# Model reduction: analysis, numerical solution and real world applications Lecture I: Model reduction, a survey


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**DFG Research Center MATHEON**

***Mathematics for key technologies***


September 7, 2015

# Outline

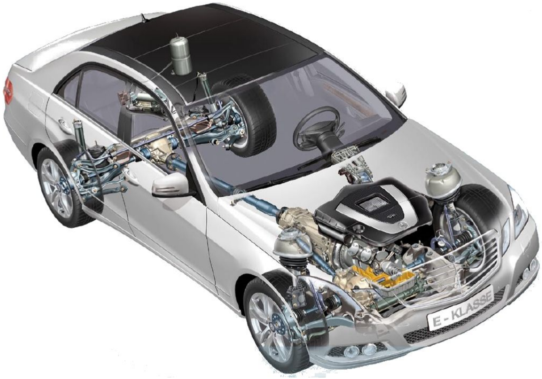
# Theses

- Key technologies require **M**odeling, **S**imulation, and **O**ptimization (MSO) of complex dynamical systems.
- Most real world systems are multi-physics systems, with different accuracies and scales in components.
- Modeling today becomes exceedingly automatized, linking subsystems together.
- Modeling, analysis, numerics, control and optimization techniques should go hand in hand.
- Most real world (industrial) models are too complicated for optimization and control. Model reduction is a key issue.
- We need to be able to quantify errors and uncertainties in the reduction process.

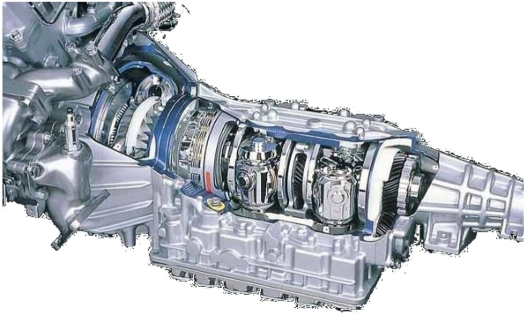
# Outline

**Model/software based control of automatic transmission.**

Project with Daimler AG



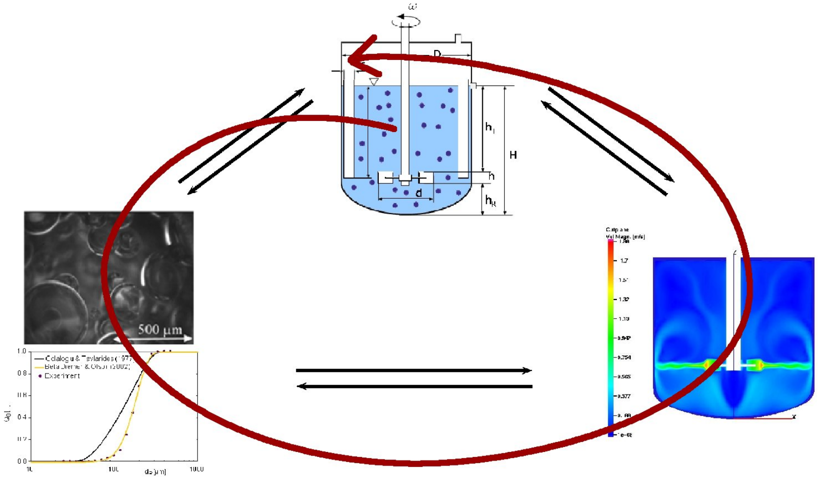
# A current half-toroid model



- Modeling of coupled dynamical multi-physics model: multi-body system, elasticity, hydraulics, friction, . . . .
- Development of control/optimization methods for coupled system.
- Model reduction to make control/optimization feasible.
- Real time control software for transmission on board computer.

**Ultimate goals: Decrease fuel consumption, save money on production, improve switching.**

with S. Schmelter and M. Kraume (Chemical Eng., TU Berlin)



# Technological Application, Tasks

**Chemical industry: pearl polymerization and extraction processes**

- Modeling of coalescence and breakage in turbulent flow.
- Numerical methods for simulation of coupled system of population balance equations/fluid flow equations.
- Development of optimal control methods for large scale coupled systems.
- Model reduction and observer design.
- Feedback control of real configurations via stirrer speed.

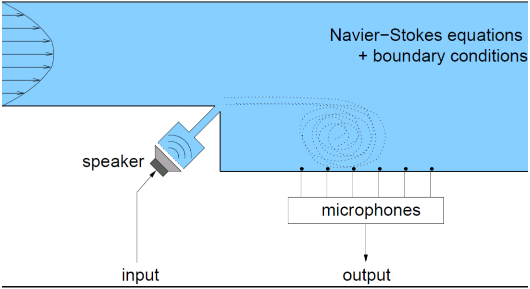
**Goal:** Achieve specified average drop diameter and small standard deviation for distribution by real time-control of stirrer-speed.

# Mathematical system components

- Navier Stokes equation (flow field) ($\rightarrow$ Film).
- Population balance equation (drop size distribution).
- One or two way coupling.
- Initial and boundary conditions.

Space discretization leads to an extremely large control system of nonlinear DAEs.

**Project in SFB 557 Control of complex shear flows**, with F. Tröltzsch, M. Schmidt



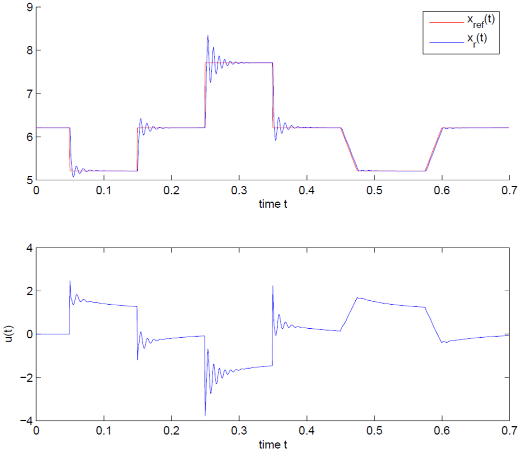
# Technological Application, Tasks

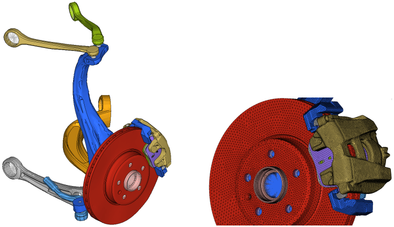
**Control of detached turbulent flow on airline wing**

- Test case (backward step to compare experiment/numerics.)
- Modeling of turbulent flow.
- Development of control methods for large scale coupled systems.
- Model reduction.
- Optimal feedback control of real configurations via blowing and sucking of air in wing.

**Ultimate goal:** Force detached flow back to wing.

Movement of recirculation bubble following reference curve.





- Disc brake squeal is a frequent and annoying phenomenon (with cars, trains, bikes).
- Important for customer satisfaction, even if not a safety risk.
- Nonlinear effect that is hard to detect in experiments.
- The car industry is trying for decades to improve this, by changing the designs of brake and disc.

# Model based approach

Interdisciplinary project with car manufacturers + SMEs
Supported by German Minist. of Economics via AIF foundation.
University: N. Gräbner, U. von Wagner, TU Berlin, Mechanics,
N. Hoffmann, TU Hamburg-Harburg, Mechanics,
S. Quraishi, C. Schröder, TU Berlin Mathematics.
Goals:

- Develop model of brake system with all effects that may cause squeal. (Friction, circulatory, gyroscopic effects, etc).
- Simulate brake behavior for many different parameters (disk speed, material geometry parameters).
- Our task: Model reduction, solution of eigenvalue problems.
- Long term: Stability/bifurcation analysis for a given parameter region.

# Outline

... replace a big complicated (computational) model with a smaller and simpler (but still accurate) one.

- Everybody does this. (Also called **Science**)
- One wants to have the most simple model for analysis, simulation, optimization and control.
- Ideally the reduced model should have good fidelity compared to reality.

# What is a small and simple?

- Small and simple model to do analysis;
- small and simple model for computational simulation (parameter studies);
- small and simple model for optimization (of design parameters);
- small and simple model to do (real time) control;
- small nonlinear vs. large linear model?

# Outline

Classical and successful approach in control engineering:

- ▷ Build prototype or high fidelity simulator for dynamical system.
- ▷ Generate input-output sequences $(u_i)_i,\ (y_i)_i$ by measurement or solving forward problem.
- ▷ Generate input-output map (transfer function in frequency domain) that interpolates input-output sequence.
- ▷ Realize input-output map as a linear finite dimensional system

$$\dot{x} = Ax + Bu,\ y = Cx$$

with (typically large) matrices $A, B, C$.
- ▷ Reduce model to small model $\dot{x}_r = A_r x_r + B_r u,\ y_r = C_r x_r$ with small error $\|y - y_r\| \leq ||u||tol$.
- ▷ Build a (feedback) controller from small linear model and apply it in the full physical model.

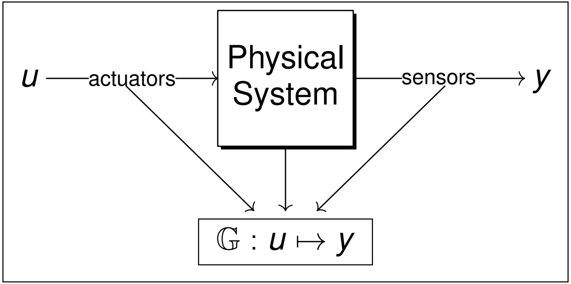



Figure: I/O map (transfer function) for physical system.

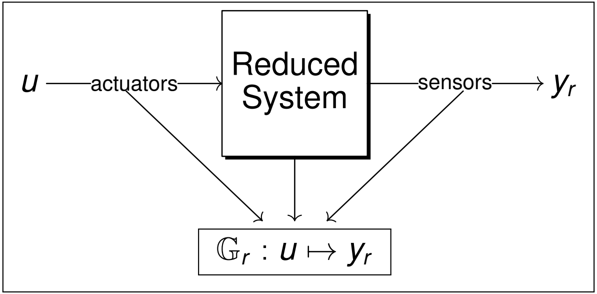



Figure: I/O map (transfer function) for reduced system.

$$\|y - y_r\| \leq \|Gu - G_r u\| \leq \|G - G_r\| \|u\|$$

So want good approximation of transfer function.

# Outline

**Physical system**

↙ ↘

**Modeling**      **Modeling**

↓      ↓

**ODE/DAE** ⟵ *semidiscr.* **PDE**

↓

**Mod. reduction**

↓

**Reduced ODE/DAE** ⟶ *Sim., Control*

- Semidiscretization in space using FV, FE, FD $\Longrightarrow$ large scale ODE/DAE-control problem.
- Model reduction to reduce state dimension.
- Computation of (feedback) control for reduced model using standard software.
- Apply computed control in large semi-discretized model infinite dimensional or real physical model.

# MOR in PDE constrained optimization

Given a PDE constraint control or optimization problem.
Different approaches.

- First semi-discretize (in space), then reduce continuous time model, then optimize and control. (POD, Balanced truncation, DEIM, IRKA, . . . ).
- Discretize (in space and time) as optimization or control problem in adaptive way (reduced basis).
- Discretize optimality conditions (forward and adjoint problem) in adaptive way (adaptive FE, FD, FV).
- Combinations of all of these.

# Outline

Replace system

$$\begin{aligned} F(t, x, \dot{x}, u) &= 0, \quad x(t_0) = x^0 \\ y(t) &= g(x) \end{aligned}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$, by a reduced model

$$\begin{aligned} F_r(t, x_r, \dot{x}_r, u) &= 0, \quad x_r(t_0) = x_r^0 \\ y_r(t) &= g_r(x_r) \end{aligned}$$

with $x_r \in \mathbb{R}^{n_r}$, $n_r << n$.

**Goals**

- Approximation error $\|y - y_r\|$ small, global error bounds;
- Preservation of physics: stability, passivity, conservation laws;
- Stable and efficient method for model reduction.

# Outline

# Model reduction techniques

- Modal truncation
- Balanced truncation (BT)
- Hankel approximation
- Principal orthogonal decomposition (POD)
- Discrete empirical interpolation (DEIM)
- Iterative rational Krylov method (IRKA)
- Moment matching
- Reduced basis methods

# Model reduction techniques

SVD (singular value decomposition) based methods

- ▷ Balanced truncation (linear) Antoulas, Benner, Li, Moore, Penzl, Stykel, Sorensen, Varga, Wang, White, ...
- ▷ Principal orthogonal decomposition (POD), (linear/nonlinear) Banks, Benner, Hinze, King, Kunisch, Tröltzsch, Volkwein, ...
- ▷ DEIM (nonlinear) Chaturantabut, Maday, Sorensen, ...

Interpolation based methods

- ▷ IRKA (linear) Antoulas, Beattie, Gugercin, ...

Krylov methods

- ▷ Moment matching, (linear) Bai, Boley, Freund, Gallivan, Gragg, Grimme, Van Dooren, ...
- ▷ Modal truncation (linear) Bampton, Craig, Guyan, Rommes...

Reduced basis methods

- ▷ (linear/nonlinear) Haasdonk, Ohlberger, Patera, Quateroni, Rozza, ...

# Outline

Replace

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(t_0) = x_0 \\ y(t) &= Cx(t) \end{aligned}$$

by

$$\begin{aligned} \dot{x}_r(t) &= A_r x_r(t) + B_r u(t), \quad x_r(t_0) = x_r^0 \\ y_r(t) &= C_r x_r(t), \end{aligned}$$

with $x_r \in \mathbb{R}^{n_r}$, $n_r << n$.

Laplace transformation and approximation in frequency domain.

$$\begin{aligned} \hat{y} &= C(sI - A)^{-1}B\hat{u} \\ &= G(s)\hat{u}, \end{aligned}$$

with rational matrix valued transfer function $G(s)$ in Hardy space of functions that are analytic and bounded in the right half of complex plane.

$$\|G - G_r\|_{H_\infty} = \sup_{\omega \in \mathbb{R}} \|G(i\omega) - G_r(i\omega)\|$$

and approximate transfer function $G_r(s) = C_r(sI - A_r)^{-1}B_r$.
($G(i\omega)$ : “frequency response matrix”)

# Outline

# Petrov-Galerkin approach

Common idea in most methods: Galerkin or Petrov-Galerkin-type projection of state-space onto low-dimensional subspace $V$ along subspace $W$ (biorthogonal)

- $x \approx VW^T x =: x_r$, where $W^T V = I_r$.
- Then, with $x_r = W^T x$, we obtain $x \approx Vx_r$ and

$$\|x - x_r\| = \|x - Vx_r\|$$

Choose $V$ as matrix of eigenvectors associated with dominant eigenvalues of $A$.

$$T^{-1}AT = \mathrm{diag}(A_1, A_2),\ T^{-1}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\ CT = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

$$\|G - G_r\|_{H_\infty} \leq cond_2(T)\|C_2\|_2\|B_2\|_2 \frac{1}{\min_{\lambda \in \sigma(A_2)} |Re(\lambda)|}$$

# Analysis of Modal Truncation

- ▷ Fast and easy to use.
- ▷ Works for very large scale problems.
- ▷ Eigenvalues contain only limited information.
- ▷ What is dominant. Dominant pole algorithm. Rommes
- ▷ Error bound hard to compute in large scale case.
- ▷ Combination with adaptive FEM possible (talk III)

## Moment matching, Pade' via Lanczos

Expand the transfer function $G(s)$ at point $s_0$

$$G(s) = M_0 + M_1(s - s_0) + M_2(s - s_0)^2 + \dots$$

and find approximate $\tilde{C}, \tilde{B}, \tilde{A}$ so that in the expansion of

$$\tilde{C}(sI - \tilde{A})^{-1}\tilde{B} = \tilde{M}_0 + \tilde{M}_1(s - s_0) + \tilde{M}_2(s - s_0)^2 + \dots$$

as many terms as possible are matched.

- $s_0 = \infty$: partial realization, Pade' approximation. Solution via Lanzcos or Arnoldi method.
- $s_0 \in \mathbb{C}$ rational interpolation. Solution via rational Lanczos.

# Analysis of Moment Matching

- Fast and easy to use.
- Works for very large scale problems.
- Very successful in practice, VLSI simulation.
- Preservation of passivity, Freund, Sorensen, Reis.
- Choice of expansion point $s_0$?
- Computation of moments is problematic.
- No global error bound.
- Possible breakdown of Lanczos.

$$\dot{x} = Ax + Bu, \quad y = Cx$$

Consider Lyapunov equations:

$$\begin{aligned} AX_B + X_B A^T &= -BB^T \quad (X_B \text{ controllab. Gramian}) \\ A^T X_C + X_C A &= -C^T C \quad (X_C \text{ observab. Gramian}). \end{aligned}$$

- If $A$ is stable and the system is controllable and observable, then $X_B, X_C$ are positive definite.
- Idea: Make the system balanced, $X_B = X_C$ diagonal, and truncate small components (hard to control and observe).
- Every controllable and observable system can be balanced by a change of basis $\tilde{x} = Tx$.

# Balanced truncation algorithm

1. Compute Gramians, Cholesky factors. $X_B = L_B L_B^T$, $X_C = L_C L_C^T$.
2. Compute the SVD of $U \Sigma V^T = L_B^T L_C$ with

$$\Sigma = \mathrm{diag}(\sigma_1, \ldots \sigma_n) = \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix},$$

   (Hankel sing. val.). $\Sigma_2 = \mathrm{diag}(\sigma_{\tilde{n}+1}, \ldots, \sigma_n)$, $\sigma_{\tilde{n}+1}, \ldots, \sigma_n \leq tol$.
3. Set $T = \Sigma^{1/2} U^T L_B^{-1} = \Sigma^{-1/2} V^T L_C^T$.
4. Set $\tilde{x} = Tx$ and partition matrices as $\Sigma$.

$$\begin{aligned} TAT^{-1} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \; Tx = \begin{bmatrix} x_r \\ \tilde{x}_r \end{bmatrix}, \\ TB &= \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \; CT^{-1} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}. \end{aligned}$$

5. Reduced system

$$\dot{x}_r = A_{11} x_r + B_1 u, \; y = C_1 x_r.$$

# Analysis of balanced truncation

- Very good approximation properties.
- Exact error estimates.

$$\|G - G_r\|_{H_\infty} = 2(\sigma_{n_r+1} + \ldots + \sigma_n).$$

- Stability is preserved. Passivity with modification.
- Energy interpretation.
- Not feasible for general large sparse problems from semi-discretized PDEs.
- Expensive to solve large scale Lyapunov equations.
- However, often the Lyapunov solution has fast decaying eigenvalues.
- Very good large scale, parallel methods for this case, ADI, or Krylov methods. Benner, Li, Penzl, Saak, ...

Iterative rational Krylov algorithm (IRKA) Antoulas, Beattie, Gugercin.

For a transfer function $G(s) = C(sI - A)^{-1}B$ and a prescribed reduced system order $r$, find a local minimizer $G_r$ for the $H_2$ model reduction problem

$$||G - G_r||_{H_2} = \min_{\dim(\tilde{G})=r} ||G - \tilde{G}||_{H_2}$$

where

$$||G(s)||_{H_2} = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} ||G(s)||_F^2 ds \right)^{1/2}.$$

$H_2$ optimal tangential interpolation.

1. Make an initial selection of shifts $\sigma_i$ that is closed under conjugation and initial tangent directions $b_1, \ldots, b_r$, $c_1, \ldots, c_r$ and a convergence tolerance *tol*.
2. Choose $\tilde{V}_r = [(\sigma_1 I - A)^{-1} Bb_1, \ldots, (\sigma_r I - A)^{-1} Bb_r]$, $\tilde{W}_r = [(\sigma_1 I - A^T)^{-1} Cc_1, \ldots, (\sigma_r I - A^T)^{-1} Cc_r]$ and biorthogonalize so that $W_r^T V_r = I$.
3. while (relative change in $\{\sigma_i\} > tol$)
   (a) $A_r = W_r^T A V_r$, $B_r = W_r^T B$, $B_r = C V_r$.
   (b) Diagonalize $Y^* A_r X = \mathrm{diag}(\lambda_i)$.
   (c) Assign $\sigma_i \leftarrow -\lambda_i$, $b_i^* \leftarrow e_i^T Y^* B_r$, $c_i \leftarrow C_r X e_i$ for $i = 1, \ldots, r$.
   (d) Choose $\tilde{V}_r = [(\sigma_1 I - A)^{-1} Bb_1, \ldots, (\sigma_r I - A)^{-1} Bb_r]$, $\tilde{W}_r = [(\sigma_1 I - A^T)^{-1} Cc_1, \ldots, (\sigma_r I - A^T)^{-1} Cc_r]$ and bi-orthogonalize so that $W_r^T V_r = I$)]

# Analysis of IRKA

- Optimal approximation in $H_2$
- Error estimates
- Convergence analysis. Beattie/Gugercin
- Trust region descent method Beattie/Gugercin.
- Feasible for large sparse problems from semi-discretized PDEs, as long as Krylov subspace methods picks up right space.
- System has to be stable, new ideas for unstable case Gugercin.

# Outline

# Proper Orthogonal Decomposition (POD)

$$
\begin{aligned}
F(t, x, \dot{x}, u) &= 0, \quad x(t_0) = x^0 \\
y(t) &= g(x)
\end{aligned}
$$

- Consider snapshots for some control $u$, i.e. determine

$$\mathcal{X} = \left[\; x(t_1) \quad x(t_2) \quad \dots \quad x(t_N) \;\right]$$

- SVD $\mathcal{X} = U_N \Sigma_N V_N^T \approx U_{n_r} \Sigma_{n_r} V_{n_r}^T$ with $\Sigma = \mathrm{diag}(\sigma_1, \dots, \sigma_N)$
- Truncate small singular values $\sigma_i$, $i = n_r, \dots, N$, $n_r << n$
- Reduced system

$$F_r(t, U_{n_r} x_r, U_{n_r} \dot{x}_r, u) = U_{n_r}^T F(t, U_{n_r} x_r, U_{n_r} \dot{x}_r, u) = 0.$$

- Requires evaluation of $F$ still on big vector.
- To reduce work, discrete empirical interpolation Chaturantabut, Maday, Sorensen.

DEIM: Consider system with reduced state:

$$F_r(\tau) := U_{n_r}^T F(t, U_{n_r} x_r, U_{n_r} \dot{x}_r, u) = 0.$$

and interpolate

$$F_r(\tau) \approx W z_r(\tau)$$

with a fixed small $m$-dimensional basis $W$.
To determine $z_r(\tau)$ select $m$ rows (by a selection matrix $P^T$) from $F_r(\tau) = WZ_r(\tau)$ such that $P^T W$ is invertible and well conditioned and approximate

$$\tilde{F}_r(t, x_r, \dot{x}_r, u) = W(P^T W)^{-1} P^T F_r(t, P^T U_{n_r} x_r, P^T U_{n_r} \dot{x}_r, u).$$

Can be combined with off-line computation.

# Analysis of POD

- Cheap and easy to use.
- ‘Works’ for nonlinear systems.
- Successful in practice.
- How to choose $u(t)$ for snapshots?
- Quite heuristic.
- DEIM necessary.
- A posteriori error estimates: Kunisch/Tröltzsch/Volkwein.
- Usually no preservation of physical properties.
- Does not work well for transport dominated problems.

# Outline

# Conclusion I

- Many aspects of model reduction;
- Linear constant coefficient case well understood;
- Approximation errors, fast methods, tunability of model quality;
- Nonlinear case essentially POD and reduced basis method.

Thank you very much
for your attention.

# Books/surveys


- A.C. Antoulas, D.C. Sorensen, and S. Gugercin. A survey of model reduction methods for large-scale systems. Structured Matrices in Operator Theory, Numerical Analysis, Control, Signal and Image Processing, Contemporary Mathematics, AMS publications, 280: 193-219, 2001.
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- U. Baur, P. Benner, L. Feng. Model order reduction for linear and nonlinear systems: a system-theoretic perspective, MPI Magdeburg, 2014
- P. Benner, V. Mehrmann, and D.C. Sorensen, (Eds). *Dimension Reduction of Large-Scale Systems*. LECTURE NOTES IN COMPUTATIONAL SCIENCE AND ENGINEERING Vol. 45, Springer Verlag, Heidelberg, 2005.

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- A. Quarteroni, A. Manzoni, F. Negri Reduced Basis Method for Partial Differential Equations Springer, Unitext Series, vol. 92, 2015.
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