Model reduction: analysis, numerical solution and real world applications Lecture III: Model reduction in real world and industrial applications

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Mathematics for key technologies

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Outline

Model based approach

Physical system

$\swarrow \quad \searrow$

Modeling		**Modeling**
$\downarrow$		$\downarrow$
ODE/DAE	$\longleftarrow$ *semidiscr.*	**PDE**
$\downarrow$		
Mod. reduction		
$\downarrow$		
Reduced ODE/DAE	$\longrightarrow$	*Sim., Control*

How to get a reduced order model depending on parameters?

- Semidiscretization in space using FV, FE, FD $\Longrightarrow$ large scale ODE/DAE-control problem (with parameters).
- Project on a subspace that captures the dynamics in a large range of the parameters.
- We can use all the methods as before, survey by Benner, Gugercin, Willcox, enriched by sampling of the parameter space.
- Reduced basis approach, books A. Quarteroni, A. Manzoni, F. Negri, A. Quarteroni, G. Rozza

Sparse representation of PDE solutions

Given PDE model that describes the space-time behavior.

- Numerical solution of PDE $Ly = f$, with differential operator L in a domain $\Omega \subset \mathbb{R}^d$ with boundary Γ and BC on Γ. Data and solution depending on parameters (controls).
- Let $\mathcal{V}$ be an ansatz function space in which we know or expect the solution to be, (depending on parameters, controls).
- Choose another (or the same) space $\mathcal{W}$ as test space.
- Classical Galerkin or Petrov-Galerkin approach: Seek solution y in some finite dimensional ansatz space $\mathcal{V}_n \subset \mathcal{V}$ (spanned by) $\mathcal{B} = \{\phi_1, \ldots, \phi_n\}$, i.e. $y = \sum_{i=1}^{n} y_i \phi_i$ and $(Ly - f, w) = 0$ or $|(Ly - f, w)| < \epsilon$ for all $w \in W$.

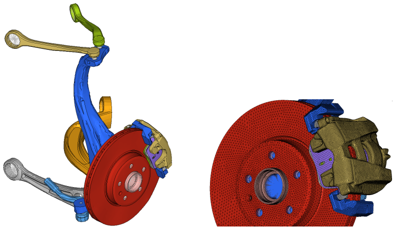
How sparse can we get?

Different Questions

- What is a good space $\mathcal{V}$, so that y can be sparsely represented/approximated in $\mathcal{V}$ (for a large parameter range)?
- Good space for forward or for optimization/control problem?
- What is a good basis of $\mathcal{V}_n$ so that u can be sparsely represented/approximated.
- What are conditions for the basis so that the finite dimensional version $L_n y_n = f_n$ is easy to solve for many parameters?
- Is there a 'eierlegende Wollmilchsau', a swiss army knife?

Can only be answered for specific application.

Outline



- Disc brake squeal is a frequent and annoying phenomenon (with cars, trains, bikes).
- Important for customer satisfaction, even if not a safety risk.
- Nonlinear effect that is hard to detect in experiments.
- The car industry is trying for decades to improve this, by changing the designs of brake and disc.

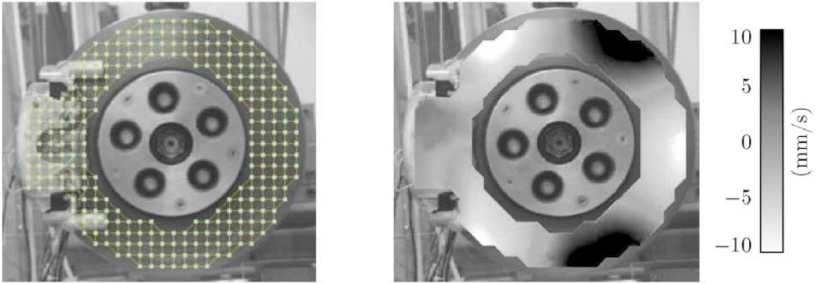
Can we do this model based?

Model based approach

Interdisciplinary project with car manufacturers + SMEs
Supported by German Minist. of Economics via AIF foundation.
University: N. Gräbner, U. von Wagner, TU Berlin, Mechanics,
N. Hoffmann, TU Hamburg-Harburg, Mechanics,
S. Quraishi, C. Schröder, TU Berlin Mathematics.

Goals:

- Develop model of brake system with all effects that may cause squeal. (Friction, circulatory, gyroscopic effects, etc).
- Simulate brake behavior for many different parameters (disk speed, material geometry parameters).
- Our task: Model reduction, solution of eigenvalue problems.
- Long term: Stability/bifurcation analysis for a given parameter region.



Gitter der Messpunkte   Betriebsschwingform (1750 Hz) [1]

- Experiments indicate nonlinear behavior (subcritical Hopf bifurcation) → film.

[1] Institute f. Mechanics, TU Berlin

Atomistic scale: Many damped harmonic oscillators: Langevin equation.

$$m\ddot{q}(t) + d\dot{q}(t) + kq(t) = \xi(t),$$

- m mass, k stiffness.
- d describes damping and dissipation effects, (very difficult to model in practice).
- ξ is the Langevin complementary force random force, d and ξ are frequency dependent.

Not a good model for simulation and definitely not for optimization.
Errors and uncertainties very hard to quantify.

Modeling in industrial practice, macroscale

Multi-body system based on Finite Element Modeling (FEM)

- Write displacements of structure $z(x,t)$ as linear combination of basis functions (e.g. but not always piecewise polynomials),

$$z(x,t) \approx \sum_{i=1}^{N} q_i(t)\phi_i(x,t).$$

- Integrate against test functions (Petrov Galerkin) $\rightarrow$ discretized model for the vibrations in weak form.
- Add friction and damping as macroscopic surrogate model fitted from experimental data.
- Simplifications: Remove some nonlinearities, asymptotic analysis for small parameters, etc.
- Produce reduced order model for large parameter set?

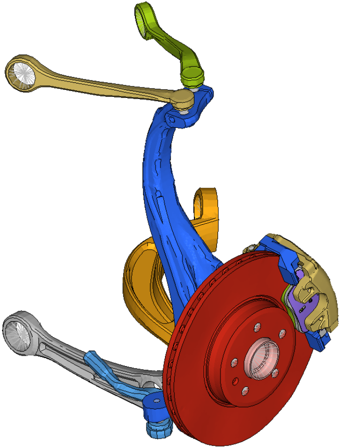


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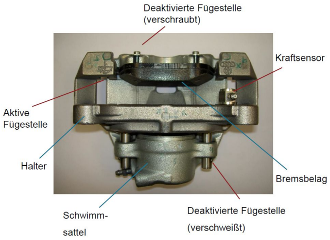
Figure: View of the brake model

[2] Institut für Mechanik, TU Berlin

Brake pad



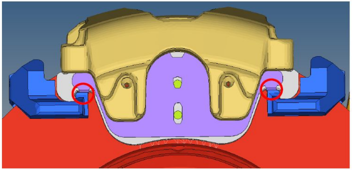

[3]

Figure: View of the brake model

[3] Institute f. Mechanics TU HH

Mathematical model details

Large differential-algebraic equation (DAE) system and evp dep. on parameters (here only disk speed displayed).

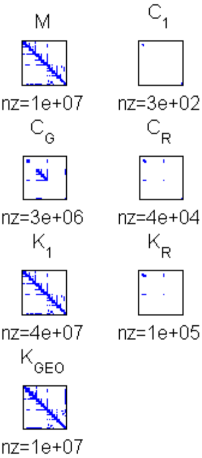
$$M\ddot{q} + (C_1 + \frac{\omega_r}{\omega} C_R + \frac{\omega}{\omega_r} C_G)\dot{q} + (K_1 + K_R + (\frac{\omega}{\omega_r})^2 K_G)q = f,$$

- M symmetric, pos. semidef., singular matrix (constraints),
- C_1 symmetric matrix, material damping,
- C_G skew-symmetric matrix, gyroscopic effects,
- C_R symmetric matrix, friction induced damping, (phenomenological)
- K_1 symmetric stiffness matrix,
- K_R nonsymmetric matrix modeling circulatory effects,
- K_G symmetric geometric stiffness matrix.
- ω rotational speed of disk with reference velocity ω_r.

Nature of FE matrices

$n = 842,638, \omega_r = 5, \omega = 17 \times 2\pi$

matrix	**pattern**	**2-norm**	**structural rank**
M	symm	5e-2	842,623
C_1	symm	1e-19	160
C_G	**skew**	**1.5e-1**	**217500**
C_R	**symm**	**7e-2**	**2120**
K_1	**symm**	**2e13**	**full**
K_R	-	3e4	2110
K_G	symm	40	842,623



Model evaluation, challenges

This is really a hierarchy and mixture of models.

- FE Model hierarchy: grid hierarchy, type of ansatz functions, component and domain decomposition.
- Coupled with surrogate model for friction and damping?

Challenges

- Are the simplifications: nonlinear/linear, expansion of small parameters justified?
- We do not really have a PDE, error estimates, adaptivity?
- Parametric reduced model for optimization, control, bifurcation analysis?
- Good subspace in function space or coordinate space?

This is a wave problem, eigenspaces seem a good choice.

Outline

- Ansatz $q(t) = e^{\lambda t} u$ gives a quadratic eigenvalue problem (QEP):
$$P_\omega(\lambda)u = (\lambda^2 M + \lambda C(\omega) + K(\omega))u = 0.$$
- Want evs with positive real part (few, ideally one, since squeal is mono-frequent) and corresponding evecs.
- Likelihood of a brake to squeal is correlated with magnitude of positive real part of eigenvalue.
- Objective: Efficient method to compute evs in right half plane for many parameter values e.g. $\omega \in (2\pi,\ 2\pi \times 20)$.

Projection approach

Determine subspace spanned by columns of matrix Q,

- Project QEP: $P_\omega(\lambda)x = (\lambda^2 M + \lambda C(\omega) + K(\omega))x = 0$ or dynamical system into small d-dimensional subspace that is independent of ω.
- projected QEP
  - $\tilde{P}_\omega(\lambda) = Q^T P_\omega(\lambda) Q = \lambda^2 Q^T M Q + \lambda Q^T C(\omega) Q + Q^T K(\omega) Q$
- How to choose Q?
  - Sufficiently accurate approximation of evs with positive real part
  - Ideally Q should contain good approximations to the desired evecs for all parameter values
  - One should be able to construct Q in a reasonable amount of computing time.

- Traditional approach to get a subspace Q:
  - Q_{TRAD}:=dominant eigenvectors (i.e. eigenvectors with smallest eigenvalues) of generalized evp $L(\lambda) = (\mu M - K_1 - K_G)$
- Advantages:
  - One only has to solve a large, sparse, symmetric, definite GEVP.
- Disadvantages:
  - Subspace does not take into account damping and parameter dependence.
  - Often poor approximation of evs/evecs of the full model.

DMD/POD

Use idea from proper orthogonal decomposition (POD) or dynamic mode decomposition (DMD).

- Compute matrices of eves $X(\omega_i)$ corresponding to right half plane evs for full QEP $P_\omega(\lambda)x = 0$ and sample parameters $\omega_1, \omega_2, \ldots, \omega_p$
- Construct measurement matrix $\widetilde{X} = [X(\omega_1),\ X(\omega_2),\ X(\omega_3) \cdots\ X(\omega_p)]$ containing computed snapshot evecs.
- Extract **dominant directions in** $\widetilde{X}$ by a truncated singular value decomposition.

Same space can also be used with other approaches.

Spectral transformation

Consider full problem $P_\omega(\lambda)x = 0$.

- Set $\lambda_\tau(\omega) = \lambda(\omega) - \tau$, where τ is such that $\det(P_\omega(\tau)) \neq 0$.
- New parametric QEP

$$P_{\omega,\tau}(\lambda(\omega))x(\omega) = (\lambda_\tau(\omega)^2 M_\tau + \lambda_\tau(\omega) C_\tau(\omega) + K_\tau(\omega))x(\omega) = 0,$$

  where $M_\tau = M$, $C_\tau = 2\tau M + C$ and $K_\tau = \tau^2 M + \tau C + K$ is nonsingular.
- Shift point τ is chosen in the right half plane, ideally near the expected eigenvalue location.
- Consider reverse polynomial, then evs near τ become large in modulus, while evs far away from τ become small.

Linearization, first order form.

We use classical companion linearization (first order form)

$$A_\tau(\omega)v(\omega) = \mu_\tau B_\tau(\omega)v(\omega)$$

with

$$\begin{bmatrix} K_\tau(\omega) & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} v(\omega) \\ \mu_\tau(\omega)v(\omega) \end{bmatrix} = \mu_\tau(\omega) \begin{bmatrix} -C_\tau(\omega) & -M_\tau \\ I_n & 0 \end{bmatrix} \begin{bmatrix} v(\omega) \\ \mu_\tau v(\omega) \end{bmatrix}.$$

Shift and invert Arnoldi

- Compute ev and evec approximations near shift τ via shift-and-invert Arnoldi method.
- Given $v_0 \in \mathbb{C}^n$ and $W \in \mathbb{C}^{n\times n}$, the Krylov subspace of $\mathbb{C}^n$ of order k associated with W is

$$\mathcal{K}_k(W, v_0) = span\{v_0, Wv_0, W^2 v_0 ..., W^{k-1} v_0\}.$$

- Arnoldi obtains orthonormal basis V_k of this space and

$$WV_k = V_k H_k + fe_k^*,$$

- Columns of V_k approx. k-dim. invariant subspace of W.
- Evs of H_k approximate evs of W associated to V_k.
- Apply with shift τ and frequency ω to $W = B_\tau(\omega)^{-1} A_\tau(\omega)$. Per step we multiply with $A_\tau(\omega)$ and solve system with $B_\tau(\omega)$.

SVD projection

- Construct measurement matrix $V \in \mathbb{R}^{n,km}$ containing 'unstable' evecs for a set of ω_i,

$$V = [V(\omega_1), V(\omega_2), V(\omega_3), \ldots V(\omega_k)]$$

- Perform (partial) SVD $V = U\Sigma Z^H$

$$V = [\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_{km}] \begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \sigma_3 & & \\ & & & \ddots & \\ & & & & \sigma_{km} \end{bmatrix} [\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_{km}]^H$$

with U, Z unitary.

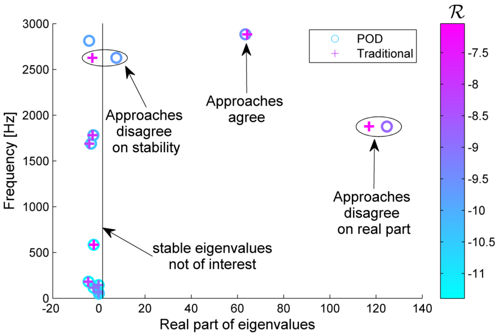
- Use approximation

$$\tilde{V} \approx [\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_d] \begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \sigma_3 & & \\ & & & \ddots & \\ & & & & \sigma_d \end{bmatrix} [\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_d]^H$$

by deleting $\sigma_{d+1}, \sigma_{d+2}, \ldots \sigma_{km}$ that are small.
(Actually these are not even computed).

- Choose $Q = [\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_d]$ to project $P_\omega(\lambda)$ or dynamical system.

Results real brake model



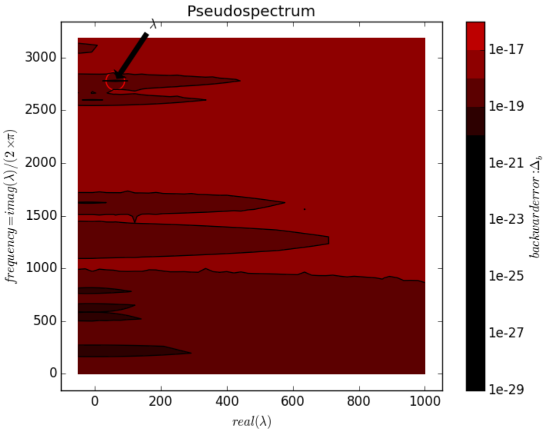
Assessing 'accuracy of evs'

Do we believe we got have a good space?

- Forward error: $\Delta_f = |\lambda_{exact} - \lambda_{computed}|$
- Backward error: smallest in norm perturbation Δ_b to M, C, K such that $\tilde{v}, \tilde{\lambda}$ satisfies QEVP defined by perturbed matrices $\tilde{M}, \tilde{C}, \tilde{K}$
- Computation of backward error: $\Delta_b(\lambda) = \frac{\|(\lambda^2 M + \lambda C + K)\|}{|\lambda|^2\|M\| + |\lambda|\|C\| + \|K\|}$
- The pseudospectrum gives the level curves of $\Delta_b(\lambda)$.

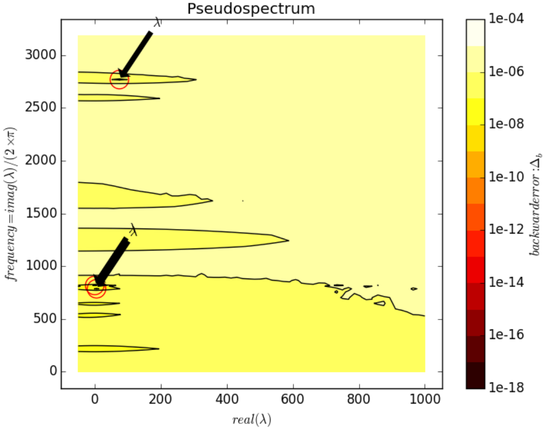
Pseudospectrum of a toy brake model

Brake model with 5000 dof, one of the springs had stiffness 10^{18}.

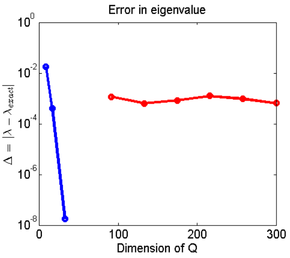


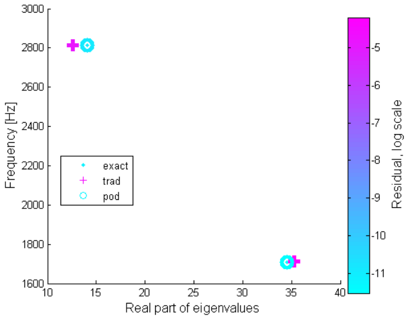
Pseudospectrum of a toy brake model

Brake model corrected with modeling high stiffness as rigid link.



Industrial model 1 million dof





- ▷ Solution for every ω
  - Solution with 300 dimensional TRAD subspace $\sim$ 30 sec
  - Solution with 100 dimensional POD subspace $\sim$ 10 sec

Intermediate Conclusions

- New POD approach captures modal information better than traditional one, but slower.
- Current numerical linear algebra methods are not efficient (in particular those in commercially codes).
- Discrete FE and quasi-uniform grids followed by expensive model reduction is really a waste.
- Can we combine FE modeling and eigenvalue computation for modal truncation or other MOR methods?
- Can we get error estimates and adaptivity? (AFEM , AMLS).

Outline

Adapative Finite Element Method

- Adaptive Finite Element methods refine the mesh where necessary, and coarsen where solution is well represented.
- They use a priori and a posteriori error estimators to get information about the discretization error.
- They are well established for PDE boundary value problems.
- But here we want to use them for PDE eigenvalue problems, which is much harder.
- And in the brake problem we do not have a PDE.
- Furthermore we have a parametric problem.

Solve $\rightarrow$ Estimate $\rightarrow$ Mark $\rightarrow$ Refine

Model problem: Elliptic PDE evp

Consider a model problem like the disk brake without damping, gyroscopic, circulatory terms and reasonable geometry.

$$\begin{aligned} \Delta u &= \lambda u \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

This is just the traditional approach that is used in industry. (Note $-\lambda^2$ in brake problem).

Weak formulation:
Determine ev/e.-function pair $(\lambda, u) \in \mathbb{R} \times V := \mathbb{R} \times H^1(\Omega; \mathbb{R})$ with $b(u,u) = 1$ and

$$a(u,v) = \lambda b(u,v) \quad \text{for all } v \in V,$$

where the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are defined by

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v \, dx, \; b(u,v) := \int_\Omega uv \, dx \quad \text{for } u,v \in V.$$

Induced norms $|||\cdot||| := |\cdot|_{H^1(\Omega)}$ on V and $\|\cdot\| := \|\cdot\|_{L^2(\Omega)}$ on $L^2(\Omega)$.

Determine ev./e.-function pair $(\lambda_\ell, u_\ell) \in \mathbb{R} \times V_\ell$ with $b(u_\ell, u_\ell) = 1$ and

$$a(u_\ell, v_\ell) = \lambda_\ell b(u_\ell, v_\ell) \quad \text{for all } v_\ell \in V_\ell.$$

Use coordinate representation to get finite-dim. generalized evp

$$A_\ell x_\ell = \lambda_\ell B_\ell x_\ell$$

with stiffness matrix $A_\ell = [a(\varphi_i, \varphi_j)]_{i,j=1,\dots,N_\ell}$, mass matrix $B_\ell = [b(\varphi_i, \varphi_j)]_{i,j=1,\dots,N_\ell}$, in nodal basis $V_\ell = \{\varphi_1, \dots, \varphi_{N_\ell}\}$.
Discrete eigenvector: $x_\ell =: [x_{\ell,1}, \dots, x_{\ell,N_\ell}]^T$.
Approximated eigenfunction:

$$u_\ell = \sum_{k=1}^{N_\ell} x_{\ell,k} \varphi_k \in V_\ell.$$

This approach includes several errors:

- ▷ Model error (PDE model vs. Physics)
- ▷ Discretization error (finite dim. subspace)
- ▷ Error in eigenvalue solver (iterative method)
- ▷ Roundoff errors in finite arithmetic.

An error estimator η_ℓ is called *efficient* and *reliable* if there exist mesh-size independent constants C_{eff} C_{rel} such that

$$C_{\text{eff}}\eta_\ell \leq \|u - u_\ell\| \leq C_{\text{rel}}\eta_\ell.$$

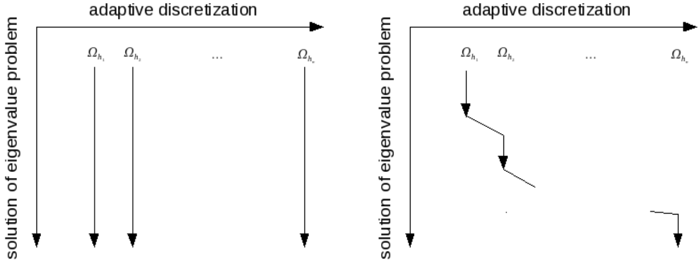
A posteriori error estimate

Estimate the error a posteriori via

$$|\lambda - \lambda_\ell| + \|u - u_\ell\|^2 \lesssim \eta_\ell^2 := \|u_{\ell-1} - u_\ell\|^2.$$

Here $\lesssim$ denotes an inequality that holds up to a multiplicative constant.
A posteriori error estimators for Laplace eigenvalue problem
Grubisic/Ovall 2009, M./Miedlar 2011, Neymeyr 2002

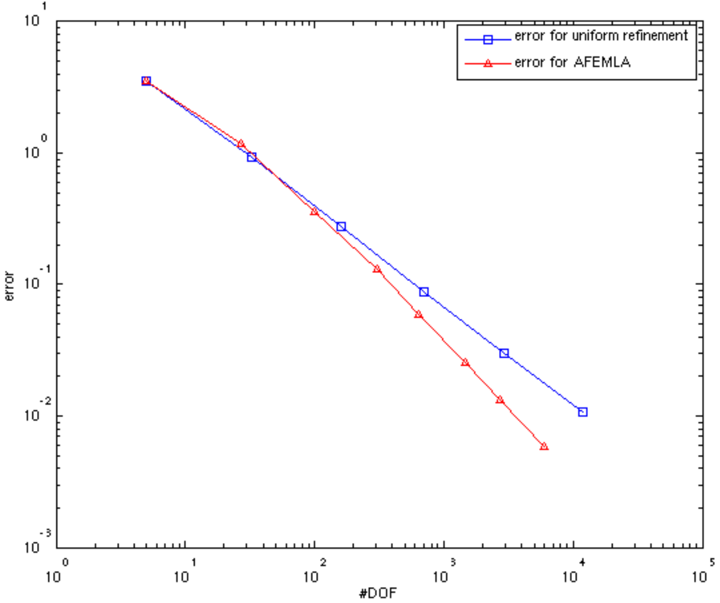
- Compute approx. eigenpair $(\tilde{\lambda}_H, \tilde{x}_H)$ on the coarse mesh,
- use iterative solver, i.e. Krylov subspace method,
- but do not solve very accurately, stop after a few steps or when tolerance *tol* is reached.
- Balance residual vector and error estimate Miedlar 2011.



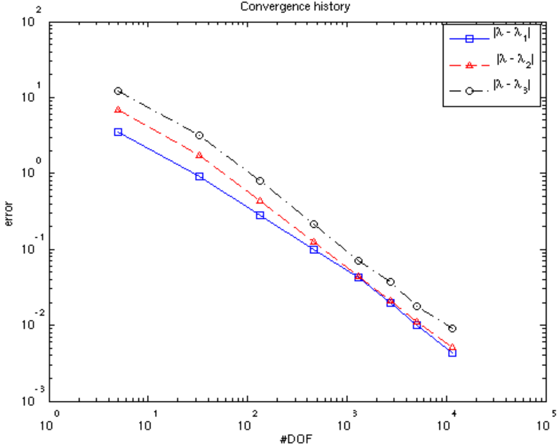
Standard AFEM

AFEMLA

Conv. history AFEMLA



Conv. first 3 evs, L-shape domain.



Intermediate Conclusion

- For purely elliptic problems we can compute evs and efunctions very efficiently.
- Can be used to compute the subspace for the traditional approach.
- We have a priori/a posteriori error estimates which allow to adapt the mesh to the solution behavior.
- With the AFEMLA approach we can even work in a purely algebraic way if the underlying PDE is not available.
- It works also for several evs at a time (invariant subspaces).
- Proof of convergence M./Miedlar 2011 if saturation property holds. Proof Carstensen/Gedicke/M./Miedlar 2013.
- So we can do the traditional approach also with adaptivity and tune in to the dominant evs.
- But we want this for the full model.

Non-selfadjoint problems

- Can we modify ideas for general problem?
- We need to deal with left and right evecs, complex evs, Jordan blocks.
- What are the right spaces and norms?
- Let us bring the nonsymmetry in via homotopy.

$$\mathcal{H}(t) = (1-t)\mathcal{L}_0 + t\mathcal{L}_1 \quad \text{for } t \in [0,1],$$

where $\mathcal{L}_0 u := -\Delta u$.
Discrete homotopy for the model eigenvalue problem:

$$\mathcal{H}_\ell(t) = (A_\ell + C_\ell)(t) = (1-t)A_\ell + t(A_\ell + C_\ell) = A_\ell + tC_\ell.$$

A non-self-adjoint model problem

Carstensen/Gedicke/M./Miedlar 2012

Convection-diffusion eigenvalue problem:

$$-\Delta u + \gamma \cdot \nabla u = \lambda u \text{ in } \Omega \qquad \text{and} \qquad u = 0 \text{ on } \partial\Omega$$

Discrete weak primal and dual problem:

$$\begin{aligned} a(u_\ell, v_\ell) + c(u_\ell, v_\ell) &= \lambda_\ell b(u_\ell, v_\ell) \quad \text{for all } v_\ell \in V_\ell, \\ a(w_\ell, u_\ell^\star) + c(w_\ell, u_\ell^\star) &= \overline{\lambda_\ell^\star} b(w_\ell, u_\ell^\star) \quad \text{for all } w_\ell \in V_\ell. \end{aligned}$$

Generalized algebraic eigenvalue problem:

$$(A_\ell + C_\ell)\mathbf{u}_\ell = \lambda_\ell B_\ell \mathbf{u}_\ell \qquad \text{and} \qquad \mathbf{u}_\ell^\star (A_\ell + C_\ell) = \lambda_\ell^\star \mathbf{u}_\ell^\star B_\ell$$

Smallest real part ev. is simple and well separated Evans '00.

Theorem (Carstensen/Gedicke/M./Miedlar 2012)

For model problem, the difference between the approx. ev. $\tilde\lambda_\ell(t)$ in the homotopy $\mathcal{H}_\ell(t)$ and the ev. $\lambda(1)$ of the original problem can be estimated via

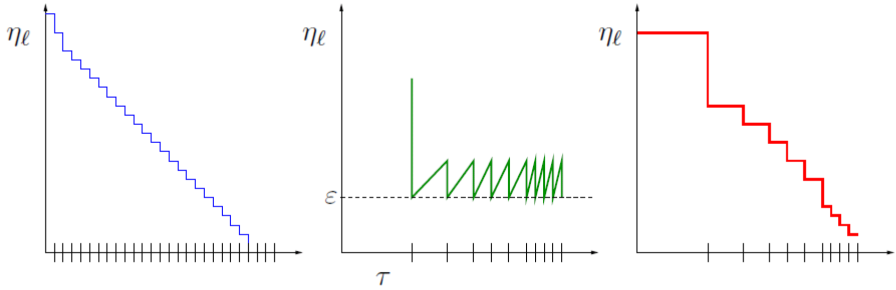
$$\begin{aligned}\|\lambda(1)-\tilde\lambda_\ell(t)\| &\lesssim \nu(\tilde\lambda_\ell(t),\tilde u_\ell(t),\tilde u_\ell^\star(t))+\eta^2(\tilde\lambda_\ell(t),\tilde u_\ell(t),\tilde u_\ell^\star(t))\\ &+ \mu^2(\tilde\lambda_\ell(t),\tilde u_\ell(t),\tilde u_\ell^\star(t))\end{aligned}$$

in terms of

$$\begin{aligned}\nu(\tilde\lambda_\ell(t),\tilde u_\ell(t),\tilde u_\ell^\star(t)) &:= (1-t)\|\gamma\|_\infty\,(|||\tilde u_\ell(t)|||+|||\tilde u_\ell^\star(t)|||)\\ &+ (1-t)\|\gamma\|_\infty\left(\eta(\tilde\lambda_\ell(t),\tilde u_\ell(t),\tilde u_\ell^\star(t))+\mu(\tilde\lambda_\ell(t),\tilde u_\ell(t),\tilde u_\ell^\star(t))\right).\end{aligned}$$

Adaptive homotopy algorithms

Algorithm 1: Balances the homotopy, discretization, iteration errors but uses fixed stepsize in homotopy.
Algorithm 2: Adaptivity in homotopy and iteration via stepsize control, discretization error is not decreased.
Algorithm 3: Adaptivity in the homotopy error, the discretization error, the iteration error including step size control.



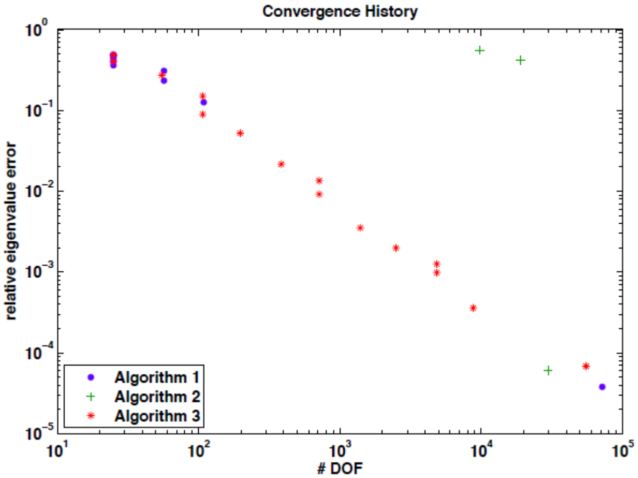


Figure: Conv. history of Algorithm 1, 2 and 3 with respect to #DOF.

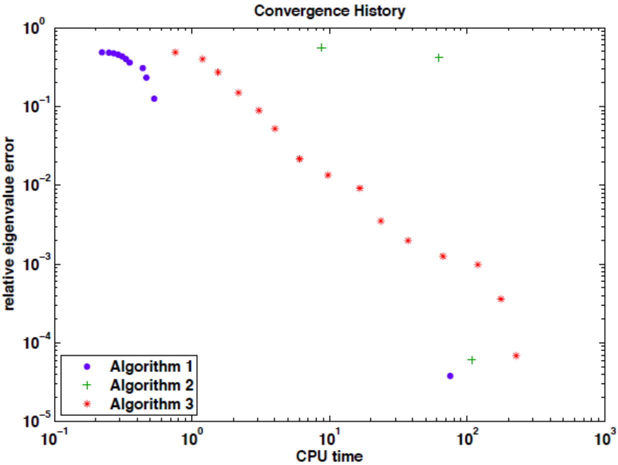


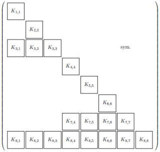
Figure: Conv. history of Algorithm 1, 2 and 3 with respect to CPU time.

Intermediate Conclusions

- Extension of backward error analysis to PDE case Miedlar 2011/2014
- Error estimates for hp-finite elements for non-self-adjoint PDE evps Giani/Grubisic/Miedlar/Ovall 2014
- Multiple evs self-adjoint case Galistil 2014
- **No results on multiple, complex evs, Jordan blocks in non-self-adjoint case**.
- Highly oscillatory eigenfunctions can only be captured with fine grids.
- Can we enrich the ansatz space with these eigenfunctions?

Outline

Compute smallest evs of self-adjoint evp $(\lambda M - K)x = 0$ with M, K pos. def. as in traditional approach. Bennighof-Lehouq 2004

- ▷ Use symmetric reordering of matrix to block form or use directly domain decomposition partition. $(\lambda \tilde{M} - \tilde{K})x = 0$, with structure 

- ▷ Compute block Cholesky factorization of $\tilde{M} = LDL^T$ and form $\hat{K} = L^{-1}\tilde{K}L^{-T}$.
- ▷ Compute smallest evs and evecs of 'substructure' evps $(\lambda D_{ii} - \hat{K}_{ii})x_i$ and project large problem (modal truncation).
- ▷ Solve projected evp.

Analysis of AMLS

- This produces locally global (spectral) ansatz functions in substructure.
- This is a domain decomposition approach, where efunctions are used in substructures.
- Substructure efunctions are sparsely represented in FE basis.
- Analysis only for self-adjoint case and real simple evs.
- Works extremely well for mechanical structures with little damping.
- Does not work for brake problem.

Conclusions III, a.

- Using fine mesh and MOR usually works, but is a waste.
- Using evecs, efunctions or singular values can be combined with other MOR approaches.
- Error estimates are needed for non-self-adjoint case, multiple evs, complex evs, in combination with MOR methods.
- Enrich FEM ansatz space with approximate/substructure eigenfunctions?

Conclusions III b.

- MOR is a topic to stay.
- Need to identify which problem we want to solve.
- Combination of adaptive FEM and AMLS type approaches with MOR methods.

Thank you very much for your attention.

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