Nečas Center for Mathematical Modeling

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ERC-CZ project LL1202 - MORE

MINISTRY OF EDUCATION, YOUTH AND SPORTS

Euler-Bernoulli Type Beam Theory for Elastic Material with Nonlinear Response in the Small Strain Range

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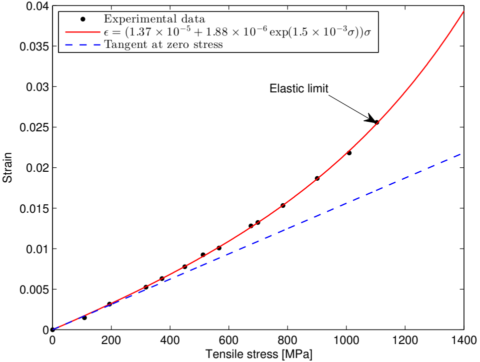


Figure : Strain-stress relation for a Gum metal alloy (Ti-12Ta-9Nb-3V-6Zr-1.5O).

Saito, T., et al. (2003). Multifunctional Alloys Obtained via a Dislocation-Free Plastic Deformation Mechanism. *Science 300* (5618), 464–467.

Euler-Bernoulli Type Beam Theory

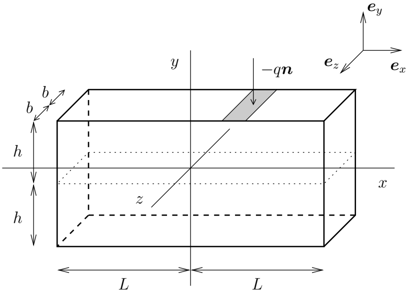


Figure : Problem geometry.

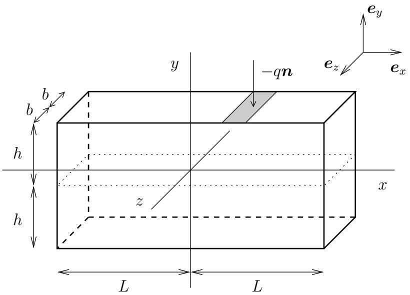
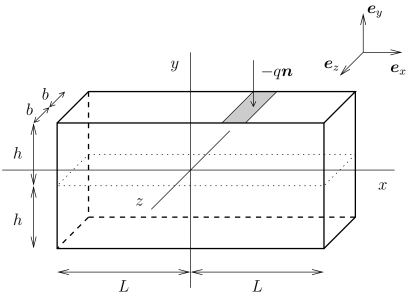


Figure : Problem geometry.

Stress field

$$\mathbb{T} = \begin{bmatrix} T_{\hat{x}\hat{x}} & T_{\hat{x}\hat{y}} & 0 \\ T_{\hat{x}\hat{y}} & T_{\hat{y}\hat{y}} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



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Governing equations

$$\frac{\mathrm{d}V}{\mathrm{d}x} - q = 0$$

$$\frac{\mathrm{d}M}{\mathrm{d}x} - V = 0$$

$V =_{\mathrm{def}} \int_{y=-h}^{h} \mathrm{T}_{\hat{x}\hat{y}} \, \mathrm{d}y$

$M =_{\mathrm{def}} \int_{y=-h}^{h} y\mathrm{T}_{\hat{x}\hat{x}} \, \mathrm{d}y$

Euler-Bernoulli Type Beam Theory

Small strain tensor

$$\varepsilon =_{\text{def}} \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top\right) \approx \begin{bmatrix} -y\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Small strain tensor

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- Two kinds of **lateral loads** acting on the beam
  - Uniform load

$$q(x) = q^* > 0$$

  - Concentrated load

$$q(x) = \tilde{q}\delta(x), \qquad \tilde{q} > 0$$

- **Linearized elasticity**

$$\mathbb{T} = \lambda\,(\operatorname{Tr}\varepsilon)\,\mathbb{I} + 2\mu\varepsilon$$

- Material with **nonlinear response**

$$\varepsilon = \lambda_1\,(\operatorname{Tr}\mathbb{T})\,\mathbb{I} + 2\lambda_2 e^{\eta\,\operatorname{Tr}\mathbb{T}}\,\mathbb{T}$$

$$\varepsilon = \lambda_1\,(\operatorname{Tr}\mathbb{T})\,\mathbb{I} + \lambda_2\left(1 + \alpha\,|\mathbb{T}|^2\right)^n \mathbb{T}$$

$$\varepsilon = \gamma_1\,(\operatorname{Tr}\mathbb{T})\,\mathbb{I} + \sinh\left[(\operatorname{Tr}\mathbb{T})^{\gamma_2}/\gamma_3\right]\mathbb{I} + \gamma_4\mathbb{T}$$

Rajagopal, K. R. (2014). On the nonlinear elastic response of bodies in the small strain range. *Acta Mech. 225* (6), 1545–1553.

Grasley, Z., R. El-Helou, M. D'Ambrosia, D. Mokarem, C. Moen, and K. R. Rajagopal (2015). Model of infinitesimal nonlinear elastic response of concrete subjected to uniaxial compression. *J. Eng. Mech. 141* (7), 04015008.

Numerical Solution

- System of **governing equations**: $\frac{\mathrm{d}^2 M}{\mathrm{d}x^2} = q$ $\quad \varepsilon_{\hat{x}\hat{x}} = -y \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\int_{y=-h}^{h} y \mathrm{T}_{\hat{x}\hat{x}} \,\mathrm{d}y\right) = q$$

$$y \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = -\left(\lambda_1 + 2\lambda_2 \mathrm{e}^{\eta \mathrm{T}_{\hat{x}\hat{x}}}\right) \mathrm{T}_{\hat{x}\hat{x}}$$

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- **Fixed ends** boundary conditions

$$w|_{x=\pm L} = \left.\frac{\mathrm{d}w}{\mathrm{d}x}\right|_{x=\pm L} = 0$$

Numerical Solution

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- **Spectral collocation method**
  - Gauss-Lobatto-Chebyshev points: $x_i = \cos\left(\frac{(i-1)\pi}{N-1}\right), \quad i = 1,\dots,N$
  - Spectral differentiation matrices
  - Clenshaw-Curtis quadrature

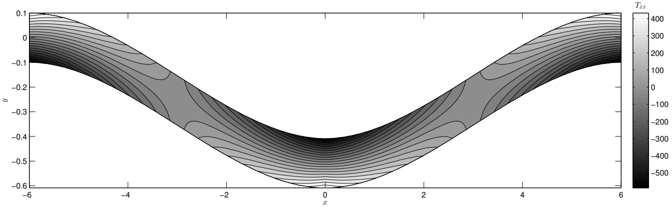
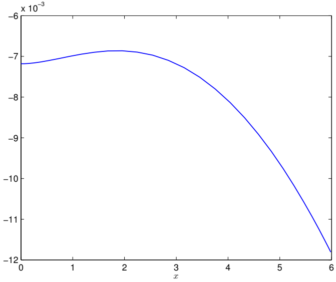
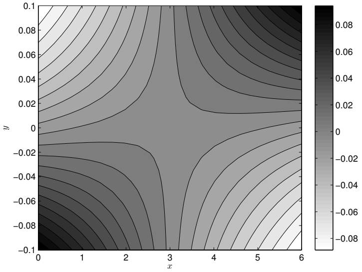


Figure : Stress field of a beam with nonlinear response under a concentrated load at $x = 0$, $L = 6\,\mathrm{m}$, $h = 0.1\,\mathrm{m}$, $\lambda_1 = 1.37 \times 10^{-5}\,\mathrm{MPa}^{-1}$, $\lambda_2 = 9.38 \times 10^{-7}\,\mathrm{MPa}^{-1}$, $\eta = 1.5 \times 10^{-3}\,\mathrm{MPa}^{-1}$, $\tilde{q} = 2.4\,\mathrm{MPa}$.



(a) Relative difference of the deflection $\left(w^{\mathrm{LIN}} - w\right)/w^{\mathrm{LIN}}$.



(b) Relative difference of the stress field $\left(\mathrm{T}_{\hat{x}\hat{x}}^{\mathrm{LIN}} - \mathrm{T}_{\hat{x}\hat{x}}\right)/\mathrm{T}_{\hat{x}\hat{x}}^{\mathrm{LIN}}$.

Figure : Comparison of the deflection and the stress field of the linearized elastic beam and a beam with nonlinear response under a concentrated load at $x = 0$, $L = 6\,\mathrm{m}$, $h = 0.1\,\mathrm{m}$, $\lambda_1 = 1.37 \times 10^{-5}\,\mathrm{MPa}^{-1}$, $\lambda_2 = 9.38 \times 10^{-7}\,\mathrm{MPa}^{-1}$, $\eta = 1.5 \times 10^{-3}\,\mathrm{MPa}^{-1}$, $\tilde{q} = 2.4\,\mathrm{MPa}$.

Thank You for Your Attention