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Preprint no. 2015-10

http://ncmm.karlin.mff.cuni.cz/

Complex wedge-shaped matrices: A generalization of Jacobi matrices☆

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Abstract

The paper [I. Hnětynková et al., Band generalization of the Golub–Kahan bidiagonalization, generalized Jacobi matrices, and the core problem, SIAM J. Matrix Anal. Appl., Vol. 36, No. 2 (2015), pp. 417–434] introduces real wedge-shaped matrices that can be seen as a generalization of Jacobi matrices, and investigates their basic properties. They are used in the analysis of the behavior of a Krylov subspace method: The band (or block) generalization of the Golub–Kahan bidiagonalization. Wedge-shaped matrices can be linked also to the band (or block) Lanczos method. In this paper, we introduce a complex generalization of wedge-shaped matrices and show some further spectral properties, complementing the already known ones. We focus in particular on nonzero components of eigenvectors.

Keywords: Eigenvalues, eigenvectors, wedge-shaped matrices, generalized Jacobi matrices, band (or block) Krylov subspace methods
2010 MSC: 15A18, 15B57, 65F15

1. Introduction

Jacobi matrices

$$T = \begin{bmatrix} \delta_1 & \xi_1 & & & \\ \xi_1 & \delta_2 & \xi_2 & & \\ & \xi_2 & \ddots & \ddots & \\ & & \ddots & \delta_{n-1} & \xi_{n-1} \\ & & & \xi_{n-1} & \delta_n \end{bmatrix} \in \mathbb{R}^{n\times n}, \qquad \xi_\ell > 0, \quad \ell = 1,\ldots,n-1, \qquad (1)$$

☆This work has been supported by the GAČR grant No. P201/13–06684S.
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[1]This author is a member of the University center for mathematical modeling, applied analysis and computational mathematics (Math MAC).
[2]The research of this author has been supported by the ESF grant CZ.1.07/2.3.00/30.0065.

i.e., symmetric tridiagonal matrices with positive sub-diagonal entries, represent thoroughly studied objects with the origin in the first half of the 19th century; see the historical note 3.4.3 in [14, Section 3.4]; see also [16, Chapter 7], [21, Section 5, §§ 36–48], and [7, Chapter 1.3]. Jacobi matrices have many interesting spectral properties such as: simple eigenvalues, strict interlacing of eigenvalues, eigenvectors with nonzero first and last entries, etc.; see, e.g., [16]. They are closely connected to the Lanczos tridiagonalization (see [13]), the Golub–Kahan bidiagonalization (see [9]), Gauss-type quadrature rules, moment problems, etc.; see [14]. Different generalizations of Jacobi matrices were proposed in different contexts; see [3, 11, 19].

In [12], the spectral properties of Jacobi matrices were used to prove fundamental properties of the so-called *core problem* (see [15]) within linear approximation problems

$$AX \approx B, \quad \text{where} \quad A \in \mathbb{R}^{n\times m},\ B \in \mathbb{R}^{n\times d},\ A^T B \neq 0, \tag{2}$$

with $d = 1$. (Note that the core problem is a useful tool in the analysis of the *total least squares* solution of (2); see [15].) In [11], it was shown that the core problem within (2) with $d > 1$ can be obtained by the *band (or block) generalization of the Golub–Kahan bidiagonalization* with *exact deflations*; see also [2]. (For further reading on finite precision computations we refer to [4, 20, 10, 18, 5, 1], or [6, Chapter 5].) Starting with an orthonormal basis $s_1, \ldots, s_\rho$ of the range of B, $\rho \equiv \mathrm{rank}(B)$, the generalized bidiagonalization yields a *block-bidiagonal matrix*, e.g.,

$$S_7^T A W_6 = \left[\begin{array}{ccc|cc|c}
\alpha_1 & & & & & \\
\beta_{2,1} & \alpha_2 & & & & \\
\beta_{3,1} & \beta_{3,2} & \alpha_3 & & & \\
\hline
\gamma_4 & \beta_{4,2} & \beta_{4,3} & \alpha_4 & & \\
 & & \gamma_5 & \beta_{5,4} & \alpha_5 & \\
\hline
 & & & \gamma_6 & \beta_{6,5} & \\
 & & & & \gamma_7 & \alpha_6
\end{array}\right] \equiv \left[\begin{array}{ccc}
\Phi_1^T & & \\
\Psi_2 & \Phi_2^T & \\
 & \Psi_3 & \Phi_3^T
\end{array}\right], \qquad \rho = 3,$$

where $\alpha_k > 0$, $\gamma_{k+\rho} > 0$, $k = 1, 2, \ldots$; Φ_ℓ, $\Psi_{\ell+1}$ are full row rank blocks in upper triangular row echelon forms; $S_k \equiv [s_1, \ldots, s_k]$, $W_k \equiv [w_1, \ldots, w_k]$, $S_k^T S_k = W_k^T W_k = I_k$; see [11]. The closely related *band (or block) Lanczos algorithm* (see [16, Chapter 13.10]) applied to AA^T with starting vectors $s_1, \ldots, s_\rho$ yields a *symmetric block-tridiagonal matrix*

$$S_7^T (AA^T) S_7 = \left[\begin{array}{ccc|cc|c|c}
\heartsuit & \heartsuit & \heartsuit & \alpha_1\gamma_4 & & & \\
\heartsuit & \heartsuit & \heartsuit & \heartsuit & & & \\
\heartsuit & \heartsuit & \heartsuit & \heartsuit & \alpha_3\gamma_5 & & \\
\hline
\alpha_1\gamma_4 & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \alpha_4\gamma_6 & \\
 & & \alpha_3\gamma_5 & \heartsuit & \heartsuit & \heartsuit & \\
\hline
 & & & \alpha_4\gamma_6 & \heartsuit & \heartsuit & \alpha_5\gamma_7 \\
\hline
 & & & & & \alpha_5\gamma_7 & \heartsuit
\end{array}\right] \equiv \left[\begin{array}{cccc}
\Delta_1 & \Xi_1^T & & \\
\Xi_1 & \Delta_2 & \Xi_2^T & \\
 & \Xi_2 & \Delta_3 & \Xi_3^T \\
 & & \Xi_3 & \Delta_4
\end{array}\right],$$

where $\heartsuit$ are in general nonzero dot-products of rows of $S_7^T A W_6$; blocks Δ_ℓ are symmetric, $\Delta_1 \in \mathbb{R}^{\rho\times\rho}$; and Ξ_ℓ are full row rank blocks in upper triangular row echelon forms. It was shown in [11] that real matrices of this block-tridiagonal form, called *wedge-shaped matrices*, can be understood as *a block generalization of Jacobi matrices*. The property of positive sub-diagonal entries of a Jacobi matrix is generalized to the property of full row rank sub-diagonal blocks.

Basic spectral properties of Jacobi matrices follow only from their nonzero structure and thus can be generalized to wedge-shaped matrices, while reflecting their block structure. It was proven in [11] that multiplicities of eigenvalues of a ρ-wedge-shaped matrix are bounded by ρ, and that the eigenvectors have nonzero leading and so-called quasi-trailing subvectors (i.e., subvectors with nonzero norms) of length ρ.

Here we extend the concept of wedge-shaped matrices to *the complex field* by preserving the block form required in the real case. Then we analyze properties of eigenvalues and eigenvectors following from the nonzero structure of wedge-shaped matrices. We give an illustrative schema for localization of nonzero subvectors of eigenvectors. We show how the results can be simplified when we consider proper band matrices.

Section 2 gives the definition of a ρ-wedge-shaped matrix and discusses some of its basic properties. Section 3 summarizes already known spectral properties of wedge-shaped matrices and reformulates them to the complex case. Section 4 discusses the structure of eigenvectors by describing a set of their nonzero subvectors, so-called running components. Section 5 comments on interlacing of eigenvalues, and Section 6 concludes the paper.

Throughout the text M^T, $M^H \equiv \overline{M}^T$, and $\mathcal{N}(M)$ denote the transposition, the conjugate transposition, and the null space of a matrix M, respectively; $M \otimes N$ denotes the Kronecker product of matrices (i.e., $m_{i,j}$, the (i,j)-th entry of M, is replaced by the block $Nm_{i,j}$ in the product); I denotes the square identity matrix, and e_k denotes the k-th column of I. The following convention concerning the entries of matrices will simplify the exposition:

- club ($\clubsuit$) stands for a nonzero entry, $\clubsuit \neq 0$;
- heart ($\heartsuit$) stands for a general entry which can also be zero;
- empty spaces in matrices always represent zero entries.

2. Definition and basic properties

For a Hermitian (self-adjoint) matrix $T \in \mathbb{C}^{n\times n}$ with entries $t_{k,j}$, we consider the following notation

$$f(k,T) = \min\{j \,:\, t_{k,j} \neq 0\} \qquad \text{and} \qquad h(k,T) = k - f(k,T), \qquad k = 1,\ldots,n, \quad (3)$$

in analogy to, e.g., [8, Chapter 4]. For simplicity, we often omit the second parameter and write $f(k)$ and $h(k)$. The number $f(k)$ is the column index of the first nonzero entry in the k-th row of T (provided it exists), and $h(k)$ is the distance between this and the diagonal entry. The number $h(k)$ is called the *k-th bandwidth of* T.[3] The following definition introduces complex wedge-shaped matrices; see also [11, Definition 4.1].

Definition 1 (ρ-wedge-shaped matrix). *Let $T \in \mathbb{C}^{n\times n}$ be a Hermitian matrix, i.e., $T = T^H$, and ρ, $1 \leq \rho < n$, an integer. If $h(k)$ is positive and non-increasing for $k = \rho+1,\ldots,n$, then we call T a ρ-wedge-shaped matrix. We denote $\mathcal{WS}_\rho^{n\times n}$ the set of all ρ-wedge-shaped matrices of order n.*

[3]Note that $h \equiv \max_{k=1,\ldots,n} h(k)$ is called the bandwidth of a Hermitian matrix; see [8]. Other authors call h the half-bandwidth and define the bandwidth as $2h+1$.

For clarity, we give an example of a 3-wedge-shaped matrix of order 9, and the corresponding values of $f(k)$ and $h(k)$:

$$T = \left[\begin{array}{ccc|cc|cc|c|c}
\heartsuit & \heartsuit & \heartsuit & \clubsuit & & & & & \\
\heartsuit & \heartsuit & \heartsuit & \heartsuit & \clubsuit & & & & \\
\heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & & & & \\ \hline
\clubsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \clubsuit & & & \\
 & \clubsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \clubsuit & & \\ \hline
 & & & \clubsuit & \heartsuit & \heartsuit & \heartsuit & & \\
 & & & & \clubsuit & \heartsuit & \heartsuit & \clubsuit & \\ \hline
 & & & & & & \clubsuit & \heartsuit & \clubsuit \\ \hline
 & & & & & & & \clubsuit & \heartsuit
\end{array}\right], \qquad (4)$$

k	4	5	6	7	8	9
$f(k)$	1	2	4	5	7	8
$h(k)$	3	3	2	2	1	1

The partitioning shows that a ρ-wedge-shaped matrix is a block-tridiagonal Hermitian matrix with full row rank sub-diagonal blocks (in upper triangular row echelon forms). Note that ρ-wedge shaped matrices satisfying

$$t_{k,k-\rho} \neq 0, \qquad \text{for} \qquad k = \rho + 1, \ldots, n, \tag{5}$$

i.e., $h(k) = \rho$, for $k = \rho + 1, \ldots, n$, are called *matrices with a constant bandwidth* (or *proper band matrices*; see [17]). They can be considered as basic building-blocks of general wedge-shaped matrices. The following examples illustrate that any wedge-shaped matrix contains overlapping principal blocks with constant bandwidth (highlighted by frames); in all three cases $\rho = 3$, $n = 7$:

$$\left[\begin{array}{ccccccc}
\heartsuit & \heartsuit & \heartsuit & \clubsuit & & & \\
\heartsuit & \heartsuit & \heartsuit & \heartsuit & & & \\
\heartsuit & \heartsuit & \heartsuit & \heartsuit & \clubsuit & & \\
\clubsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \clubsuit & \\
 & & \clubsuit & \heartsuit & \heartsuit & \heartsuit & \clubsuit \\
 & & & \clubsuit & \heartsuit & \heartsuit & \heartsuit \\
 & & & & \clubsuit & \heartsuit & \heartsuit
\end{array}\right], \left[\begin{array}{ccccccc}
\heartsuit & \heartsuit & \heartsuit & & & & \\
\heartsuit & \heartsuit & \heartsuit & \clubsuit & & & \\
\heartsuit & \heartsuit & \heartsuit & \heartsuit & \clubsuit & & \\
 & \clubsuit & \heartsuit & \heartsuit & \heartsuit & & \\
 & & \clubsuit & \heartsuit & \heartsuit & \clubsuit & \\
 & & & & \clubsuit & \heartsuit & \clubsuit \\
 & & & & & \clubsuit & \heartsuit
\end{array}\right], \left[\begin{array}{ccccccc}
\heartsuit & \heartsuit & \heartsuit & & & & \\
\heartsuit & \heartsuit & \heartsuit & & & & \\
\heartsuit & \heartsuit & \heartsuit & \clubsuit & & & \\
 & & \clubsuit & \heartsuit & \clubsuit & & \\
 & & & \clubsuit & \heartsuit & \clubsuit & \\
 & & & & \clubsuit & \heartsuit & \clubsuit \\
 & & & & & \clubsuit & \heartsuit
\end{array}\right]. \tag{6}$$

k	4	5	6	7
$f(k)$	1	3	4	5
$h(k)$	3	2	2	2

k	4	5	6	7
$f(k)$	2	3	5	6
$h(k)$	2	2	1	1

k	4	5	6	7
$f(k)$	3	4	5	6
$h(k)$	1	1	1	1

The following lemma gives some basic properties of wedge-shaped matrices.

Lemma 2. *Let* $T \in \mathcal{WS}_\rho^{n\times n}$.

(a) *If* $\rho < n - 1$, *then* $T \in \mathcal{WS}_{\rho+1}^{n\times n}$. *Consequently,*

$$\mathcal{WS}_\rho^{n\times n} \subset \mathcal{WS}_{\rho+1}^{n\times n} \subset \ldots \subset \mathcal{WS}_{n-1}^{n\times n} \subset \mathbb{C}^{n\times n}.$$

(b) *If* $f(\rho + 1) = 1$, *then* T *is a* $(2\rho + 1)$*-diagonal matrix.*

(c) *If* $f(\rho + 1) > 1$, *then* T *is at most a* $(2\rho - 1)$*-diagonal matrix.*

Proof. Property (a) follows directly from Definition 1. If $f(\rho+1) = 1$, then $h(\rho+1) = \rho$, i.e., $h(k) \leq \rho$, for $k = \rho + 1, \ldots, n$. If $f(\rho + 1) > 1$, then $h(\rho + 1) < \rho$, i.e., $h(k) < \rho$, for $k = \rho + 1, \ldots, n$. Since the bandwidth of the leading principal block of order ρ is trivially smaller than ρ, then T satisfying the assumption of (b) or (c) is a $(2\rho + 1)$- or a $(2\rho - 1)$-diagonal matrix, respectively. □

Note that we are not necessarily interested in the smallest value of ρ for which a given T is a wedge-shaped matrix. For example, in [11, Section 4.2], the aim is to verify that T is wedge-shaped for one particularly prescribed value of ρ. The following lemma on submatrices of wedge-shaped matrices will be useful later.

Lemma 3. *Let $T \in \mathcal{WS}_\rho^{n\times n}$ with the following (n_1, n_2)-partitioning,*

$$T \;=\; \underbrace{\left[\begin{array}{cc} T_1 & L \\ L^H & T_2 \end{array}\right]}_{n_1 \quad n_2} \begin{array}{l} \} \, n_1 \\ \} \, n_2 \end{array}, \qquad n_1 + n_2 = n, \quad 0 \le n_1, n_2 \le n. \tag{7}$$

(a) *If $n_1 > \rho$, then T_1 is a ρ-wedge-shaped matrix, $T_1 \in \mathcal{WS}_\rho^{n_1\times n_1}$.*

(b) *If $n_2 > h(n, T)$, then T_2 is a π-wedge-shaped matrix, $T_2 \in \mathcal{WS}_\pi^{n_2\times n_2}$. The smallest possible value of π is defined as*

$$\pi \equiv \pi(n_1, n_2) \equiv \Big(\arg \min_{\rho<k\le n} \{f(k, T) > n_1\}\Big) - n_1 - 1, \quad \pi \le \rho\,. \tag{8}$$

Proof. Assertion (a) follows directly from Definition 1. Let us focus on assertion (b). If $n_2 > h(n, T)$, then

$$f(n, T) = n - h(n, T) > n_1,$$

i.e., the first nonzero entry in the last row of T is placed in the block T_2. Let $k_{\min}$, $\rho < k_{\min} \le n$, be the first row of T for which $f(k_{\min}, T) > n_1$. Since $h(k, T) = k - f(k, T)$ is positive, then

$$k_{\min} = f(k_{\min}, T) + h(k_{\min}, T) > n_1 + 1,$$

i.e., the entry $t_{k_{\min}, f(k_{\min}, T)}$ is placed in the block T_2. By employing

$$f(n_1 + \ell, T) = f(\ell, T_2) + n_1, \quad h(n_1 + \ell, T) = h(\ell, T_2), \quad \text{for} \quad \ell = k_{\min} - n_1, \ldots, n_2,$$

we obtain $T_2 \in \mathcal{WS}_\pi^{n_2\times n_2}$ (and $T_2 \notin \mathcal{WS}_{\pi-1}^{n_2\times n_2}$), where $\pi \equiv k_{\min} - n_1 - 1$.

Clearly, $1 \le \pi < n_2$. If $n_2 \le \rho$, then $\pi < \rho$. If $n_2 > \rho$, then $h(n_1 + \rho + 1, T) \le \rho$ gives

$$f(n_1 + \rho + 1, T) \ge n_1 + 1, \quad \text{i.e.} \quad f(\rho + 1, T_2) \ge 1.$$

Consequently, $T_2 \in \mathcal{WS}_\rho^{n_2\times n_2}$ and the minimality of π results in $\pi \le \rho$. $\square$

The lemma directly implies that *any sufficiently large principal block* of a wedge-shaped matrix is again a wedge-shaped matrix. Let us illustrate how to determine the value of π from assertion (b). Consider, e.g., the $(5, 4)$-partitioning of (4). One can see that

$$T_2 = \left[\begin{array}{cccc} \heartsuit & \heartsuit & & \\ \heartsuit & \heartsuit & \clubsuit & \\ & \clubsuit & \heartsuit & \clubsuit \\ & & \clubsuit & \heartsuit \end{array}\right], \qquad \begin{array}{c||l|l} k & 3 = 8 - n_1 & 4 = 9 - n_1 \\ \hline f(k, T_2) & 2 = 7 - n_1 & 3 = 8 - n_1 \\ h(k, T_2) & 1 & 1 \end{array}, \tag{9}$$

is 2-wedge-shaped; i.e., $n_1 = 5$, $n_2 = 4$ yield $\pi = 2$.

Finally note that if $n_1 \leq \rho$ or $n_2 \leq h(n,T)$, then T_1 or T_2 have no particular structure, respectively. Moreover, for the given wedge-shaped matrix and the given (n_1, n_2)-partitioning it may happen that none of the conditions in assertions (a) and (b) of Lemma 3 is satisfied; see, e.g., the following 2-wedge-shaped matrix:

$$T = \left[\begin{array}{cc|cc} \heartsuit & \heartsuit & \clubsuit & \\ \heartsuit & \heartsuit & \heartsuit & \clubsuit \\ \hline \clubsuit & \heartsuit & \heartsuit & \heartsuit \\ & \clubsuit & \heartsuit & \heartsuit \end{array}\right], \qquad \rho = 2, \quad h(n,T) = 2, \quad \text{and} \quad n_1 = n_2 = 2.$$

3. Spectral properties of wedge-shaped matrices

Since ρ-wedge-shaped matrices are Hermitian block-tridiagonal with full row rank sub-diagonal blocks, they can be seen as a block generalization of real symmetric tridiagonal matrices with nonzero sub-diagonal entries (including the special case of the Jacobi matrices). It is well known that a symmetric tridiagonal matrix with nonzero sub-diagonal entries has distinct eigenvalues and its eigenvectors have nonzero first and last entries; see, e.g., [16, Lemma 7.7.1 and Theorem 7.9.3 (7.9.5 in the original Prentice-Hall edition)]. These properties fully follow from the nonzero pattern of the matrix, allowing their extension to wedge-shaped matrices.

The property of nonzero first entry can be generalized in two ways. We can either stay with a single eigenvector and study its leading subvectors, or we can look at the whole eigenspace corresponding to the given λ. We start with the first approach. The proofs are straightforward generalizations of proofs in [11] for real wedge-shaped matrices. Thus we give only ideas.

Theorem 4. *Let $T \in \mathcal{WS}_\rho^{n\times n}$ and let $\lambda \in \mathbb{R}$, $v = [\nu_1, \ldots, \nu_n]^T \in \mathbb{C}^n$ be an eigenpair of T, i.e., $Tv = \lambda v$, $v \neq 0$. Then the subvector*

$$v^{\downarrow} \equiv [\nu_1, \ldots, \nu_\rho]^T \in \mathbb{C}^\rho, \tag{10}$$

called the leading component of the eigenvector v, is nonzero.

Assuming $[\nu_1, \ldots, \nu_\rho]^T = 0$, the comparison of the left and right-hand sides of the first row of $Tv = \lambda v$ gives $\nu_{\rho+1} = 0$. Repeating the argument gives $\nu_k = 0$ for $k = \rho+2, \ldots, n$ which contradicts $v \neq 0$; see [11, Theorem 4.2] for details.

The following result for the whole eigenspace clearly reflects the block structure of the wedge-shaped matrix.

Corollary 5. *Let $T \in \mathcal{WS}_\rho^{n\times n}$ and let $\lambda \in \mathbb{R}$ be an eigenvalue of T with multiplicity r. Let $v_\ell = [\nu_{1,\ell}, \ldots, \nu_{n,\ell}]^T \in \mathbb{C}^n$, $\ell = 1, \ldots, r$, be an arbitrary basis of the corresponding eigenspace, i.e., $TV = \lambda V$, where $V = [v_1, \ldots, v_r] \in \mathbb{C}^{n\times r}$. Then the leading $\rho \times r$ block of V,*

$$V^{\downarrow} \equiv \begin{bmatrix} \nu_{1,1} & \cdots & \nu_{1,r} \\ \vdots & \ddots & \vdots \\ \nu_{\rho,1} & \cdots & \nu_{\rho,r} \end{bmatrix} \in \mathbb{C}^{\rho\times r}, \tag{11}$$

is of full column rank r.

Proof. Since $Vw \equiv [\omega_1, \ldots, \omega_n]^T$ represents an eigenvector of T for any $w \neq 0 \in \mathbb{C}^r$, then $V^{\downarrow}w = [\omega_1, \ldots, \omega_\rho]^T$ is nonzero, by Theorem 4. Thus $V^{\downarrow}$ has linearly independent columns, which gives the assertion; see also [11, Corollary 4.3]. □

Another corollary bounds the dimension of eigenspaces of a wedge-shaped matrix.

Corollary 6. *An eigenvalue of $T \in \mathcal{WS}_\rho^{n\times n}$ has multiplicity at most ρ.*

The proof follows directly from Corollary 5; see also [11, Corollary 4.4]. It can also be derived *independently* of Theorem 4 as follows: Consider $S \in \mathbb{C}^{(n-\rho)\times(n-\rho)}$ a submatrix of $T - \lambda I$, $\lambda \in \mathbb{R}$, formed by rows $\rho+1, \ldots, n$ and columns $f(\rho+1), \ldots, f(n)$. Since $f(k) < k$, for $k = \rho+1, \ldots, n$, S is upper triangular with nonzero entries $t_{k,f(k)}$ on the diagonal. Thus it is nonsingular for any λ, giving $\dim(\mathcal{N}(T-\lambda I)) \leq \rho$.

Generalization of the property of nonzero last entry is more complicated. The particular structure of the band of the given wedge-shaped matrix has to be taken into account. The following theorem states the result on trailing subvectors of eigenvectors.

Theorem 7. *Let $T \in \mathcal{WS}_\rho^{n\times n}$ and let $\lambda \in \mathbb{R}$, $v = [\nu_1, \ldots, \nu_n]^T \in \mathbb{C}^n$ be an eigenpair of T, i.e., $Tv = \lambda v$, $v \neq 0$. Denote*

$$\mathcal{I}^{\uparrow}(T) \equiv \{s_1, \ldots, s_\rho\} \equiv \{1, \ldots, n\} \setminus \{f(k,T) \,:\, k = \rho+1, \ldots, n\} \quad , \qquad s_1 < s_2 < \ldots < s_\rho\,, \tag{12}$$

where $f(k,T)$ is given by (3). *Then the subvector*

$$v^{\uparrow} \equiv [\nu_{s_1}, \ldots, \nu_{s_\rho}]^T \in \mathbb{C}^\rho, \tag{13}$$

called the quasi-trailing component of the eigenvector v, is nonzero.

The proof is similar to the proof of Theorem 4. Assuming $[\nu_{s_1}, \ldots, \nu_{s_\rho}] = 0$, the comparison of the left and right-hand sides of the last row of $Tv = \lambda v$ gives $\nu_{f(n)} = 0$. Repeating this argument gives $\nu_{f(k)} = 0$ for $k = n-1, n-2, \ldots, \rho+1$ which contradicts $v \neq 0$; see [11, Theorem 4.5] for details.

Note that the vector (13) always contains the last entry of the eigenvector v, i.e., $s_\rho = n$, but in general it does not represent the trailing part of v. See the nonzero quasi-trailing components of the above given examples of wedge-shaped matrices:

- $[\nu_3, \nu_6, \nu_9]^T \in \mathbb{C}^3$ of an eigenvector $v \in \mathbb{C}^9$ of (4),
- $[\nu_2, \nu_6, \nu_7]^T \in \mathbb{C}^3$ of an eigenvector $v \in \mathbb{C}^7$ of the first matrix in (6),
- $[\nu_1, \nu_4, \nu_7]^T \in \mathbb{C}^3$ of an eigenvector $v \in \mathbb{C}^7$ of the second matrix in (6),
- $[\nu_1, \nu_2, \nu_7]^T \in \mathbb{C}^3$ of an eigenvector $v \in \mathbb{C}^7$ of the third matrix in (6).

A simplified assertion can be obtained for ρ-wedge-shaped matrices with a constant bandwidth. Here $f(k,T) = k - h(k,T) = k - \rho$ giving

$$\mathcal{I}^{\uparrow}(T) = \{s_1, \ldots, s_\rho\} = \{n-\rho+1, \ldots, n\}.$$

Thus $v^{\uparrow} = [\nu_{n-\rho+1}, \ldots, \nu_n]^T \in \mathbb{C}^\rho$ is the trailing part of v of length ρ.

Denote, similarly to (12),

$$\mathcal{I}^{\downarrow}(T) \equiv \{1, \ldots, \rho\}. \tag{14}$$

The sets $\mathcal{I}^{\downarrow}(T)$ and $\mathcal{I}^{\uparrow}(T)$ of indices describing components $v^{\downarrow}$ and $v^{\uparrow}$, respectively, can be observed from the pattern of T. See for example the matrix (4) for which $\mathcal{I}^{\downarrow}(T) = \{1, 2, 3\}$ and $\mathcal{I}^{\uparrow}(T) = \{3, 6, 9\}$:

$$\begin{array}{ccccccccc}
\nu_1 & \nu_2 & \nu_3 & \nu_4 & \nu_5 & \nu_6 & \nu_7 & \nu_8 & \nu_9 \\
\downarrow & \downarrow & \downarrow & & & & & &
\end{array}$$

$$T = \left[\begin{array}{ccccccccc}
\heartsuit & \heartsuit & \heartsuit & \clubsuit & & & & & \\
\heartsuit & \heartsuit & \heartsuit & \heartsuit & \clubsuit & & & & \\
\heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & & & & \\
\clubsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \clubsuit & & & \\
 & \clubsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \clubsuit & & \\
 & & \uparrow & \clubsuit & \heartsuit & \heartsuit & \heartsuit & & \\
 & & \vdots & & \clubsuit & \heartsuit & \heartsuit & \clubsuit & \\
 & & \vdots & & & \uparrow & \clubsuit & \heartsuit & \clubsuit \\
 & & \vdots & & & \vdots & & \clubsuit & \heartsuit
\end{array}\right]. \tag{15}$$

$$\begin{array}{ccccccccc}
 & & \vdots & & & \vdots & & & \uparrow \\
\nu_1 & \nu_2 & \nu_3 & \nu_4 & \nu_5 & \nu_6 & \nu_7 & \nu_8 & \nu_9
\end{array}$$

The numbers $s_1, \ldots, s_{\rho-1} \in \mathcal{I}^{\uparrow}(T)$ in Theorem 7 represent row (and column) indices where the effective bandwidth of T is reduced.

Theorem 7 has a corollary analogous to Corollary 5 dealing with the whole eigenspace, which reflects the block structure of the wedge-shaped matrix.

Corollary 8. *Let $T \in \mathcal{WS}_{\rho}^{n\times n}$ and let $\lambda \in \mathbb{R}$ be an eigenvalue of T with multiplicity r. Let $v_\ell = [\nu_{1,\ell}, \ldots, \nu_{n,\ell}]^T \in \mathbb{C}^n$, $\ell = 1, \ldots, r$, be an arbitrary basis of the corresponding eigenspace, i.e., $TV = \lambda V$, where $V = [v_1, \ldots, v_r] \in \mathbb{C}^{n\times r}$. Let $\mathcal{I}^{\uparrow}(T) = \{s_1, \ldots, s_\rho\}$. Then the $\rho \times r$ submatrix of V,*

$$V^{\uparrow} \equiv \left[\begin{array}{ccc} \nu_{s_1,1} & \cdots & \nu_{s_1,r} \\ \vdots & \ddots & \vdots \\ \nu_{s_\rho,1} & \cdots & \nu_{s_\rho,r} \end{array}\right] \in \mathbb{C}^{\rho\times r}, \tag{16}$$

is of full column rank r.

Proof. Since $Vw \equiv [\omega_1, \ldots, \omega_n]^T$ represents an eigenvector of T for any $w \neq 0 \in \mathbb{C}^r$, then $V^{\uparrow}w = [\omega_{s_1}, \ldots, \omega_{s_\rho}]^T$ is nonzero, by Theorem 7. Thus $V^{\uparrow}$ has linearly independent columns, which gives the assertion. □

4. Running nonzero components of eigenvectors

In this section we focus on the characterization of a set of nonzero subvectors of eigenvectors of wedge-shaped matrices. We start with another well-known property of symmetric tridiagonal matrices with nonzero sub-diagonal entries, that will be demonstrated on the Jacobi matrix T in (1). We include the derivation in order to motivate further steps. Let $\lambda \in \mathbb{R}$, $v = [\nu_1, \ldots, \nu_n]^T \in \mathbb{C}^n$ be an eigenpair of T, i.e., $Tv = \lambda v$, $v \neq 0$. Then

$$(\delta_1 - \lambda)\nu_1 + \xi_1 \nu_2 = 0, \tag{17}$$
$$\xi_{\ell-1}\nu_{\ell-1} + (\delta_\ell - \lambda)\nu_\ell + \xi_\ell \nu_{\ell+1} = 0, \qquad \ell = 2, \ldots, n-1, \tag{18}$$
$$\xi_{n-1}\nu_{n-1} + (\delta_n - \lambda)\nu_n = 0. \tag{19}$$

Assume that $\nu_\ell = \nu_{\ell+1} = 0$ for some $1 \leq \ell < n$. Then from (18) it successively follows that $\nu_1 = \cdots = \nu_n = 0$ contradicting $v \neq 0$. Thus *two subsequent entries of an eigenvector of a symmetric tridiagonal matrix with nonzero sub-diagonal entries* cannot be zero. In other words, violating this property would imply that *either* the leading principal submatrix of T of order ℓ has an eigenvector with zero last component, *or* the trailing principal submatrix of T of order $(n - \ell)$ has an eigenvector with zero first component. However, none of these situations can occur.

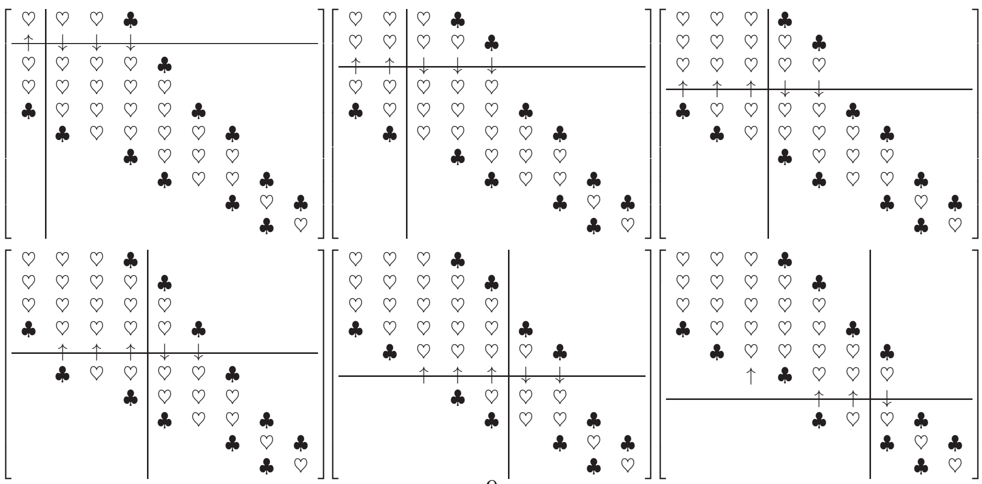
Now we use similar ideas to generalize this property to a wedge-shaped matrix T. Consider the *nontrivial* (n_1, n_2)-partitioning (7), i.e. satisfying $0 < n_1, n_2 < n$. Define two corresponding independent eigenvalue problems

$$T_1 \widetilde{v}_1 = \lambda_1 \widetilde{v}_1, \quad \widetilde{v}_1 \neq 0, \qquad \text{and} \qquad T_2 \widetilde{v}_2 = \lambda_2 \widetilde{v}_2, \quad \widetilde{v}_2 \neq 0, \tag{20}$$

where, $T_1 \in \mathbb{C}^{n_1 \times n_1}$, $T_2 \in \mathbb{C}^{n_2 \times n_2}$. If $n_1 > \rho$, then T_1 represents a ρ-wedge-shaped matrix and indices $\mathcal{I}^{\uparrow}(T_1)$ form the nonzero quasi-trailing component $\widetilde{v}_1^{\uparrow} \in \mathbb{C}^\rho$ of $\widetilde{v}_1$; see Lemma 3 (a) and Theorem 7. Analogously, if $n_2 > h(n, T)$, then T_2 represents a π-wedge-shaped matrix for $\pi \equiv \pi(n_1, n_2) \leq \rho$ defined in (8) and indices $\mathcal{I}^{\downarrow}(T_2) = \{1, \ldots, \pi\}$ form the nonzero leading component $\widetilde{v}_2^{\downarrow} \in \mathbb{C}^\pi$ of $\widetilde{v}_2$; see Lemma 3 (b) and Theorem 4. Otherwise, if $n_1 \leq \rho$ or $n_2 \leq h(n, T)$, then T_1 or T_2 are general square Hermitian matrices, respectively. For a *general square Hermitian matrix* $H \in \mathbb{C}^{k \times k}$ and its eigenvector $w \neq 0$ we formally define

$$\mathcal{I}^{\uparrow}(H) \equiv \mathcal{I}^{\downarrow}(H) \equiv \{1, \ldots, k\} \qquad \text{and} \qquad w^{\uparrow} \equiv w^{\downarrow} \equiv w. \tag{21}$$

The following schema shows all possible nontrivial (n_1, n_2)-partitionings of (4); the up- and down-arrows denote positions of entries belonging to components $\widetilde{v}_1^{\uparrow}$ and $\widetilde{v}_2^{\downarrow}$, respectively, similarly to (15):



$$
\left[\begin{array}{ccccccc|cc}
\heartsuit & \heartsuit & \heartsuit & \clubsuit & & & & & \\
\heartsuit & \heartsuit & \heartsuit & \heartsuit & \clubsuit & & & & \\
\heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & & & & \\
\clubsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \clubsuit & & & \\
 & \clubsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \clubsuit & & \\
 & & \uparrow & \clubsuit & \heartsuit & \heartsuit & \heartsuit & & \\
 & & & & \clubsuit & \heartsuit & \heartsuit & \clubsuit & \\
 & & & & & \uparrow & \uparrow & \downarrow & \\ \hline
 & & & & & & \clubsuit & \heartsuit & \clubsuit \\
 & & & & & & & \clubsuit & \heartsuit
\end{array}\right]
\left[\begin{array}{cccccccc|c}
\heartsuit & \heartsuit & \heartsuit & \clubsuit & & & & & \\
\heartsuit & \heartsuit & \heartsuit & \heartsuit & \clubsuit & & & & \\
\heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & & & & \\
\clubsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \clubsuit & & & \\
 & \clubsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \clubsuit & & \\
 & & \uparrow & \clubsuit & \heartsuit & \heartsuit & \heartsuit & & \\
 & & & & \clubsuit & \heartsuit & \heartsuit & \clubsuit & \\
 & & & & & \uparrow & \clubsuit & \heartsuit & \clubsuit \\
 & & & & & & & \uparrow & \downarrow \\ \hline
 & & & & & & & \clubsuit & \heartsuit
\end{array}\right]. \tag{22}
$$

Note that $n_1 \leq \rho$ holds in the first three partitionings, and $n_2 \leq h(n,T)$ in the last one. Denote

$$
\mathcal{I}^{(n_1,n_2)}(T) \equiv \mathcal{I}^{\uparrow}(T_1) \cup \Big(\mathcal{I}^{\downarrow}(T_2) + n_1\Big), \tag{23}
$$

where the set $(\mathcal{I}^{\downarrow}(T_2) + n_1)$ contains indices from $\mathcal{I}^{\downarrow}(T_2)$ increased by n_1. Let $v = [\nu_1, \ldots, \nu_n]^T \in \mathbb{C}^n$ be an eigenvector of T. Denote by $v^{(n_1,n_2)}$ a subvector (component) of v containing entries with the indices $\mathcal{I}^{(n_1,n_2)}$. The following set lists all $v^{(n_1,n_2)}$ components of v for the matrix (4):

$$
\left\{
\underset{v^{(1,8)}}{\underset{\parallel\!|}{\begin{bmatrix} \nu_1 \\ \hline \nu_2 \\ \nu_3 \\ \nu_4 \end{bmatrix}}},
\underset{v^{(2,7)}}{\underset{\parallel\!|}{\begin{bmatrix} \nu_1 \\ \nu_2 \\ \hline \nu_3 \\ \nu_4 \\ \nu_5 \end{bmatrix}}},
\underset{v^{(3,6)}}{\underset{\parallel\!|}{\begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ \hline \nu_4 \\ \nu_5 \end{bmatrix}}},
\underset{v^{(4,5)}}{\underset{\parallel\!|}{\begin{bmatrix} \nu_2 \\ \nu_3 \\ \nu_4 \\ \hline \nu_5 \\ \nu_6 \end{bmatrix}}},
\underset{v^{(5,4)}}{\underset{\parallel\!|}{\begin{bmatrix} \nu_3 \\ \nu_4 \\ \nu_5 \\ \hline \nu_6 \\ \nu_7 \end{bmatrix}}},
\underset{v^{(6,3)}}{\underset{\parallel\!|}{\begin{bmatrix} \nu_3 \\ \nu_5 \\ \nu_6 \\ \hline \nu_7 \end{bmatrix}}},
\underset{v^{(7,2)}}{\underset{\parallel\!|}{\begin{bmatrix} \nu_3 \\ \nu_6 \\ \nu_7 \\ \hline \nu_8 \end{bmatrix}}},
\underset{v^{(8,1)}}{\underset{\parallel\!|}{\begin{bmatrix} \nu_3 \\ \nu_6 \\ \nu_8 \\ \hline \nu_9 \end{bmatrix}}}
\right\}.
$$

The horizontal lines separate the entries determined by the indices of the quasi-trailing and leading components $\widetilde{v}_1^{\uparrow}$ and $\widetilde{v}_2^{\downarrow}$ of $\widetilde{v}_1$ and $\widetilde{v}_2$, respectively. We prove that $v^{(n_1,n_2)}$ are nonzero components of v, i.e. *indices* of nonzero subvectors of eigenvectors of submatrices T_1 and T_2 define indices of nonzero subvectors of eigenvectors of the matrix T.

Theorem 9. *Let $T \in \mathcal{WS}_{\rho}^{n\times n}$ and let $\lambda \in \mathbb{R}$, $v = [\nu_1, \ldots, \nu_n]^T \in \mathbb{C}^n$ be an eigenpair of T, i.e., $Tv = \lambda v$, $v \neq 0$. Consider the nontrivial (n_1,n_2)-partitioning* (7) *of T, $0 < n_1, n_2 < n$, and the conformal partitioning of v,*

$$
T = \begin{bmatrix} T_1 & L \\ L^H & T_2 \end{bmatrix}, \; v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad T_\ell \in \mathbb{C}^{n_\ell \times n_\ell},\; v_\ell \in \mathbb{C}^{n_\ell},\; \ell = 1,2, \quad n_1 + n_2 = n.
$$

Denote

$$
\begin{aligned}
\mathcal{I}^{(n_1,n_2)}(T) \equiv \{t_1, \ldots, t_\mu\} \equiv \mathcal{I}^{\uparrow}(T_1) \cup (\mathcal{I}^{\downarrow}(T_2) + n_1), \\
t_1 < t_2 < \ldots < t_\mu\,,
\end{aligned}
\qquad (\textit{see } (23)),
$$

where

$$
\mathcal{I}^{\uparrow}(T_1) = \begin{cases} \{s_1, \ldots, s_\rho\}, & \textit{when} \quad n_1 > \rho \qquad (\textit{see } (12)), \\ \{1, \ldots, n_1\}, & \textit{when} \quad n_1 \leq \rho \qquad (\textit{see } (21)), \end{cases}
$$

$$
\mathcal{I}^{\downarrow}(T_2) = \begin{cases} \{1, \ldots, \pi(n_1,n_2)\}, & \textit{when} \quad n_2 > h(n,T) \qquad (\textit{see } (14)), \\ \{1, \ldots, n_2\}, & \textit{when} \quad n_2 \leq h(n,T) \qquad (\textit{see } (21)), \end{cases}
$$

and where $\pi(n_1, n_2)$ is given in (8)*, and $h(n, T)$ in* (3)*. Then subvectors*

$$v^{(n_1,n_2)} \equiv [\nu_{t_1}, \ldots \nu_{t_\mu}]^T, \tag{24}$$

called running components of the eigenvector v, are nonzero.

Proof. The eigenvalue problem

$$Tv = \begin{bmatrix} T_1 & L \\ L^H & T_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda v, \quad v \neq 0 \tag{25}$$

yields

$$T_1 v_1 = \lambda v_1 - L v_2 \qquad \text{and} \qquad T_2 v_2 = \lambda v_2 - L^H v_1. \tag{26}$$

Assume that $v^{(n_1,n_2)} = 0$. The following situations may occur:

- If $n_1 > \rho$, then $\mathcal{I}^{\uparrow}(T_1) = \{1, \ldots, n_1\} \setminus \{f(k, T_1) : k = \rho + 1, \ldots, n_1\}$, and, in particular, entries of v_1 with indices $f(n_1, T_1) + 1, \ldots, n_1$ are zero. Since $h(k, T)$ is positive and non-increasing for $k = \rho + 1, \ldots, n$, we get

$$h(n_1, T) \geq h(n_1 + 1, T) \qquad \text{and}$$
$$f(n_1, T_1) = f(n_1, T) < f(n_1 + 1, T),$$

  i.e., the first $f(n_1, T_1)$ columns of L^H are zero. Consequently $L^H v_1 = 0$.
- If $n_1 \leq \rho$, then $\mathcal{I}^{\uparrow}(T_1) = \{1, \ldots, n_1\}$ giving $v_1 = 0$, and $L^H v_1 = 0$ independently of the structure of L.
- If $n_2 > h(n, T)$, then $\mathcal{I}^{\downarrow}(T_2) = \{1, \ldots, \pi(n_1, n_2)\}$ giving the first $\pi(n_1, n_2)$ entries of v_2 zero. Because T_2 is a $\pi(n_1, n_2)$-wedge-shaped matrix, we get

$$f(k, T_2) = f(k + n_1, T) - n_1, \qquad \text{for} \qquad k = \pi(n_1, n_2) + 1, \ldots, n_2,$$

  i.e., the first nonzero entry of the $(k + n_1)$-th row of T is localized in the block T_2 while the k-th row of L^H is zero. Consequently $Lv_2 = 0$.
- If $n_2 \leq h(n, T)$, then $\mathcal{I}^{\downarrow}(T_2) = \{1, \ldots, n_2\}$ giving $v_2 = 0$, and $Lv_2 = 0$ independently of the structure of L.

Summarizing, in any case all nonzero entries of L are multiplied in (26) by the leading zeros in v_2, and all nonzero entries of L^H are multiplied by the trailing zeros in v_1 giving

$$Lv_2 = 0 \qquad \text{and} \qquad L^H v_1 = 0; \tag{27}$$

see also (22) for an example. Thus (26) becomes

$$T_1 v_1 = \lambda v_1 \qquad \text{and} \qquad T_2 v_2 = \lambda v_2.$$

Since $v \neq 0$, then at least one of the vectors v_1, v_2 is nonzero and thus represents an eigenvector of T_1 giving $v_1 = \widetilde{v}_1$ or of T_2 giving $v_2 = \widetilde{v}_2$, respectively; see (20). Using (23) the assumption $v^{(n_1,n_2)} = 0$ contradicts the property that $\widetilde{v}_1^{\uparrow} \neq 0$ and $\widetilde{v}_2^{\downarrow} \neq 0$. □

Note that the theorem can be extended to the *trivial partitionings* with $n_1 = 0$, $n_2 = n$ (i.e., $T_1 \equiv [\,]$ is an empty matrix and $T_2 \equiv T$) and with $n_1 = n$, $n_2 = 0$ (i.e., $T_1 \equiv T$ and $T_2 \equiv [\,]$). Here the running components are identical to the leading and quasi-trailing components, respectively, i.e.,

$$v^{(0,n)} \equiv v^{\downarrow} \qquad \text{and} \qquad v^{(n,0)} \equiv v^{\uparrow}.$$

Although the statement of the theorem is complicated, schemata (15) and (22) give a simple and illustrative approach to localize nonzero components of eigenvectors based on the structure of the matrix. Finally note that Theorem 9 has a corollary analogous to Corollaries 5 and 8 that we do not formulate explicitly.

In the special case of a ρ-wedge shaped matrix $T \in \mathbb{C}^{n\times n}$ satisfying (5), i.e., with a constant bandwidth, and $n > 2\rho$, we get

$$v^{(\ell,n-\ell)} \in \begin{cases} \mathbb{C}^{\rho+\ell}, & \text{for} \quad \ell = 0,\dots,\rho-1, \\ \mathbb{C}^{2\rho}, & \text{for} \quad \ell = \rho,\dots,n-\rho, \\ \mathbb{C}^{\rho+n-\ell}, & \text{for} \quad \ell = n-\rho+1,\dots,n. \end{cases}$$

The running components have the constant length 2ρ (except for several leading and trailing components). Consequently, Theorems 4, 7, and 9 have the following corollary.

Corollary 10. *Let $T \in \mathcal{WS}_\rho^{n\times n}$ and let $\lambda \in \mathbb{R}$, $v = [\nu_1,\dots,\nu_n]^T \in \mathbb{C}^n$ be an eigenpair of T, i.e., $Tv = \lambda v$, $v \neq 0$. If*

$$t_{k,k-\rho} \neq, 0 \quad \textit{for} \quad k = \rho+1,\dots,n,$$

then:

(a) *The leading component $v^{(0,n)} = v^{\downarrow} = [\nu_1,\dots,\nu_\rho]^T \in \mathbb{C}^\rho$ of v is nonzero.*
(b) *The trailing component $v^{(n,0)} = v^{\uparrow} = [\nu_{n-\rho+1},\dots,\nu_n]^T \in \mathbb{C}^\rho$ of v is nonzero.*
(c) *Provided $n \geq 2\rho$, any running component $v^{(\ell,n-\ell)} = [\nu_{\ell-\rho+1},\dots,\nu_{\ell+\rho}]^T \in \mathbb{C}^{2\rho}$ for $\ell = \rho,\dots,n-\rho$ of v is nonzero.*

5. Note on the interlacing property

It is well-known that eigenvalues of Jacobi (and all 1-wedge-shaped) matrices have the so-called *strict interlacing* property. Let T_n be a 1-wedge-shaped matrix of order n, and T_j its leading principal submatrices of orders j, $j = 1,\dots,n-1$, i.e.,

$$T_1 = [\delta_1] \in \mathbb{C}^{1\times 1}, \qquad \delta_1 = \overline{\delta}_1,$$

$$T_j = \begin{bmatrix} T_{j-1} & e_{j-1}\overline{\xi}_{j-1} \\ \xi_{j-1}e_{j-1}^T & \delta_j \end{bmatrix} \in \mathbb{C}^{j\times j}, \qquad \delta_j = \overline{\delta}_j,\ \xi_{j-1} \neq 0,\ j = 2,\dots,n,$$

$$T_n = \begin{bmatrix} \delta_1 & \overline{\xi}_1 & & & \\ \xi_1 & \delta_2 & \overline{\xi}_2 & & \\ & \xi_2 & \ddots & \ddots & \\ & & \ddots & \delta_{n-1} & \overline{\xi}_{n-1} \\ & & & \xi_{n-1} & \delta_n \end{bmatrix} \in \mathbb{C}^{n\times n}. \tag{28}$$

The eigenvalues λ_ℓ, $\ell = 1, \dots, j$, of T_j are strictly interlaced by the eigenvalues λ'_s, $s = 1, \dots, j-1$, of T_{j-1},

$$\lambda_1 < \lambda'_1 < \lambda_2 < \lambda'_2 < \ldots < \lambda_{j-1} < \lambda'_{j-1} < \lambda_j\,;$$

see, e.g., [16, section 7.10].

The ρ-wedge-shaped matrices have multiplicities of eigenvalues bounded by ρ, by Corollary 6. Employing the 1-wedge-shaped matrix (28) yields

$$\boldsymbol{T}_n \equiv T_n \otimes I_\sigma = \begin{bmatrix} I_\sigma\delta_1 & I_\sigma\overline{\xi}_1 & & & \\ I_\sigma\xi_1 & I_\sigma\delta_2 & I_\sigma\overline{\xi}_2 & & \\ & I_\sigma\xi_2 & \ddots & \ddots & \\ & & \ddots & I_\sigma\delta_{n-1} & I_\sigma\overline{\xi}_{n-1} \\ & & & I_\sigma\xi_{n-1} & I_\sigma\delta_n \end{bmatrix} \in \mathbb{C}^{n\sigma\times n\sigma},$$

a σ-wedge shaped matrix satisfying (5), i.e., with a constant bandwidth. Matrices T_n and $\boldsymbol{T}_n$ have the same spectra; since all eigenvalues of (28) are simple, all eigenvalues of $\boldsymbol{T}_n$ have multiplicities σ, i.e., multiplicities of all eigenvalues reach the maximal bound given by Corollary 6. The *multiple* eigenvalues of $\boldsymbol{T}_j$ are strictly interlaced by *multiple* eigenvalues of its leading principal submatrix $\boldsymbol{T}_{j-1}$. Spectra of all $\sigma - 1$ interjacent leading principal submatrices of $\boldsymbol{T}_j$, having $\boldsymbol{T}_{j-1}$ as the leading principal submatrix, are fully given by the standard (not the strict) interlacing, as illustrated on the example in Table 1. The *strong interlacing property therefore cannot hold* for general wedge-shaped matrices. Note that the wedge-shaped matrix $\boldsymbol{T}_n$ is for $\sigma > 1$ *reducible*, whereas Jacobi matrices are always irreducible.

Table 1: Interlacing of eigenvalues of *leading* principal submatrices of $\boldsymbol{T}_3 = T_3 \otimes I_\sigma$ of order k. Eigenvalues λ_ℓ of $T_3 \in \mathbb{C}^{3\times 3}$ are strictly interlaced by eigenvalues λ'_s of its leading principal submatrix $T_2 \in \mathbb{C}^{2\times 2}$, i.e., $\lambda_1 < \lambda'_1 < \lambda_2 < \lambda'_2 < \lambda_3$.

Submatrix of order k	Characteristic polynomial				
$\boldsymbol{T}_3 \in \mathbb{C}^{k\times k}$, $k = 3\sigma$	$(\lambda-\lambda_1)^\sigma$		$(\lambda-\lambda_2)^\sigma$		$(\lambda-\lambda_3)^\sigma$
$k = 3\sigma-1$	$(\lambda-\lambda_1)^{\sigma-1}$	$(\lambda-\lambda'_1)^1$	$(\lambda-\lambda_2)^{\sigma-1}$	$(\lambda-\lambda'_2)^1$	$(\lambda-\lambda_3)^{\sigma-1}$
$k = 3\sigma-2$	$(\lambda-\lambda_1)^{\sigma-2}$	$(\lambda-\lambda'_1)^2$	$(\lambda-\lambda_2)^{\sigma-2}$	$(\lambda-\lambda'_2)^2$	$(\lambda-\lambda_3)^{\sigma-2}$
$k = 3\sigma-3$	$(\lambda-\lambda_1)^{\sigma-3}$	$(\lambda-\lambda'_1)^3$	$(\lambda-\lambda_2)^{\sigma-3}$	$(\lambda-\lambda'_2)^3$	$(\lambda-\lambda_3)^{\sigma-3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$k = 2\sigma+3$	$(\lambda-\lambda_1)^3$	$(\lambda-\lambda'_1)^{\sigma-3}$	$(\lambda-\lambda_2)^3$	$(\lambda-\lambda'_2)^{\sigma-3}$	$(\lambda-\lambda_3)^3$
$k = 2\sigma+2$	$(\lambda-\lambda_1)^2$	$(\lambda-\lambda'_1)^{\sigma-2}$	$(\lambda-\lambda_2)^2$	$(\lambda-\lambda'_2)^{\sigma-2}$	$(\lambda-\lambda_3)^2$
$k = 2\sigma+1$	$(\lambda-\lambda_1)^1$	$(\lambda-\lambda'_1)^{\sigma-1}$	$(\lambda-\lambda_2)^1$	$(\lambda-\lambda'_2)^{\sigma-1}$	$(\lambda-\lambda_3)^1$
$\boldsymbol{T}_2 \in \mathbb{C}^{k\times k}$, $k = 2\sigma$		$(\lambda-\lambda'_1)^\sigma$		$(\lambda-\lambda'_2)^\sigma$	

6. Conclusion

We have extended some of the well-known spectral properties of symmetric tridiagonal matrices with nonzero sub-diagonal entries (including Jacobi matrices) to the class of complex matrices called wedge-shaped. In particular, we have characterized a set of nonzero subvectors (running components) of eigenvectors of wedge-shaped matrices and described an illustrative schema for their localization based on the structure of the matrix. We have shown how the presented properties can be reformulated when we consider a wedge-shaped matrix with a constant bandwidth (a proper band matrix). The concept of (real) wedge-shaped matrices has been already used in [11] in the analysis of the band (or block) generalization of the Golub–Kahan bidiagonalization closely connected to the band (or block) Lanczos algorithm, and also in the analysis of core problems within linear approximation problems (2) with multiple right hand sides, i.e., with $d > 1$. Thus, we believe that the presented results will be useful, e.g, in further study of band and block Krylov subspace methods for complex data and related topics.

Acknowledgements

We wish to thank the anonymous referee for her or his comments which have led to improvements of our manuscript.

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