A, t) \Rightarrow J_K(p_s, \vec{v}_s, t) = const$$

Asymptotic and structural stability of the Diffusive capacity with respect to boundary Data

Theorem . Dirichlet BC *Assume degree condition* $\deg(g) \le \frac{4}{n-2}$

Let $\phi(x,t) \underset{H}{\Rightarrow} \phi_0(x)$

$$H : t^{\kappa} \| \phi'(t) \|_{W^{1,2}(\Gamma_w)}, t^{\kappa} \| \Delta_\phi(t) \|_{W^{1,2}(\Gamma_w)} \Rightarrow 0 \quad \text{as } t \to \infty, \text{ for some } k$$

Then $$J_K(p_A, \vec{v}_A, t) \Rightarrow J_K(p_s, \vec{v}_s, t) = const$$

Convergence follows from the inequality $$|Q(t) - Q| = \left| \int_U p_t ds - Q \right| \le C \|q\|_{L_2}$$

and asymptotical regularity with respect to time derivative:

Theorem on "asymptotical regularity". *If H condition is satisfied then* $\int_U \left(p - p_s \right)_t^2 dx \to 0$

q

Proof:

$$\frac{1}{2}\frac{d}{dt}\int_U q^2 dx \le -(1-a)\int_U K(|\nabla p|)|\nabla q|^2 dx + \int_U |\Psi_{tt}||q| dx + C\int_U |\nabla \Psi_t|^2 dx$$

*L.Hoang, **A.I.**" Structural stab. of gen. Forchheimer equations for compressible fluids in porous media", Nonlinearity, v.24, 1 , 2011*

*L.Hoang, **A.I.**" Qualitative study of Gen. Forchheimer flow with flux BC " Advances in Differential Equations, # 5-6, 2012*

*E. Aulisa, L. Bloshanskaya, **A. I**. "Long-term dynamics for well productivity index for nonlinear flows ", J. Math. Phys. 52 (2011)*

In case of total Flux condition solution is not unique

Class of the traces on the boundary is introduced in terms of the deviation from the averages on the boundary

$$p(x,t)\big|_{\Gamma} = \psi_0(x,t)$$

Average of the trace on the boundary

$$\gamma(t) = \bar{\psi}_0(x,t)$$

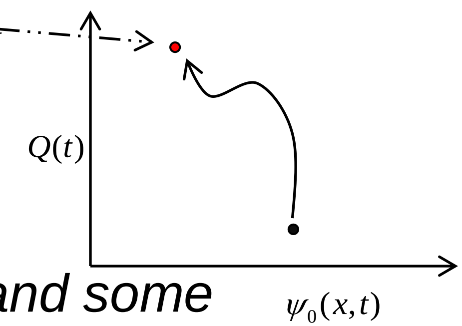
$$\psi(x,t) = \psi_0(x,t) - \gamma(t)$$

Then the trace

$$p(x,t)\big|_{\Gamma} = \psi(x,t) + \gamma(t)$$

Assume $\psi(x,t)$ is stabilising at time infinity to some $\phi(x)$ and total flux $Q(t)$ is stabilising to constant Q.

Then pair $(\phi(x), Q)$ generates a PSS problem, with time independent diffusive capacity



The following Theorem is true under degree constraint and some assumptions on deviation from the average:

Theorem

$$J_K(p_q, \vec{v}_q, t) \Rightarrow J_K(p_s, \vec{v}_s, t) = const$$

No explicite conditions on $\gamma(t)$!

For example if Dirchlet Data are split as $f(x,t) + r(t)$, and $f(x,t) \to f_0(x)$, and total flux $Q(t) \to Q$

*Aulisa, E., Bloshanskaya, L., **A.I.** (The time asymptotic of non-Darcy flows controlled by total flux, Journal of Mathematical Science, New York, Springer, Vol. 184, No. 4, July, 2012, 399-430*

Obtained results reveal several interesting observations, we list some of them:

- Generalized non-linear potential flows can be used as an effective framework to study non-linear flows in porous media.

- PSS regime serves as a global attractor for a class of IBVP with arbitral initial pressure distribution, and diffusive capacitance characterizes the deviation between regimes of production

- Diffusive capacity can be used as an criteria in up scaling procedure.

- Degree condition is not essential for long –time dynamics IBVP for Forchheimer flow with smooth boundary data.

Homogenization for Horizontally stratified reservoir

$$k_i; \beta_i; \phi_i$$

$$w = 0; w^* = 0$$

$$\vec{v} \cdot \vec{n} = 0; \vec{v}^* \cdot \vec{n} = 0$$

$$\frac{Q}{|U|}\phi(x) = \nabla\left(K(x, \nabla w)\nabla w\right) \qquad \frac{Q}{|U|}\phi^*(x) = \nabla\left(K^*(\nabla w^*)\nabla w^*\right)$$

$$(k^*)^{-1} = \sum k_i^{-1}\frac{h_i}{H}\left(\frac{\phi_i}{\phi^*}\right)^2; \; \frac{\beta^*}{k^*} = \sum \frac{\beta_i h_i}{k_i H}\left(\frac{\phi_i}{\phi^*}\right)^3$$

$$\phi^* = \frac{1}{|U|}\int_U \phi dx$$