Model reduction: analysis, numerical solution and real world applications Lecture II: Preservation of physical properties in model reduction methods

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Outline

Model based approach

Physical system

↙ ↘

Modeling **Modeling**

↓ ↓

ODE/DAE ⟵ *semidiscr.* **PDE**

↓

Mod. reduction

↓

Reduced ODE/DAE ⟶ *Sim., Control*

Replace system

$$\begin{aligned} F(t, x, \dot{x}, u) &= 0, \quad x(t_0) = x^0 \\ y(t) &= g(x) \end{aligned}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$, by a reduced model

$$\begin{aligned} F_r(t, x_r, \dot{x}_r, u) &= 0, \quad x_r(t_0) = x_r^0 \\ y_r(t) &= g_r(x_r) \end{aligned}$$

with $x_r \in \mathbb{R}^{n_r}$, $n_r << n$.

Ultimate Goals

- Small approximation error in input/output map;
- Small approximation error in outputs;
- Error bounds, estimates;
- preservation of physical properties such as stability, passivity, conservation of constraints (energy, impulse, ...);
- Cheap method to produce reduced order model;
- Integration of model reduction in multi-physics modeling, control and optimization;
- ...

Open loop vs. closed loop control

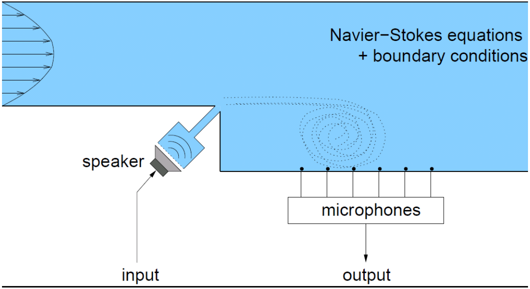
Open loop control:

- In open loop control we consider the problem as an optimization problem.
- We either discretize everything and then solve a large scale optimization problem or we derive optimality conditions for the infinite dimensional problem and then discretize the necessary optimality conditions.
- This is great because one can use e.g. gradient information from codes, but it is not good for fast nonlinear dynamics

Closed loop control:

- In closed loop control we derive feedback solutions $u = g(x)$.
- This works very well if one can get a reduced model that captures the dynamics and can be implemented for real time control.

Project in SFB 557 Control of complex shear flows, with F. Tröltzsch, M. Schmidt



Control of Stokes/Navier-Stokes

$$\begin{aligned} \frac{\partial v}{\partial t} &= \nabla(K(\nabla v)) + \nabla p + Bu(t), \\ 0 &= \text{div } v. \end{aligned}$$

Formally looking at semi-discretization in space gives nonlinear descriptor system

$$\begin{aligned} \frac{dv_h}{dt} &= \nabla_h(K_h \nabla_h v_h(t)) + \nabla_h p_h(t) + B_h u(t), \\ 0 &= \text{div}_h v_h(t), \end{aligned}$$

where v_h is the semi-discretized vector of velocities and p_h is the semi-discretized vector of pressures.
Linearization and robust H_∞ control to take care of nonlinearity.

- Preservation of constraints $0 = \mathrm{div}\ v$, $0 = \mathrm{div}_h v_h(t)$.
- Transfer function?
- Error measures?
- Feedback or optimization?

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \quad x(t_0) = x_0 \\ y(t) &= Cx(t) \end{aligned}$$

Replace by

$$\begin{aligned} E_r\dot{x}_r(t) &= A_r x_r(t) + B_r u(t), \quad x(t_0) = x_r^0 \\ y_r(t) &= C_r x_r(t), \end{aligned}$$

If E is singular (but $sE - A$ regular), then

$$G(s) = C(sE - A)^{-1}B = G_p(s) + P(s),$$

where $G_p(s)$ is the proper rational part and $P(s)$ is the polynomial part, associated with the singular part of E. (H_∞ norm not defined.)

Gramians for descriptor systems

Stykel, Diss. '02 Let P_l, P_r be left, right spectral projectors onto deflating subspace of $\lambda E - A$ to finite eigenvalues.

- $EX_{pc}A^T + AX_{pc}E^T = -P_lBB^TP_l^T, \quad X_{pc} = P_rX_{pc}$
  proper controllability Gramian.
- $E^TX_{po}A + A^TX_{po}E = -P_r^TC^TCP_r \quad X_{pc} = X_{pc}P_l$
  proper observability Gramian.
- $AX_{ic}A^T - EX_{ic}E^T = (I - P_l)BB^T(I - P_l)^T, \quad P_rX_{ic} = 0$
  improper controllability Gramian.
- $A^TX_{io}A - E^TX_{io}E = (I - P_r)^TC^TC(I - P_r) \quad X_{pc}P_l = 0$
  improper observability Gramian.

Proper Hankel singular values:
$\xi_j = \sqrt{\lambda_j(X_{pc}E^TX_{po}E)}, j = 1, \ldots, n_f$.
Improper Hankel singular values:
$\theta_j = \sqrt{\lambda_j(X_{ic}A^TX_{io}A)}, j = 1, \ldots, n_\infty$.

- Compute (low rank) Cholesky factors

$$X_{pc} = R_p R_p^T,\ X_{po} = L_p^T L_p,\ X_{ic} = R_i R_i^T,\ X_{io} = L_i^T L_i$$

- Form singular value decompositions

$$L_p E R_p = \begin{bmatrix} U_0 & U_1 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_0 \end{bmatrix} \begin{bmatrix} V_0 & V_1 \end{bmatrix}^T$$

with $\Sigma_1 = \mathrm{diag}(\xi_1, \ldots, \xi_{\tilde{n}_f})$, $\tilde{n}_f << n_f$ and

$$L_i E R_i = \begin{bmatrix} U_2 & U_3 \end{bmatrix} \begin{bmatrix} \Theta_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_2 & V_3 \end{bmatrix}^T$$

with $\Theta_1 = \mathrm{diag}(\theta_1, \ldots, \theta_{\tilde{n}_\infty})$ invertible.

- $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}) = (W_l^T E T_l, W_l^T A T_l, W_l^T B, C T_l)$, where
  $W_l = [L_p^T U_1 \Sigma_1^{-1/2}, L_i^T U_2 \Theta_1^{-1/2}]$, $T_l = [R_p^T V_1 \Sigma_1^{-1/2}, R_i^T V_2 \Theta_1^{-1/2}]$

- Balancing for dynamic and algebraic part.
- Reduction for dynamic and deflation of nullspace for algebraic part.
- Good approximation properties.
- Exact error estimates if polynomial part (constraint) is exactly preserved $P(s) = P_r(s)$:

$$\|G - G_r\|_{H_\infty} = \|G_p - G_{r,p}\|_{H_\infty} 2(\xi_{\tilde{n}_f+1} + \ldots + \xi_{n_f}).$$

- Stability is preserved. Passivity with modification.
- Low Rank methods for large scale generalized Lyapunov equations Stykel '04.

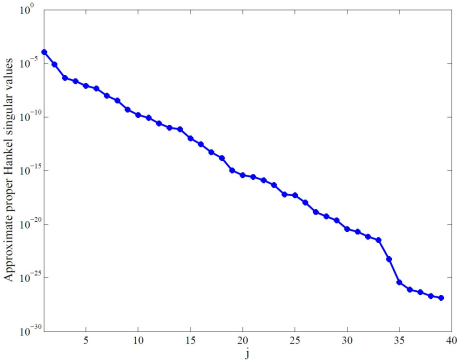
Discretization with FEM.

$$
\begin{aligned}
E &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} \Delta_h & \nabla_h \\ \mathrm{div}_h & 0 \end{bmatrix}, \\
B &= \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} 0 & C_2 \end{bmatrix}, \\
P_r &= P_l^T = \begin{bmatrix} \Pi & 0 \\ -(\nabla_h^T \mathrm{div}_h)^{-1} \nabla_h^T \Delta_h \Pi & 0 \end{bmatrix}, \\
\Pi &= I - \mathrm{div}_h (\nabla_h^T \mathrm{div}\ _h)^{-1} \nabla_h^T
\end{aligned}
$$

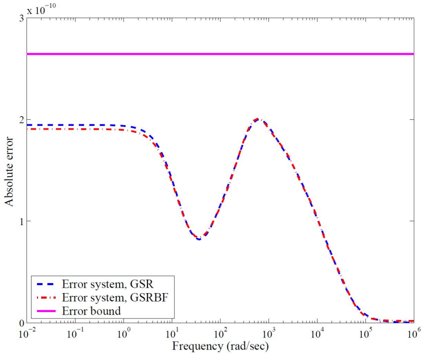
- We need only solutions with discrete Laplace Δ_h.
- Projectors P_l, P_r are easy to get.
- Reduced models are ‘discretizations’ of Stokes.
- Recent work avoids projection Heinkenschloss Sorensen

Semidiscretized model with $n = 19520$, $n_f = 6400$ and $n_\infty = 13120$.
Approximation with $\tilde{n} = 11$, $\tilde{n}_f = 10$, $\tilde{n}_\infty = 1$.

Approximate proper Hankel singular values for the semidiscretized Stokes equation.



Absolute error plots and error bound for the semi-discretized Stokes equation.



Limits of the approach

- Experiments are costly or not feasible
- Simulators are typically for the forward problem, they usually use very fine grids.
- Commercial codes cannot always be used well.
- Adaptive methods adapt for the error in the forward simulation.
- Space discretization leads to a very large dynamical system.
- Model reduction is expensive.
- Preservation of physical properties is difficult.

Outline

Common scheme

- Multibody dynamics
- quantum mechanics
- Electrical circuit simulation
- Optimality systems in optimal control of ODEs/DAEs/PDEs
- fluid dynamics
- ...

Variational principle, Hamiltonian like system with dissipation, ...
Survey Van der Schaft 2013.

$$\dot{\mathbf{x}} = (\mathbf{J} - \mathbf{R})\,\nabla_{\mathbf{x}} H(\mathbf{x}) + (\mathbf{B} - \mathbf{P})\mathbf{u}(t),$$
$$\mathbf{y}(t) = (\mathbf{B} + \mathbf{P})^T \nabla_{\mathbf{x}} H(\mathbf{x}) + (\mathbf{S} + \mathbf{N})\mathbf{u}(t).$$

- $H : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ continuously differentiable - the Hamiltonian,
- $\mathbf{J} = -\mathbf{J}^T \in \mathbb{R}^{n \times n}$ is the structure matrix describing the interconnection of energy storage elements in the system;
- $\mathbf{R} = \mathbf{R}^T$ is the $n \times n$ dissipation matrix describing energy dissipation/loss in the system,
- $\mathbf{B} \pm \mathbf{P} \in \mathbb{R}^{n \times m}$ are the port matrices, describing how energy enters and exits the system, and
- $\mathbf{S} + \mathbf{N}$ with $\mathbf{S} = \mathbf{S}^T, \mathbf{N} = -\mathbf{N}^T \in \mathbb{R}^{m,m}$ feed-through term.

Further properties

- Typically the matrix

$$\mathbf{K} = \begin{bmatrix} \mathbf{R} & \mathbf{P} \\ \mathbf{P}^T & \mathbf{S} \end{bmatrix}$$

  is symmetric positive-semidefinite;
- Port-Hamiltonian systems are stable and passive.
- The connection of port-Hamiltonian systems is again port-Hamiltonian.

Conservation of energy

- Standard port Hamiltonian systems generalize the classical notion of Hamiltonian systems, in our notation $\dot{\mathbf{x}} = \mathbf{J}\nabla_{\mathbf{x}} H(\mathbf{x})$,
- The analog of the conservation of energy for standard Hamiltonian systems takes the form of a dissipation inequality

$$H(\mathbf{x}(t_1)) - H(\mathbf{x}(t_0)) \leq \int_{t_0}^{t_1} \mathbf{y}(t)^T \mathbf{u}(t)\, dt,$$

  which means that the change in internal energy of the system H, is bounded by the total work done on the system.
- The dissipation inequality holds also if **J**, **R**, **B**, **P**, **M** and **D** depend on $\mathbf{x}$ or explicitly on time, t.

MOR for port-Hamiltonian systems

Beattie/Gugercin/Polyuga/van der Schaft 2009

Goal: Reduce state space dimension without degrading input-output response;

- Keep advantageous system features (port-Hamiltanion structure.)
- Maintain high fidelity and physical consistency (structure).
- Error estimates.

Determine subspaces V_r and W_r so that $x(t) \approx V_r x_r(t)$ and $\nabla_x H(x(t)) \approx W_r h_r(t)$ which implies

$$V_r^T W_r h_r(t) \approx V^T \nabla_x H(V_r x(t)) = \nabla_{x_r} H_r(x_r(t))$$

with reduced energy:

$$H_r(x_r(t)) = H(V_r x(t)).$$

So, if biorthogonal bases for V_r and W_r are chosen ($V_r^T W_r = I$) then

$$h_r(t) = \nabla_{x_r} H_r(x_r(t))$$

and port-Hamiltonian structure is preserved.

1. Generate trajectory $x(t)$, and snapshot matrix:

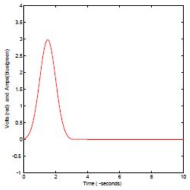
$$\mathcal{X} = [x(t_0), x(t_1), x(t_2), ..., x(t_N)].$$

2. Truncate SVD of snapshot matrix, to get POD basis, $\tilde{V}_r$, for the state variables. Then approximate $x(t) \approx \tilde{V}_r \tilde{x}_r(t)$.
3. Collect associated force snapshots:

$$\mathcal{F} = [\nabla_x H(x(t_0)), \nabla_x H(x(t_1)), \ldots, \nabla_x H(x(t_N))]$$

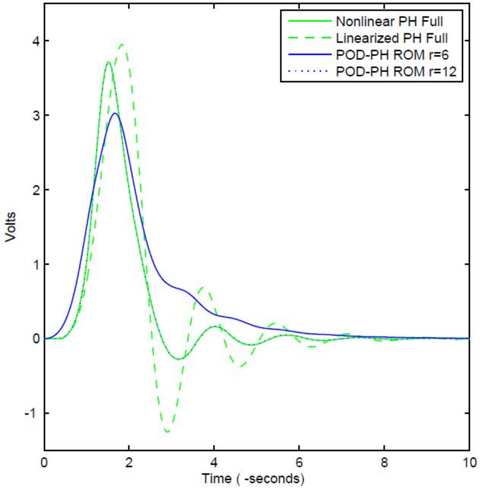
4. Truncate SVD of $\mathcal{F}$ to get a second POD basis, $\tilde{W}_r$.
5. Change to bi-orthogonal bases W_r and V_r such that $W_r^T V_r = I$.

Example of Beattie et al MOR for nonlinear ladder network.



Input:
Gaussian pulse (3V pk)

ROM w/order r=12
accurate to 3.e-3



- Energy based modeling leads to port-Hamiltonian structure.
- POD for port-Hamiltonian structure easy.
- Stability, passivity, structure preserved.
- Balanced truncation, IRKA etc can be done analogously.
- But still one first discretizes, then reduces.

Outline

Heiland/M./Schmidt 2012

- Input space $\mathcal{U}$, Output space $\mathcal{Y}$, State space $\mathcal{Z}$.
- System governed by instationary linear or nonlinear PDE

$$\begin{aligned} \partial_t z(t) &= Az + Bu(t), \quad \text{in } \Omega \times [0, T], \\ z(0) &= z^0 + \text{boundary conditions}, \\ y(t) &= Cz(t), \end{aligned}$$

with operators

$$\begin{aligned} &B \in \mathcal{L}(\mathcal{U}, \mathcal{Z}), \ C \in \mathcal{L}(\mathcal{Z}, \mathcal{Y}) \\ &A = \mathcal{Z} \to \mathcal{Z}, \end{aligned}$$

$$u \in \mathcal{U} = L^2([0, T], U), \ y \in \mathcal{Y} = L^2([0, T], Y)$$

and Hilbert spaces for the spacial dependence U, Z, Y.
- System maps inputs u to outputs y.

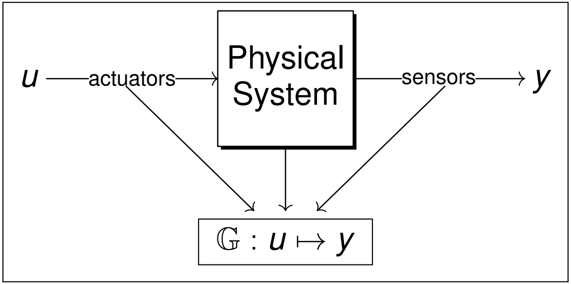


Figure: I/O map for a physical system, mathematical model.

Classical frequency domain approach usually cannot be applied.

Discretization of I/O maps

Suppose we have a solution formula.

$$y(t) = (\mathbb{G}u)(t) = \int_0^T CS(t-s)Bu(s)\,ds$$

with kernel

$$K(t-s) = CS(t-s)B \in \mathcal{L}(U, Y)$$

Here S is the solution operator for the PDE.
Approximate $\mathbb{G} \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ in two steps.

Two step procedure

1. Approximation of input-output signals, by restricting to finite dimensional subspaces in U, Y.
2. Approximation of the dynamics/kernel

$$K(t) \approx \tilde{K}_{\text{tol}}$$

by approximate solution of PDEs for a basis in input space to the needed tolerance.

Discretization of I/O signals

$$u \in \mathcal{U}, \; y \in \mathcal{Y}$$

Finite dimensional subspaces, tensor product approach.

$$\begin{aligned}
U_{h_1} &= \mathrm{span}(\mu_1(\Theta), \ldots, \mu_p(\Theta)) \subset U \\
\mathcal{R}_{\tau_1} &= \mathrm{span}(\phi_1(t), \ldots, \phi_r(t)) \\
\mathcal{U}_{h_1,\tau_1} &= \mathrm{span}(\mu_i(\Theta)\phi_j(t)) \\
Y_{h_2} &= \mathrm{span}(\nu_1(\xi), \ldots, \nu_p(\xi)) \subset Y \\
\mathcal{Y}_{h_2,\tau_2} &= \mathrm{span}(\nu_i(\xi)\psi_j(t))
\end{aligned}$$

leads to approximation

$$\mathbb{G}_S(h_1, \tau_1, h_2, \tau_2) = \mathbb{P}_{\mathcal{Y},h_2,\tau_2}\mathbb{G}\mathbb{P}_{\mathcal{U},h_2,\tau_2}.$$

Approximation error when $\mathbb{G}$ is approximated by $\mathbb{G}_{DS}$

1.

$$u \in \mathcal{U} \to \mathbb{P}_{\mathcal{U},h_1,\tau_1} u \in \mathcal{U}_{h_1,\tau_1} \to \mathbf{u} \in \mathbb{R}^{pr}$$

$$y \in \mathcal{Y} \to \mathbb{P}_{\mathcal{Y},h_2,\tau_2} y \in \mathcal{Y}_{h_2,\tau_2} \to \mathbf{y} \in \mathbb{R}^{ms}$$

2.

$$K(t) \approx \tilde{K}_{\text{tol}}$$

via discretization of PDE solution operator

Global approximation error:

$$\|\mathbb{G} - \mathbb{G}_{DS}(h, \text{tol})\| \leq e_S(h_1, h_2, \tau_1, \tau_2) + e_D(\text{tol}).$$

The two errors can be balanced.

Application to PDE const. optimization

Determine (open loop) control u that give outputs y such that

$$\begin{aligned} &\mathcal{J}(y,u) \rightarrow \min \\ &\text{subject to } F(y,u) = 0, \end{aligned}$$

where F is a PDE constraint.
If the system is represented by an I/O operator $\mathbb{G}$ via $y = \mathbb{G}u$, then we can turn the problem into an unconstrained optimization problem.
Determine controls u such that

$$\mathcal{J}(\mathbb{G}u, u) \rightarrow \min .$$

Discretization of signals

Matrix representation of $\mathbb{G}_S = \mathbb{P}_\mathcal{Y}\mathbb{G}\mathbb{P}_\mathcal{U}$. Setting

$$\mathbb{P}_\mathcal{U}(u)(t;\Theta) = \sum_{j=1}^{r}\sum_{\ell=1}^{p} \mathbf{u}_j^\ell \phi_j(t)\mu_\ell(\Theta),$$

$$\mathbb{G}_S(u)(t;\xi) = \sum_{i=1}^{s}\sum_{k=1}^{q} \mathbf{y}_i^k \psi_i(t)\nu_j(\xi),$$

and testing against $(\psi_m\nu_n)$ we obtain

$$\sum_{i=1}^{s}\sum_{k=1}^{q} \mathbf{y}_i^k (\psi_i\nu_k, \psi_m\nu_n)_{(0,T)\times Y}\mathbb{G}_S(u)(t;\xi)$$

$$= \sum_{i=1}^{r}\sum_{\ell=1}^{p} \mathbf{u}_j^\ell (\mathbb{G}(\phi_j\mu_\ell, \psi_m\nu_n)_{(0,T)\times Y}.$$

The weak formulation can be written as

$$\mathbf{M}_{\mathcal{Y}} y = \mathbf{H}\mathbf{u}, \;\; \mathbf{G}_h = \mathbf{M}_{\mathcal{Y}}^{-1}\mathbf{H},$$

with block-structured matrices

$$\mathbf{M}_{\mathcal{Y}} = \mathbf{M}_Y \otimes \mathbf{M}_{\mathcal{S}}, \;\; \mathbf{H}_{ij}^{kl} = \big(\nu_k \psi_i, \mathbb{G}(\mu_\ell \phi_j)_{(0,T)\times Y}\big).$$

$$
\begin{aligned}
\mathbf{H}_{ij}^{kl} &= (\nu_k \psi_i, \mathbb{G}(\mu_\ell \phi_j))_{(0,T)\times Y} \\
&= \int_0^T (\nu_k \psi_i(t), \int_0^T K(t-s)\phi_j(s)\mu_\ell \, ds)_Y \, dt \\
&= \int_0^T \int_0^T \psi_i(t)\phi_j(s)(\nu_k, K(t-s)\mu_\ell \, ds \, dt \\
&= \int_0^T \mathbf{W}_{ij}(t) K_{kl}(t) \, dt
\end{aligned}
$$

$$\mathbf{H} = \mathbf{M}_{\mathcal{Y}}\mathbf{G}_h = \int_0^T \mathbf{K}(t) \otimes \mathbf{W}(t)\, dt$$

with matrix valued functions

$$W_{ij}(t) = \int_0^{T-t} \psi_i(t+s)\phi_j(s)\, ds$$

that we can calculate exactly and

$$\mathbf{K}_{kl}(t) = (\nu_k, K(t)\mu_\ell)_Y$$

which we have to approximate numerically.

The operator norm

$$\|\mathbb{G}\|_{\mathcal{L}(\mathcal{U},\mathcal{Y})}$$

can be computed via weighted norm of matrix representation

$$\begin{aligned}\|\mathbb{G}\|_{\mathcal{L}(\mathcal{U},\mathcal{Y})} &= \|\mathbf{G}_h\|_{\mathbf{h}} := \sup_{\mathbf{u}\in\mathbb{R}^{pr}} \frac{\|\mathbf{G}_h\mathbf{u}\|_{qs;w}}{\|\mathbf{u}\|_{pr;w}} \\ &= \|\mathbf{M}_{\mathcal{Y},h_2,\tau_2}\mathbf{G}_h\mathbf{M}_{\mathcal{U},h_1,\tau_1}\|_{qs,pr}\end{aligned}$$

Observations

- $K(t)$ can be calculated column-wise,
- Parallelization is easy.
- No storage of state trajectories is necessary.
- Accuracy only needed in the observations of excited states not in full state.
- One can easily deal with non-smooth initial transients.
- Approximate error estimation is possible, e.g. via *Dual-Weighted Residuals*

System dynamics error

Lemma

The system dynamics error $\epsilon_D = \|\mathbb{G}_S - \mathbb{G}_{DS}\|_{\mathcal{L}(\mathcal{U},\mathcal{Y})}$ *satisfies*

$$\epsilon_D \leq \sqrt{T}\|\mathbf{K} - \tilde{\mathbf{K}}\|_{L^2([0,T],\mathbb{R}^{p,q})}$$

Error in the observations may be very small, even if state error is large.
Dual weighted residual method to control the output error, if one can solve the adjoint error equation.

Further reduction

- We can further reduce the I/O dimension by computing the singular value decomposition (SVD) of the transfer matrix.
- Delete the inputs/outputs associated to negligible singular values.
- SVD approximation error can be incorporated in error bounds.
- Can be interpreted as POD Baumann/Heiland/Schmidt 2015.

Extension to flow control

- The techniques can be applied to linear flow systems Stokes, Oseen, linearized Navier-Stokes.
- The error bounds are more difficult, but partially available.
- We need a semigroup representation.
- For Navier-Stokes the theory is open, the methods work well for moderate Reynolds numbers.

Linearized Navier-Stokes

Linearization of Navier-Stokes along a reference velocity V_∞

$$
\begin{aligned}
V_t + (V_\infty \cdot \nabla) V + (V \cdot \nabla) V_\infty + \nabla P - \frac{1}{Re} \triangle V &= \\
(V_\infty \cdot \nabla) V_\infty + f + \mathcal{B} u, & \\
\nabla \cdot V &= 0, \\
y &= \mathcal{C} V.
\end{aligned}
$$

together with appropriate initial and boundary conditions.

This linear model, together with discrete input and output spaces, enables the construction of a finite dimensional discrete linear I/O-operator.

Space discretized linearized Navier Stokes equation.

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} v(t) \\ p(t) \end{bmatrix} + \begin{bmatrix} D & -J^T \\ J & Q \end{bmatrix} \begin{bmatrix} v(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}, \quad \text{for } t \in (0, T],$$
$$v(0) = v_0 \in \mathbb{R}^{n_v}.$$

Set

$$\hat{\mathcal{E}} = \begin{bmatrix} E_{11} & 0 \\ E_{21} & 0 \end{bmatrix} := \begin{bmatrix} (I - D^{-1}J^T S^{-1} J)D^{-1}M & 0 \\ -S^{-1}JD^{-1}M & 0 \end{bmatrix}.$$

with the Schur complement $S := Q + JD^{-1}J^T$.

Explicit solution operator

Theorem (Emmrich/M. 2013)

$$
\begin{bmatrix} v(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} \exp(-E_{11}^D t) E_{11}^D E_{11} q_v \\ E_{21} \exp(-E_{11}^D t)(E_{11}^D)^2 E_{11} q_v \end{bmatrix}
+ \int_0^t \begin{bmatrix} \exp(-E_{11}^D (t-s)) E_{11}^D \hat{f}_1(s) \\ E_{21} \exp(-E_{11}^D (t-s))(E_{11}^D)^2 \hat{f}_1(s) \end{bmatrix} ds
+ \begin{bmatrix} [I - E_{11}^D E_{11}] \hat{f}_1(t) \\ -E_{21} E_{11}^D \hat{f}_1(t) + \hat{f}_2(t) \end{bmatrix} + \begin{bmatrix} [E_{11} - E_{11}^D E_{11}^2] \hat{f}_1^{(1)}(t) \\ [E_{21} - E_{21} E_{11}^D E_{11}] \hat{f}_1^{(1)}(t) \end{bmatrix},
$$

assuming that the vector q_v belongs to a given consistent initial value v_0. Here ^{D} is the Drazin inverse.

Explicit I/O map

$$
\begin{aligned}
y(t) = \\
& C\Big\{ \begin{bmatrix} \exp(-E_{11}^D t) E_{11}^D E_{11} q_v \\ E_{21}\exp(-E_{11}^D t)(E_{11}^D)^2 E_{11} q_v \end{bmatrix} + \int_0^t \begin{bmatrix} \exp(-E_{11}^D (t-s)) E_{11}^D \hat{f}_1(s) \\ E_{21}\exp(-E_{11}^D (t-s))(E_{11}^D)^2 \hat{f}_1(s) \end{bmatrix} \mathrm{d}s \\
& + \begin{bmatrix} [E_{11} - E_{11}^D E_{11}^2][M^{-1} f_1(t) + [I - E_{11}E_{11}^D] R_0 f_2(t)] \\ E_{21}[I - E_{11}^D E_{11}] M^{-1} f_1(t) + [S^{-1} - E_{21}E_{11}^D R_0] f_2(t)] \end{bmatrix} \\
& + \begin{bmatrix} [E_{11} - E_{11}^D E_{11}^2] R_0 \dot{f}_2(t) \\ E_{21}[E_{11} - E_{11}^D E_{11}^2] M^{-1} \dot{f}_1(t) + [E_{21} - E_{21}E_{11}^D E_{11}] R_0 \dot{f}_2(t) \end{bmatrix} \Big\} := y_0 \\
& + C\Big\{ \int_0^t \begin{bmatrix} \exp(-E_{11}^D (t-s)) E_{11}^D E_{11} M^{-1} B_1 u(s) \\ E_{21}\exp(-E_{11}^D (t-s))(E_{11}^D)^2 E_{11} M^{-1} B_1 u(s) \end{bmatrix} \mathrm{d}s + \\
& + \begin{bmatrix} [E_{11} - E_{11}^D E_{11}^2] M^{-1} B_1 u(t) \\ E_{21}[I - E_{11}^D E_{11}] M^{-1} B_1 u(t) \end{bmatrix} + \begin{bmatrix} 0 \\ E_{21}[E_{11} - E_{11}^D E_{11}^2] M^{-1} (B_1 u)^{(1)}(t) \end{bmatrix} \Big\} := Gu(t).
\end{aligned}
$$

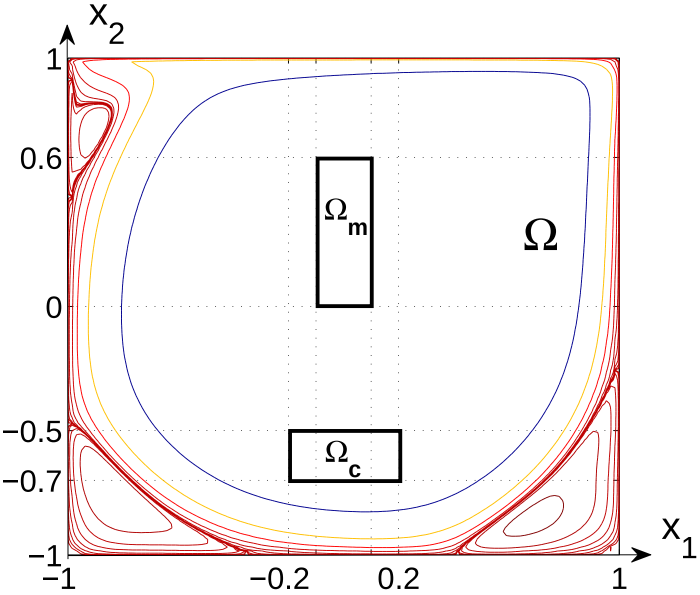
The linear I/O map is defined via $G : \mathcal{U} \to \mathcal{Y},\ u \mapsto Gu$, by subtracting the vector y_0.

To obtain a well-defined I/O map, one needs

- $B_1 u(0)$ has to be consistent with the initial condition v_0,
- the function $B_1 u : [0, T] \to \mathbb{R}^{n_v}$ has to be sufficiently smooth.
- $\mathcal{U} \subset \mathcal{C}^1([0, T], U)$ in the case that the nilpotency index of $E_{11} = 2$ or
- $\mathcal{U} \subset \mathcal{C}([0, T], U)$ if the nilpotency index $E_{11} = 1$ or if only the velocity is considered for the output .

In both cases the output space $\mathcal{Y}$ is a subspace of $\mathcal{C}([0, T], Y)$.

Outline


x2
1
0.6
0
−0.5
−0.7
−1
−1
−0.2
0.2
1
x1
Ωm
Ωc
Ω

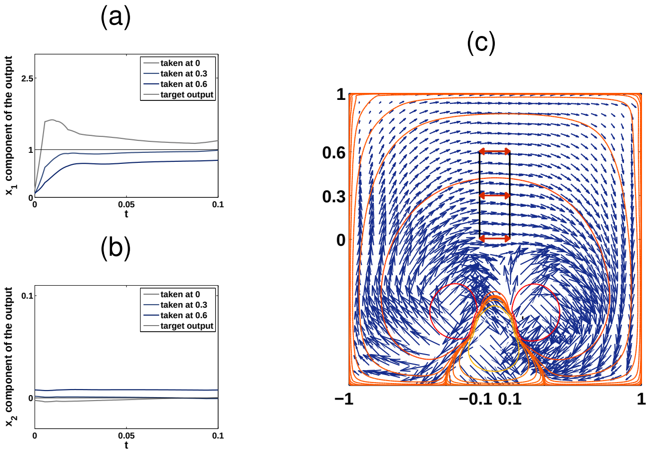


Figure: System response for input $\bar{u}$ computed with IFISS to match output $y^* = [1\ 0]^T$. (a) and (b) show time evolution. Plot (c) shows velocities and streamlines at $t = 0.1$.

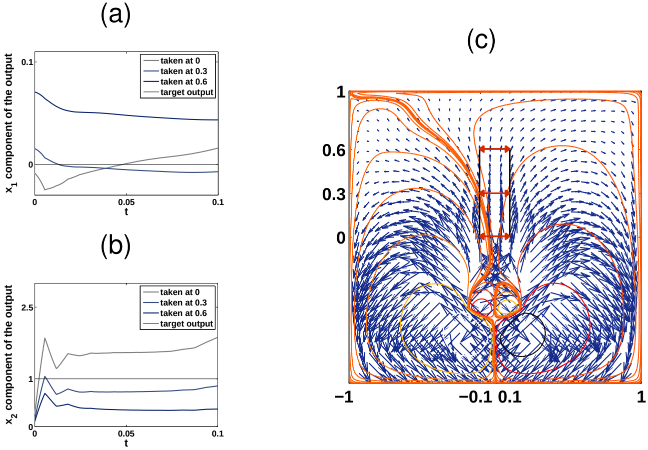
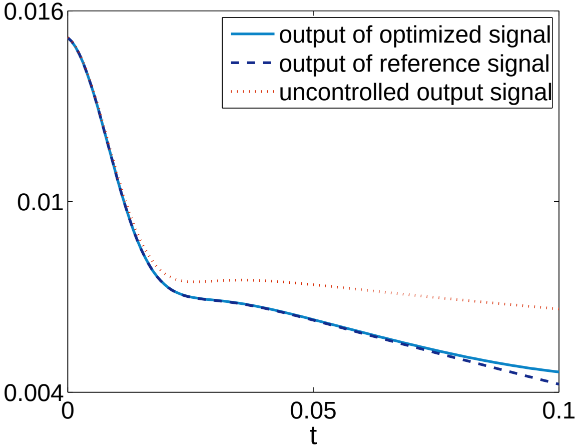
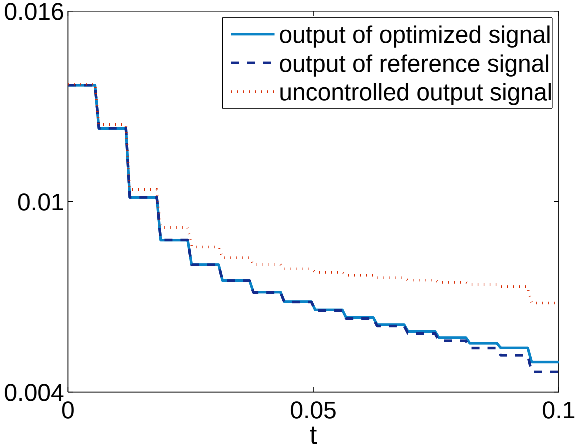


Figure: System response for input $\bar{u}$ computed with IFISS to match output $y^* = [0\ 1]^T$. (a) and (b) show time evolution of the output signal. Plot (c) shows velocities and streamlines at $t = 0.1$.

Optimal control, Smooth





0.016
0.01
0.004
0
0.05
0.1
t
output of optimized signal
output of reference signal
uncontrolled output signal

- Direct discretization of I/O map possible if transfer map is available/computable.
- Allows open loop, optimal control, optimization.
- Adaptivity in input and output space is possible and can be balanced with error in transfer operator.

Outline

Conclusion II

- Incorporation of constraints and structure into model reduction techniques;
- Port Hamiltonian structure is easy to preserve;
- Direct discretization of transfer function I/O map.

Thank you very much
for your attention.

References

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