Incompressible limit of planar compressible ideal magnetohydrodynamic equations

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Outline

As is well-known, one can derive formally incompressible models such as the incompressible Navier-Stokes equations from the compressible ones when the Mach number is small.

Aim of this talk: **justify rigorously the low Mach number limit (singular limit) for the planar compressible ideal magnetohydrodynamic equations.**

• Outline:

1. Formulation of the problem

2. Results on the MHD equations

3. Planar ideal MHD flows, general data

1. Formulation of the problem

Magnetohydrodynamics (MHD) studies the dynamics of compressible quasi-neutrally ionized fluids under the influence of electromagnetic fields.

The applications of MHD cover a very wide range of physical objects: liquid metals, astrophysics, geophysics, plasma physics, etc.

The three-dimensional full MHD equations for ideal gases ($P = R\rho\theta$, $e = c_v\theta$, setting $R = c_v = 1$ for simplicity), after a suitable scaling, can be written in the following form of the Mach number ϵ:

$$
\begin{cases}
\rho_t + \mathrm{div}(\rho u) = 0, \\
\rho(u_t + u \cdot \nabla u) + \dfrac{\nabla(\rho\theta)}{\epsilon^2} = (\nabla \times H) \times H + \mathrm{div}\Psi, \\
\rho(\theta_t + u \cdot \nabla\theta) + \rho\theta \mathrm{div} u \\
\qquad = \epsilon^2 \nu |\nabla \times H|^2 + \epsilon^2 \Psi : \nabla u + \kappa \Delta\theta, \\
H_t - \nabla \times (u \times H) = -\nabla \times (\nu \nabla \times H), \quad \mathrm{div} H = 0,
\end{cases} \tag{1}
$$

where ρ: Density, $u \in \mathbb{R}^3$: Velocity, $H \in \mathbb{R}^3$: Magnetic field, θ: Temperature, ϵ: Mach number, $\nu > 0$: Magnetic diffusivity,

$$\Psi = \mu(\nabla u + \nabla u^T) + \lambda\, \mathrm{div} u\, \mathrm{Id}: \quad \text{Viscous stress tensor,}$$

λ, μ: Viscosity coefficients, $\kappa > 0$: Heat conductivity,

Aim: to study limit as $\epsilon \to 0$.

Two approaches in the study of the limit as $\epsilon \to 0$:

1) Global weak solutions, global convergence in the weak sense;

2) Local smooth solutions, local convergence in the strong sense.

Here we shall employ the 2nd approach.

Main features for system (1):

- Highly oscillations (in time) due to acoustic waves
- Strong coupling of the fluid and magnetic fields
- Require uniform in ϵ local existence, uniform estimates of higher order derivatives

2. Results on MHD equations (non-exhaustive)

Isentropic case:

- **Klainerman-Majda '81: strong convergence for smooth local solutions with well-prepared initial data, by using theory of symmetric hyperbolic systems in delicate weighted norms and applying a fixed point argument.**

- **Wang-Hu '09: weak convergence for global weak solutions with general data by a weak convergence argument.**

- **J-Li-Ju '10: Weak solutions $\rightarrow$ strong solutions for general data, explicit convergence rate for well-prepared data by using modulated energy method and refined energy analysis.**

Full MHD equations

- **Feireisl, Novotný, Kukučka, Ruzicka, Thäter, Kwon, Trivisa '10-'11, $\cdots\cdots$: Convergence in the framework of variational solutions.**

- **Strong convergence for smooth solutions**

(i) small variations on ρ and θ, well-prepared data

Ansatz:

$$\rho = 1 + \epsilon q^\epsilon, \quad \theta = 1 + \epsilon \phi^\epsilon, \quad u = u^\epsilon, \quad H = H^\epsilon, \tag{2}$$

and rewrite system (1) in the form of $(q^\epsilon, \phi^\epsilon, u^\epsilon, H^\epsilon)$:

$$q_t^\epsilon + u^\epsilon \cdot \nabla q^\epsilon + \frac{1}{\epsilon}(1+\epsilon q^\epsilon)\mathrm{div} u^\epsilon = 0, \tag{3}$$

$$\begin{aligned}(1+\epsilon q^\epsilon)(u_t^\epsilon + u^\epsilon \cdot \nabla u^\epsilon) + \frac{1}{\epsilon}\left[(1+\epsilon q^\epsilon)\nabla\phi^\epsilon + (1+\epsilon\phi^\epsilon)\nabla q^\epsilon\right] \\ -H^\epsilon \cdot \nabla H^\epsilon + \frac{1}{2}\nabla(|H^\epsilon|^2) = 2\mu \mathrm{div}(\mathbb{D}(u^\epsilon)) + \lambda\nabla(\mathrm{tr}\mathbb{D}(u^\epsilon)),\end{aligned} \tag{4}$$

$$\begin{aligned}(1+\epsilon q^\epsilon)(\phi_t^\epsilon + u^\epsilon \cdot \nabla\phi^\epsilon) + \frac{1}{\epsilon}(1+\epsilon q^\epsilon)(1+\epsilon\phi^\epsilon)\mathrm{div} u^\epsilon \\ = \kappa\Delta\phi^\epsilon + \epsilon\left\{2\mu|\mathbb{D}(u^\epsilon)|^2 + \lambda(\mathrm{tr}\mathbb{D}(u^\epsilon))^2\right\} + \epsilon\nu|\nabla \times H^\epsilon|^2,\end{aligned} \tag{5}$$

$$H_t^\epsilon + u^\epsilon \cdot \nabla H^\epsilon + \mathrm{div} u^\epsilon H^\epsilon - H^\epsilon \cdot \nabla u^\epsilon = \nu\Delta H^\epsilon, \quad \mathrm{div} H^\epsilon = 0, \tag{6}$$

where $\mathbb{D}(u) := (\nabla u + \nabla u^T)/2$.

The **formal limit** of (3)–(6) as $\epsilon \to 0$, assuming $(u^\epsilon, H^\epsilon) \to (w, B)$, reads as (incompressible MHD):

$$\begin{cases} \partial_t w + w \cdot \nabla w + \nabla \pi + \frac{1}{2}\nabla(|B|^2) - B \cdot \nabla B = \mu \Delta w, \\ \partial_t B + w \cdot \nabla B - B \cdot \nabla w = \nu \Delta B, \\ \mathrm{div} w = 0, \qquad \mathrm{div} B = 0. \end{cases} \tag{7}$$

Consider systems (3)–(6) and (7) in the torus $\mathbb{T}^3$ or in the whole space $\mathbb{R}^3$ with initial data

$$(q^\epsilon, u^\epsilon, H^\epsilon, \phi^\epsilon)|_{t=0} = (q_0^\epsilon, u_0^\epsilon, H_0^\epsilon, \phi_0^\epsilon), \quad \text{and} \tag{8}$$

$$(w, B)|_{t=0} = (w_0, B_0), \quad \mathrm{div} w_0 = \mathrm{div} B_0 = 0, \tag{9}$$

respectively. One can prove

J-Ju-Li '11: *Let $s > 2 + \frac{3}{2}$, (w, B): a smooth solution to (7), (9) with $(w, B) \in C([0, T^*], H^{s+2}) \cap C^1([0, T^*], H^s)$, $T^* > 0$. Assume*

$$\|q_0^\epsilon, u_0^\epsilon - w_0, H_0^\epsilon - B_0, \phi_0^\epsilon\|_{H^s} = O(\epsilon).$$

Then $\exists\, \epsilon_0 > 0$, such that for $\epsilon \le \epsilon_0$, the system (3)–(6), (8) has a unique smooth solution $(q^\epsilon, u^\epsilon, H^\epsilon, \phi^\epsilon) \in C([0, T^*], H^s)$ satisfying

$$\sup_{t\in[0,T^*]} \left\| \left\{ (q^\epsilon, u^\epsilon, H^\epsilon, \phi^\epsilon) - \left(\frac{\epsilon}{2}\pi, w, B, \frac{\epsilon}{2}\pi \right) \right\}(t) \right\|_{H^s} \le K\epsilon.$$

Remarks: **i) From the above theorem, we see that the full MHD equations (3)–(6), (8) admits a unique smooth solution on the same time interval where a smooth solution of the incompressible MHD equations exists.**

ii) The approach is still valid for the ideal non-isentropic compressible MHD equations, e.g., $\mu, \lambda, \nu, \kappa = 0$**.**

Proof ideas: **The main ingredients in the proof:**

energy estimates for symmetrizable quasilinear hyperbolic-parabolic systems

compact arguments and a convergence-stability lemma due to Brenier-Yong □

(ii) Large variations on ρ and θ, general data

Consider the full MHD system (1) in the physical regime:

$$P \sim 1 + O(\epsilon), \quad u \sim O(\epsilon), \quad H \sim O(\epsilon). \quad \nabla\theta \sim O(1),$$

Introduce the transforms to ensure positivity of P and θ (Métivier-Schochet):

$$P(x,t) = e^{\epsilon p^\epsilon(x,\epsilon t)}, \quad \theta(x,t) = e^{\theta^\epsilon(x,\epsilon t)},$$

which gives $\rho(x,t) = e^{\epsilon p^\epsilon(x,\epsilon t) - \theta^\epsilon(x,\epsilon t)}$

Taking the scalings:

$$H(x,t) = \epsilon H^\epsilon(x,\epsilon t), \quad u(x,t) = \epsilon u^\epsilon(x,\epsilon t),$$
$$\mu = \epsilon\mu^\epsilon, \quad \lambda = \epsilon\lambda^\epsilon, \quad \nu = \epsilon\nu^\epsilon, \quad \kappa = \epsilon\kappa^\epsilon.$$

Thus, system (1) takes the form:

$$
\begin{aligned}
& p_t^\epsilon + (u^\epsilon \cdot \nabla) p^\epsilon + \frac{1}{\epsilon} \mathrm{div}(2u^\epsilon - \kappa^\epsilon e^{-\epsilon p^\epsilon + \theta^\epsilon} \nabla \theta^\epsilon) \\
& = \epsilon e^{-\epsilon p^\epsilon} [\nu^\epsilon |\mathrm{curl} H^\epsilon|^2 + \Psi(u^\epsilon) : \nabla u^\epsilon] + \kappa^\epsilon e^{-\epsilon p^\epsilon + \theta^\epsilon} \nabla p^\epsilon \cdot \nabla \theta^\epsilon,
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
& e^{-\theta^\epsilon} [u_t^\epsilon + (u^\epsilon \cdot \nabla) u^\epsilon] + \frac{\nabla p^\epsilon}{\epsilon} \\
& \quad = e^{-\epsilon p^\epsilon} [(\mathrm{curl} H^\epsilon) \times H^\epsilon + \mathrm{div} \Psi(u^\epsilon)],
\end{aligned} \tag{11}
$$

$$
H_t^\epsilon - \mathrm{curl}(u^\epsilon \times H^\epsilon) - \nu^\epsilon \Delta H^\epsilon = 0, \quad \mathrm{div} H^\epsilon = 0, \tag{12}
$$

$$
\begin{aligned}
& \theta_t^\epsilon + (u^\epsilon \cdot \nabla) \theta^\epsilon + \mathrm{div} u^\epsilon = \kappa^\epsilon e^{-\epsilon p^\epsilon} \mathrm{div}(e^{\theta^\epsilon} \nabla \theta^\epsilon) \\
& \quad + \epsilon^2 e^{-\epsilon p^\epsilon} [\nu^\epsilon |\mathrm{curl} H^\epsilon|^2 + \Psi(u^\epsilon) : \nabla u^\epsilon].
\end{aligned} \tag{13}
$$

Formally, as $\epsilon \to 0$, assuming $(u^\epsilon, H^\epsilon, \theta^\epsilon) \to (w, B, \vartheta)$ and $(\mu^\epsilon, \lambda^\epsilon, \nu^\epsilon, \kappa^\epsilon) \to (\bar{\mu}, \bar{\lambda}, \bar{\nu}, \bar{\kappa})$, then system (10)–(13) tends to

$$\mathrm{div}(2w - \bar{\kappa}\, e^{\vartheta} \nabla \vartheta) = 0, \tag{14}$$

$$e^{-\vartheta}[w_t + (w \cdot \nabla) w] + \nabla \pi = (\mathrm{curl} B) \times B + \mathrm{div}\Phi(w), \tag{15}$$

$$B_t - \mathrm{curl}(w \times B) - \bar{\nu} \Delta B = 0, \quad \mathrm{div} B = 0, \tag{16}$$

$$\vartheta_t + (w \cdot \nabla)\vartheta + \mathrm{div} w = \bar{\kappa}\, \mathrm{div}(e^{\vartheta} \nabla \vartheta), \tag{17}$$

with some function π.

Supplement system (14)–(17) with initial data

$$(p^\epsilon, u^\epsilon, H^\epsilon, \theta^\epsilon)|_{t=0} = (p^\epsilon_{\mathrm{in}}, u^\epsilon_{\mathrm{in}}, H^\epsilon_{\mathrm{in}}, \theta^\epsilon_{\mathrm{in}})(x), \quad x \in \mathbb{R}^3, \tag{18}$$

and for simplicity, assume $(\mu^\epsilon, \nu^\epsilon, \kappa^\epsilon, \lambda^\epsilon) \equiv (\bar{\mu}, \bar{\nu}, \bar{\kappa}, \bar{\lambda})$.

Notation: denote

$$\|(p^\epsilon, u^\epsilon, H^\epsilon, \theta^\epsilon - \bar\theta)(t)\|_{s,\epsilon}$$

$$:= \sup_{\tau\in[0,t]} \left\{ \|(p^\epsilon, u^\epsilon, H^\epsilon)(\tau)\|_{H^s} + \|(\epsilon p^\epsilon, \epsilon u^\epsilon, \epsilon H^\epsilon, \theta^\epsilon - \bar\theta)(\tau)\|_{H^{s+2}_\epsilon} \right\}$$

$$+ \left\{ \int_0^t [\|\nabla(p^\epsilon, u^\epsilon, H^\epsilon)\|^2_{H^s} + \|\nabla(\epsilon u^\epsilon, \epsilon H^\epsilon, \theta^\epsilon)\|^2_{H^{s+2}_\epsilon}](\tau)d\tau \right\}^{1/2},$$

where $\|v\|_{H^\sigma_\epsilon} := \|v\|_{H^{\sigma-1}} + \epsilon\|v\|_{H^\sigma}$

J-Ju-Li-Xin '12 **(1) Uniform solutions.** *Let $s \geq 4$ and*

$$\|(p_{\rm in}^{\epsilon}, u_{\rm in}^{\epsilon}, H_{\rm in}^{\epsilon}, \theta_{\rm in}^{\epsilon} - \bar{\theta})(t)\|_{s,\epsilon} \leq L_0, \qquad \forall \epsilon \in (0,1]$$

for some constants $\bar{\theta}$, L_0. *Then,* $\exists$ T_0 *and* $\epsilon_0 < 1$*, such that for* $\forall \epsilon \in (0, \epsilon_0]$*, the problem* (10)–(13), (18) *has a unique solution* $(p^{\epsilon}, u^{\epsilon}, H^{\epsilon}, \theta^{\epsilon})$ *on* $[0, T_0]$ *satisfying*

$$\|(p^{\epsilon}, u^{\epsilon}, H^{\epsilon}, \theta^{\epsilon} - \bar{\theta})(t)\|_{s,\epsilon} \leq L, \quad \forall t \in [0, T_0], \epsilon \in (0, \epsilon_0].$$

(2) Convergence. *Assume further that*

$$|\theta_0^{\epsilon}(x) - \bar{\theta}| \leq N_0|x|^{-1-\zeta}, \ |\nabla \theta_0^{\epsilon}(x)| \leq N_0|x|^{-2-\zeta}, \ \forall \epsilon \in (0,1],$$
$$\left(p_{\rm in}^{\epsilon}, curl(e^{-\theta_{\rm in}^{\epsilon}} u_{\rm in}^{\epsilon}), H_{\rm in}^{\epsilon}, \theta_{\rm in}^{\epsilon}\right) \to (0, w_0, B_0, \vartheta_0) \quad \text{in } H^s,$$

where N_0 and ζ are constants.

Then, $(p^\epsilon, u^\epsilon, H^\epsilon, \theta^\epsilon) \to (0, w, B, \vartheta)$ *weakly in* $L^\infty(0, T_0; H^s)$ *and strongly in* $L^2(0, T_0; H^{s_2}_{\rm loc})$ $\forall\, 0 \le s_2 < s$, *where* (w, B, ϑ) *solves* (14)–(18) *with* $(w, B, \vartheta)|_{t=0} = (w_0, B_0, \vartheta_0)$.

Proof ideas: $\star$ **Exploit effect of dissipation (viscosity, magnetic diffusion, heat conductivity)**

$\star$ **Div-Curl decomposition of** u

$\star$ **Refined energy estimates**

$\star$ **Weak compactness arguments**

$\star$ **Detailed analysis of the oscillation equations (dispersive estimates)**

(Some ideas from Métivier-Schochet'01, Alazard'06, Levermore-Sun-Trivisa'12 are used)

Ideal MHD flows, $\mu, \lambda, \nu, \kappa = 0$

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ \rho(u_t + u \cdot \nabla u) + \dfrac{\nabla(\rho\theta)}{\epsilon^2} = (\nabla \times H) \times H, \\ \rho(\theta_t + u \cdot \nabla\theta) + \rho\theta \mathrm{div} u = 0, \\ H_t - \nabla \times (u \times H) = 0, \quad \mathrm{div} H = 0. \end{cases} \tag{19}$$

If omitting H, (19) reduces to Euler equations, the limit of which was investigated by Métivier-Schochet'01 (exploiting structure of the system, e.g., control curl$u$, ...)

When the magnetic effect is taken into account, these arguments do not work directly, since nice structure used to control some cross-terms is destroyed. Limit as $\epsilon \to 0$ still open !

3. Planar ideal MHD flows, general data

However, in the planar flow case, the cross terms can be carefully controlled, using some weighted estimates, and we can show the low Mach number limit for general data.

This is the next goal of this talk.

Planar ideal compressible MHD equations:

Take the pressure p as an unknown, write $\rho = R(p, S)$ where S: Entropy, $u \in \mathbb{R}$: Longitudinal velocity, $\vec{w} \in \mathbb{R}^2$: Transverse velocity, $\vec{z} \in \mathbb{R}^2$: Transverse magnetic field,
then, the planar ideal compressible MHD equations read as

$$\begin{cases} A(S,p)(p_t + up_x) + u_x = 0, \\ R(S,p)(u_t + uu_x) + p_x + \vec{z} \cdot \vec{z}_x = 0, \\ R(S,p)(\vec{w}_t + u\vec{w}_t) - \vec{z}_t = 0, \\ \vec{z}_t + (u\vec{z})_x - \vec{w}_x = 0, \\ S_t + uS_x = 0, \end{cases}$$

with

$$A(S,p) := \frac{1}{R(S,p)} \frac{\partial R(S,p)}{\partial p}.$$

Introduce the scalings

$$p(x,t) = p^\epsilon(x,\epsilon t),\ \ S(x,t) = S^\epsilon(x,\epsilon t),$$
$$u(x,t) = \epsilon u^\epsilon(x,\epsilon t),\ \ \vec{w}(x,t) = \epsilon \vec{w}^\epsilon(x,\epsilon t),\ \ \vec{z}(x,t) = \epsilon \vec{z}^\epsilon(x,\epsilon t),$$

and the transform $p^\epsilon(x,\epsilon t) = \underline{p} e^{\epsilon q^\epsilon(x,\epsilon t)}$ for some positive constant $\underline{p}$ to arrive at

$$a(S^\epsilon, \epsilon q^\epsilon)(q_t^\epsilon + u^\epsilon q_x^\epsilon) + \frac{1}{\epsilon} u_x^\epsilon = 0, \tag{20}$$

$$r(S^\epsilon, \epsilon q^\epsilon)(u_t^\epsilon + u^\epsilon u_x^\epsilon) + \frac{1}{\epsilon} q_x^\epsilon + \vec{z}^\epsilon \cdot \vec{z}_x^\epsilon = 0, \tag{21}$$

$$r(S^\epsilon, \epsilon q^\epsilon)(\vec{w}_t^\epsilon + u^\epsilon \vec{w}_x^\epsilon) - \vec{z}_x^\epsilon = 0, \tag{22}$$

$$\vec{z}_t^\epsilon + (u^\epsilon \vec{z}^\epsilon)_x - \vec{w}_x^\epsilon = 0, \tag{23}$$

$$S_t^\epsilon + u^\epsilon S_x^\epsilon = 0, \tag{24}$$

where

$$a(S^\epsilon, \epsilon q^\epsilon) := A(S^\epsilon, \underline{p}e^{\epsilon q^\epsilon})\underline{p}e^{\epsilon q^\epsilon}$$

$$= \frac{\underline{p}e^{\epsilon q^\epsilon}}{R(S^\epsilon, \underline{p}e^{\epsilon q^\epsilon})} \cdot \frac{\partial R(S^\epsilon, \xi)}{\partial \xi}\Big|_{\xi = \underline{p}e^{\epsilon q^\epsilon}}, \quad r(S^\epsilon, \epsilon q^\epsilon) := \frac{R(S^\epsilon, \underline{p}e^{\epsilon q^\epsilon})}{\underline{p}e^{\epsilon q^\epsilon}}.$$

Suppose $(S^\epsilon, q^\epsilon, u^\epsilon, \vec{w}^\epsilon, \vec{z}^\epsilon) \to (\bar{S}, \bar{q}, \bar{u}, \bar{\vec{w}}, \bar{\vec{z}})$. Formally, as $\epsilon \to 0$ $\Rightarrow$ $\bar{q}_x = \bar{u}_x = 0$. The limiting equations of (22)-(24) take the form

$$r(\bar{S}, 0)(\bar{\vec{w}}_t + \bar{u}\bar{\vec{w}}_x) - \bar{\vec{z}}_x = 0, \tag{25}$$

$$\bar{\vec{z}}_t + \bar{u}\bar{\vec{z}}_x - \bar{\vec{w}}_x = 0, \tag{26}$$

$$\bar{S}_t + \bar{u}\bar{S}_x = 0. \tag{27}$$

We want to to establish this limit rigorously in a one-dimensional torus $\mathbb{T}^1$. Supplement system (20)–(24) with initial data:

$$(S^\epsilon, q^\epsilon, u^\epsilon, \vec{w}^\epsilon, \vec{z}^\epsilon)|_{t=0} = \left(S_0^\epsilon, q_0^\epsilon, u_0^\epsilon, \vec{w}_0^\epsilon, \vec{z}_0^\epsilon\right), \ x \in \mathbb{T}^1. \tag{28}$$

Thm. 1.1. *Let* $\|(S_0^\epsilon, q_0^\epsilon, u_0^\epsilon, \vec{w}_0^\epsilon, \vec{z}_0^\epsilon)\|_{H^2(\mathbb{T}^1)} \le M_0$. *Then, there is a* $T > 0$, *such that for any* $\epsilon \in (0,1]$, $\exists!$ *solution* $(q^\epsilon, u^\epsilon, \vec{w}^\epsilon, \vec{z}^\epsilon, S^\epsilon) \in C^0([0,T], H^2(\mathbb{T}^1))$ *to* (20)–(23), (28), *satisfying*

$$\|(q^\epsilon, u^\epsilon, \vec{w}^\epsilon, \vec{z}^\epsilon, S^\epsilon)(t)\|_{H^2(\mathbb{T}^1)}, \ \|(\vec{w}_t^\epsilon, \vec{z}_t^\epsilon, S_t^\epsilon)(t)\|_{H^1(\mathbb{T}^1)} \le N$$

for $\forall\, t \in [0,T]$. *Moreover, if*

$$(q_0^\epsilon, u_0^\epsilon, \vec{w}_0^\epsilon, \vec{z}_0^\epsilon, S_0^\epsilon) \to (q_0^0, u_0^0, \vec{w}_0^0, \vec{z}_0^0, S_0^0) \quad in \ \ H^2(\mathbb{T}^1),$$

then $(q^\epsilon, u^\epsilon, \vec{w}^\epsilon, \vec{z}^\epsilon, S^\epsilon)$ *converges to the limit* $(\bar{q}, \bar{u}, \bar{\vec{w}}, \bar{\vec{z}}, \bar{S},)$, *where*

$$\bar{q} \equiv \frac{\int_{\mathbb{T}^1} a(S_0^0(x),0) q_0^0(x)\mathrm{d}x}{\int_{\mathbb{T}^1} a(S_0^0(x),0)\mathrm{d}x}, \quad \bar{u} \equiv \frac{\int_{\mathbb{T}^1} r(S_0^0(x),0) u_0^0(x)\mathrm{d}x}{\int_{\mathbb{T}^1} r(S_0^0(x),0)\mathrm{d}x},$$

and $(\bar{S}, \bar{\vec{w}}, \bar{\vec{z}})$ *is the unique solution of* (25)–(27) *with initial data*

$$(\bar{S}, \bar{\vec{w}}, \bar{\vec{z}})|_{t=0} = (S_0^0, \vec{w}_0^0, \vec{z}_0^0).$$

Proof Steps: **Main ingredients:**

- **uniform estimates of higher order derivatives**
- **averaging analysis to treat acoustic wave (no dispersive property)**

Step I. Uniform estimates

If one directly works on system (20)–(24), then this involves 2nd-order derivatives of S which we cannot control uniformly. Instead, we rewrite (20)–(23) to avoid this.

⋆ For 1st-order derivative estimate, denote

$$q_1^\epsilon = q_x^\epsilon / r(S^\epsilon, \epsilon q^\epsilon), \quad u_1^\epsilon = u_x^\epsilon / a(S^\epsilon, \epsilon q^\epsilon),$$
$$\vec{w}_1^\epsilon = \vec{w}_x^\epsilon / a(S^\epsilon, \epsilon q^\epsilon), \quad \vec{z}_1^\epsilon = \vec{z}_x^\epsilon / r(S^\epsilon, \epsilon q^\epsilon),$$

and write (20)–(23) by differentiating as

$$r(S^\epsilon, \epsilon q^\epsilon)(\partial_t q_1^\epsilon + u^\epsilon \partial_x q_1^\epsilon) + \frac{1}{\epsilon}\partial_x u_1^\epsilon = f_1^\epsilon, \quad (29)$$

$$a(S^\epsilon, \epsilon q^\epsilon)(\partial_t u_1^\epsilon + u^\epsilon \partial_x u_1^\epsilon) + \frac{1}{\epsilon}\partial_x q_1^\epsilon + \partial_x\{\vec{z}^\epsilon \cdot \vec{z}_1^\epsilon\} = f_2^\epsilon, \quad (30)$$

$$a(S^\epsilon, \epsilon q^\epsilon)(\partial_t \vec{w}_1^\epsilon + u^\epsilon \partial_x \vec{w}_1^\epsilon) - \partial_x \vec{z}_1^\epsilon = \vec{f}_3^\epsilon, \quad (31)$$

$$\frac{r(S^\epsilon, \epsilon q^\epsilon)}{a(S^\epsilon, \epsilon q^\epsilon)}\partial_t \vec{z}_1^\epsilon + \partial_x\left\{u_1^\epsilon \vec{z}^\epsilon + \frac{u^\epsilon \vec{z}_x^\epsilon}{a(S^\epsilon, \epsilon q^\epsilon)}\right\} - \partial_x \vec{w}_1^\epsilon = \vec{f}_4^\epsilon, \quad (32)$$

with $f_1^\epsilon = \cdots$, $f_2^\epsilon = \cdots$, ...

⋆ **For 2nd-order derivative estimate**, denote

$$q_2^\epsilon = \partial_x q_1^\epsilon / a(S^\epsilon, \epsilon q^\epsilon), \quad u_2^\epsilon = \partial_x u_1^\epsilon / r(S^\epsilon, \epsilon q^\epsilon),$$
$$\vec{w}_2^\epsilon = \partial_x \vec{w}_1^\epsilon / r(S^\epsilon, \epsilon q^\epsilon), \quad \vec{z}_2^\epsilon = \partial_x \vec{z}_1^\epsilon / a(S^\epsilon, \epsilon q^\epsilon),$$

and write (29)–(32), by differentiating, as

$$a(S^\epsilon, \epsilon q^\epsilon)(\partial_t q_2^\epsilon + u^\epsilon \partial_x q_2^\epsilon) + \frac{1}{\epsilon}\partial_x u_2^\epsilon = g_1^\epsilon, \tag{33}$$
$$r(S^\epsilon, \epsilon q^\epsilon)(\partial_t u_2^\epsilon + u^\epsilon \partial_x u_2^\epsilon) + \frac{1}{\epsilon}\partial_x q_2^\epsilon + h_1^\epsilon = g_2^\epsilon, \tag{34}$$
$$r(S^\epsilon, \epsilon q^\epsilon)(\partial_t \vec{w}_2^\epsilon + u^\epsilon \partial_x \vec{w}_2^\epsilon) - \partial_x \vec{z}_2^\epsilon = \vec{g}_3^\epsilon, \tag{35}$$
$$\partial_t \vec{z}_2^\epsilon - \partial_x \vec{w}_2^\epsilon + \vec{h}_2^\epsilon = \vec{g}_4^\epsilon, \tag{36}$$

with $g_1^\epsilon = \cdots$, $g_1^\epsilon = \cdots$, ..., $h_1^\epsilon = \cdots$, $\vec{h}_2^\epsilon = \cdots$, ...

Systems (29)–(32), and (33)–(36) still keep good structure, then we can use careful energy estimates to control the 1st- and 2nd-order derivatives (only 1st-order derivatives of S are involved), such that we can close the estimates to get finally

Lemma 1. $\exists$ *an increasing function* $C(\cdot) : [0, \infty) \mapsto [0, \infty)$ *independent of* ϵ*, such that*

$$\mathcal{M}_\epsilon(T) \leq C_0 + TC(\mathcal{M}_\epsilon(T)), \tag{37}$$

where

$$\mathcal{M}_\epsilon(T) = \sup_{t\in[0,T]} \|(q^\epsilon, u^\epsilon, \vec{w}^\epsilon, \vec{z}^\epsilon, S^\epsilon)(t)\|_{H^2(\mathbb{T}^1)}.$$

$\Rightarrow$ **Uniform a priori estimates in the theorem**

$\Rightarrow$ **Local existence uniform in** ϵ

Step II. Take limit (averaging analysis)

$$(q^\epsilon, u^\epsilon, \vec{w}^\epsilon, \vec{z}^\epsilon) \rightharpoonup (\bar{q}, \bar{u}, \bar{\vec{w}}, \bar{\vec{z}}) \text{ weakly-} * \text{ in } L^\infty(0, T; H^2(\mathbb{T}^1)),$$
$$(S^\epsilon, \vec{w}^\epsilon, \vec{z}^\epsilon) \to (\bar{S}, \bar{\vec{w}}, \bar{\vec{z}}) \text{ strongly in } C([0, T], H^{s'}(\mathbb{T}^1)), \ \forall\, s' < 2.$$

thus, (20)–(24) $\Rightarrow$ $\partial_x \bar{q} = \partial_x \bar{u} = 0$ (i.e., $\bar{q}(x, t) = \bar{q}(t)$, $\bar{u}(x, t) = \bar{u}(t)$), and $\bar{S}, \bar{\vec{w}}, \bar{\vec{z}}$ satisfy

$$r(\bar{S}, 0)(\partial_t \bar{\vec{w}} + \bar{u}\partial_x \bar{\vec{w}}) - \partial_x \bar{\vec{z}} = 0, \tag{38}$$
$$\partial_t \bar{\vec{z}} + \bar{u}\partial_x \bar{\vec{z}} - \partial_x \bar{\vec{w}} = 0, \tag{39}$$
$$\partial_t \bar{S} + \bar{u}\partial_x \bar{S} = 0, \qquad \text{in } \mathcal{D}. \tag{40}$$

Averaging analysis $\Rightarrow$ $\frac{d}{dt}\bar{q} = 0$. In fact, by equations (20), (24), we have

$$\partial_t a(S^\epsilon, \epsilon q^\epsilon) = -u^\epsilon \partial_x a(S^\epsilon, \epsilon q^\epsilon) - d(S^\epsilon, \epsilon q^\epsilon)\partial_x u^\epsilon. \qquad (41)$$

where

$$d(S^\epsilon, \epsilon q^\epsilon) := \frac{1}{a(S^\epsilon, \epsilon q^\epsilon)} \frac{\partial a(S^\epsilon, \xi)}{\partial \xi}|_{\xi=\epsilon q^\epsilon}.$$

Denoting

$$\langle\!\langle \cdot \rangle\!\rangle := \frac{1}{|\mathbb{T}^1|}\int_{\mathbb{T}^1} \cdot \, dx,$$

and taking the average of (41) and integrating by parts ⇒

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\!\langle a(S^\epsilon, \epsilon q^\epsilon) \rangle\!\rangle = \langle\!\langle [a(S^\epsilon, \epsilon q^\epsilon) - d(S^\epsilon, \epsilon q^\epsilon)]\partial_x u^\epsilon \rangle\!\rangle \qquad (42)$$

Taking limit ⇒

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\!\langle a(\bar{S}, 0) \rangle\!\rangle = 0.$$

Now, equations (20) and (41) gives

$$\partial_t\{a(S^\epsilon, \epsilon q^\epsilon)q^\epsilon\} + u^\epsilon \partial_x\{a(S^\epsilon, \epsilon q^\epsilon)q^\epsilon\} + \frac{1}{\epsilon}\partial_x u^\epsilon$$
$$+d(S^\epsilon, \epsilon q^\epsilon)q^\epsilon \partial_x u^\epsilon = 0. \tag{43}$$

Taking the average of (43) and using (20) to eliminate $\partial_x u^\epsilon$ $\Rightarrow$

$$\frac{d}{dt}\langle\langle\, a(S^\epsilon, \epsilon q^\epsilon)q^\epsilon \,\rangle\rangle$$
$$= -\frac{\epsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle\langle\, [a(S^\epsilon, \epsilon q^\epsilon) - d(S^\epsilon, \epsilon q^\epsilon)]a(S^\epsilon, \epsilon q^\epsilon)(q^\epsilon)^2 \,\rangle\rangle$$
$$+\frac{\epsilon}{2}\langle\langle\, (q^\epsilon)^2 \partial_t\{[a(S^\epsilon, \epsilon q^\epsilon) - d(S^\epsilon, \epsilon q^\epsilon)]a(S^\epsilon, \epsilon q^\epsilon)\} \,\rangle\rangle$$
$$-\epsilon\langle\langle\, [a(S^\epsilon, \epsilon q^\epsilon) - d(S^\epsilon, \epsilon q^\epsilon)]a(S^\epsilon, \epsilon q^\epsilon)q^\epsilon u^\epsilon \partial_x q^\epsilon \,\rangle\rangle$$
$$\rightharpoonup 0. \tag{44}$$

$\Rightarrow$ (recalling $\frac{d}{dt}\langle\!\langle\, a(\bar{S},0)\,\rangle\!\rangle = 0, \quad a(\bar{S},0) > 0$)

$$\frac{d}{dt}\langle\!\langle\, a(\bar{S},0)\bar{q}\,\rangle\!\rangle = 0 \quad \Rightarrow \frac{d}{dt}\bar{q} = 0. \qquad \square$$

$\Rightarrow$

$$\langle\!\langle\, a(\bar{S}(x,0),0)\bar{q}(0)\,\rangle\!\rangle = \langle\!\langle\, a(\bar{S}(x,t),0)\bar{q}(t)\,\rangle\!\rangle$$
$$= \bar{q}(t)\langle\!\langle\, a(\bar{S}(x,t),0)\,\rangle\!\rangle = \bar{q}(0)\langle\!\langle\, a(\bar{S}(x,0),0)\,\rangle\!\rangle$$

$\Rightarrow$

$$\bar{q}(t) \equiv \bar{q}(0) = \frac{\langle\!\langle\, a(\bar{S}(x,0),0)\bar{q}(0)\,\rangle\!\rangle}{\langle\!\langle\, a(\bar{S}(x,0),0)\,\rangle\!\rangle} = \frac{\int_{\mathbb{T}^1} a(S_0^0(x),0)q_0^0(x)dx}{\int_{\mathbb{T}^1} a(S_0^0(x),0)dx}.$$

Similarly,

$$\bar{u}(t) \equiv \bar{u}(0) = \frac{\int_{\mathbb{T}^1} r(S_0^0(x),0)u_0^0(x)dx}{\int_{\mathbb{T}^1} r(S_0^0(x),0)dx}.$$

Since the limiting system (38)–(40) has a unique solution $(S^*, \vec{w}^*, \vec{z}^*) \in C([0,T], H^2(\mathbb{T}^1)) \Rightarrow$ the convergence results hold for the full sequence of $(S^\epsilon, \vec{w}^\epsilon, \vec{z}^\epsilon)$. □

Remark. In numerical simualtion of MHD with small ϵ, traditional numerical schemes require the cell size $h = O(\epsilon)$ in order to resolve the solution. Now, a new scheme, Asymptotic Preserving (AP) scheme, is under development, for which h can be independent of ϵ, and which is valid for $\epsilon \to 0$. (1D case, OK). The successful construction of a AP scheme requires the knowledge on the limiting equations and propagation of oscillations. The study of zero Mach number limit will contribute to this.

Future study:

Multi-dimensional case

THANK YOU !