Implicitly constituted materials: from modelling towards PDE-analysis

Josef Málek

Nečas Center for Mathematical Modeling
and
Mathematical Institute of Charles University in Prague
Faculty of Mathematics and Physics

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Project MORE

• Implicitly constituted material models: from theory through model reduction to efficient numerical methods (5 year ERC-CZ project MORE financed by the Ministry of Education, Youth and Sports since September 2012)

- Team members: M. Bulíček, E. Feireisl, J. Hron, V. Pruša, I. Pultarová, O. Souček, E. Süli (Oxford), Z. Strakoš, M. Vohralík (INRIA), M. Tuma
- Advisory board members: M. Benzi (Atlanta), K.R. Rajagopal (Texas A&M University), R. Rannacher (Heidelberg), G. Seregin (Oxford, St. Petersburg)
- two postdoc positions since February 2013 taken by 6 postdocs in 2013: Y. Lu, G. Tierra, M. Pokojovy
- six Ph.D. scholarships: J. Blechta, T. Gergelits, A. Janečka, J. Papež, M. Řehoř, J. Žabenský [M. Netušil, V. Orava, K. Tuma]

Fluid and Solid Mechanics

- concept of continuum
- balance of equations
  - conservation of mass, energy
  - principles of classical Newtonian mechanics applied to subsets of the body: $\frac{d}{dt}(m\mathbf{v}) = \mathbf{F}$ with $\mathbf{v} = \frac{d\chi}{dt}$
- boundary conditions
- initial conditions

Insufficient to predict the deformation/flow/evolution of the body

Contents

- Examples of standard linear constitutive equations
- Implicit constitutive theory and its advantages
- Commercial break
- Impact of implicit constitutive theory on PDE analysis of initial and boundary value poblems

A. Compressible Navier-Stokes equations

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0$$

$$\frac{\partial(\rho \mathbf{v})}{\partial t} + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v}) = \operatorname{div} \mathbb{T} + \rho \mathbf{b}$$

$$\mathbb{T} = -p(\rho)\mathbb{I} + 2\nu\mathbb{D} + \lambda(\operatorname{div} \mathbf{v})\mathbb{I}$$

$\mathbb{T}$ is the Cauchy stress $\qquad \mathbb{D} = \frac{1}{2}\left[(\nabla \mathbf{v}) + (\nabla \mathbf{v})^T\right]$

$$m := \frac{1}{3} \operatorname{Tr} \mathbb{T} \qquad \mathbb{S} := \mathbb{T} - \frac{1}{3}(\operatorname{Tr} \mathbb{T})\mathbb{I} \qquad \mathbb{A}_\delta := \mathbb{A} - \frac{1}{3}(\operatorname{Tr} \mathbb{A})\mathbb{I}$$

$$m + p(\rho) = \frac{2\nu + 3\lambda}{3} \operatorname{div} \mathbf{v} \qquad m + p(\rho) \sim \operatorname{div} \mathbf{v}$$

$$\mathbb{S} = 2\nu\mathbb{D}_\delta \qquad \mathbb{S} \sim \mathbb{D}_\delta$$

B. Incompressible Navier-Stokes equations

$$\operatorname{div} \mathbf{v} = 0$$

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \operatorname{div}(\mathbf{v}\otimes\mathbf{v})\right) = \operatorname{div}\mathbb{T} + \rho\mathbf{b} = \nabla m + \operatorname{div}\mathbb{S} + \rho\mathbf{b}$$

$$\mathbb{S} = 2\nu\mathbb{D} \qquad \mathbb{S} \sim \mathbb{D}$$

Boundary conditions

- $\mathbf{v}\cdot\mathbf{n} = 0$ on $\partial\Omega$
- constitutive equation involving $\mathbf{v}_\tau$ and/or $(-\mathbb{T}\mathbf{n})_\tau$

$\mathbf{s} := (-\mathbb{T}\mathbf{n})_\tau \qquad \mathbf{z}_\tau := \mathbf{z} - (\mathbf{z}\cdot\mathbf{n})\mathbf{n}$

$\mathbb{T}_\delta = 2\nu_*\mathbb{D}$ with $\nu_* > 0$	$\mathbb{S} \sim \mathbb{D}$ Navier-Stokes fluid
$\mathbf{s} = \gamma_*\mathbf{v}_\tau$ with $\gamma_* > 0$	$\mathbf{s} \sim \mathbf{v}_\tau$ Navier's slip boundary condition

C. Compressible elastic neo-Hookean solid

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0$$

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v}) = \operatorname{div} \mathbb{T} + \rho \mathbf{b}$$

$$\mathbb{T} = \mu(\mathbb{B} - \mathbb{I}) \qquad \mathbb{T} \sim \mathbb{B}$$

$$\frac{\partial \mathbb{B}}{\partial t} + \nabla \mathbb{B} \cdot \mathbf{v} - (\nabla \mathbf{v})\mathbb{B} - \mathbb{B}(\nabla \mathbf{v})^T = \mathbb{O}$$

$\mathbb{B}$ is the left Cauchy-Green stretch tensor $\qquad \mathbb{B} = (\mathbb{I} + \nabla \mathbf{u})\left(\mathbb{I} + (\nabla \mathbf{u})^T\right)$

For static problems assuming $|\nabla \mathbf{u}| = \delta \ll 1$

$$\operatorname{div} \mathbb{T} = \mathbf{0}$$

$$\mathbb{T} = 2\mu\, \varepsilon(\mathbf{u}) + \sigma(\operatorname{div} \mathbf{u})\mathbb{I} \qquad \varepsilon(\mathbf{u}) = \tfrac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right)$$

D. Flows through porous rigid media

$$\frac{\partial \rho^f}{\partial t} + \operatorname{div}\left(\rho^f \mathbf{v}^f\right) = 0$$
$$\frac{\partial \rho^s}{\partial t} + \operatorname{div}\left(\rho^s \mathbf{v}^s\right) = 0$$
$$\frac{\partial (\rho^f \mathbf{v}^f)}{\partial t} + \operatorname{div}\left(\rho^f \mathbf{v}^\mathbf{f} \otimes \mathbf{v}^\mathbf{f}\right) = \operatorname{div} \mathbb{T}^f + \rho^f \mathbf{b} + \mathbf{m}$$
$$\frac{\partial (\rho^s \mathbf{v}^s)}{\partial t} + \operatorname{div}\left(\rho^s \mathbf{v}^\mathbf{s} \otimes \mathbf{v}^\mathbf{s}\right) = \operatorname{div} \mathbb{T}^s + \rho^s \mathbf{b} - \mathbf{m}$$

Simplifications leading to Brinkman-Darcy model

- solid is rigid
- density of the fluid is constant

$$\operatorname{div} \mathbf{v} = 0$$
$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \operatorname{div}\left(\mathbf{v} \otimes \mathbf{v}\right)\right) = \nabla m + \operatorname{div} \mathbb{S} + \mathbf{m} + \rho \mathbf{b}$$
$$\mathbb{S} = 2\nu_* \mathbb{D} \text{ with } \nu_* > 0 \qquad \mathbb{S} \sim \mathbb{D}$$
$$\mathbf{m} = \alpha_* \mathbf{v} \text{ with } \alpha_* > 0 \qquad \mathbf{m} \sim \mathbf{v}$$

Constitutive relations

Linear constitutive relations

- compressible Navier-Stokes fluids $m + p(\varrho) \sim \operatorname{div} \mathbf{v}$ $\mathbb{T}_\delta \sim \mathbb{D}_\delta$
- incompressible Navier-Stokes fluids $\mathbb{T}_\delta \sim \mathbb{D}$
- Navier's slip boundary conditions $\mathbf{s} \sim \mathbf{v}_\tau$
- compressible neo-Hookean solid $\mathbb{T} \sim \mathbb{B}$
- Brinkman-Darcy's equation $\mathbb{T}_\delta \sim \mathbb{D}$ $\mathbf{m} \sim \mathbf{v}$

Implicit constitutive relations

- compressible fluids $g(m + p(\varrho), \operatorname{div} \mathbf{v}) = 0$ $\mathfrak{g}(\mathbb{T}_\delta, \mathbb{D}_\delta) = \mathbb{O}$
- incompressible fluids $\mathfrak{g}(\mathbb{T}_\delta, \mathbb{D}) = \mathbb{O}$
- boundary conditions $\mathbf{h}(\mathbf{s}, \mathbf{v}_\tau) = \mathbf{0}$
- compressible solid $\mathfrak{g}(\mathbb{T}, \mathbb{B}) = \mathbb{O}$
- flows through porous media $g(\mathbb{T}_\delta, \mathbb{D}) = \mathbb{O}$ $\mathbf{h}(\mathbf{m}, \mathbf{v}) = \mathbf{0}$

Is this apparently simple generalization useful?

Implicit constitutive theory includes two explicit

Standard approach:
Stress/force is a function of kinematical variables.

$$\mathbb{T} = \mathfrak{g}(\mathbb{B}) \qquad \mathbb{S} = \mathfrak{f}(\mathbb{D}) \qquad \mathbf{s} = \mathbf{h}(\mathbf{v}_\tau)$$

Approach via implicit constitutive theory provides a useful alternative:
Kinematical variable is a function of stress/force.

$$\mathbb{B} = \mathfrak{g}(\mathbb{T}) \qquad \mathbb{D} = \mathfrak{f}(\mathbb{S}) \qquad \mathbf{v}_\tau = \mathbf{h}(\mathbf{s})$$

K. R. Rajagopal: On implicit constitutive theories, *Appl. Math.*, Vol. 48, pp. 279–319 (2003)

K. R. Rajagopal: Elasticity of elasticity, *Z. Angew. Math. Phys.*, Vol. 58, pp. 309–417 (2007)

K. R. Rajagopal: On the nonlinear elastic response of bodies in the small strain range, *Acta Mechanica*, Vol. 225, pp. 1545–1553 (2014)

K. R. Rajagopal, A. R. Srinivasa: On the thermodynamics of fluids defined by implicit constitutive relations, *Z. Angew. Math. Phys.*, Vol. 59, pp. 715–729 (2008)

J. Málek, K. R. Rajagopal: Compressible generalized Newtonian fluids, *Z. Angew. Math. Phys.*, Vol. 61, pp. 1097–1110 (2010)

S. Srinivasan, K. R. Rajagopal: A thermodynamic basis for the derivation of the Darcy, Forchheimer and Brinkman models for flows through porous media and their generalizations, *International J. Non-linear Mechanics*, Vol. 58, pp. 162–166 (2014)

Examples where an alternative approach is very useful

- incompressible fluids with pressure dependent viscosity (and porosity)
- solids with `bounded` linearized strain
- compressible fluids with `bounded` divergence of the velocity
- activated materials such as Bingham fluids
- stick-slip boundary conditions

Further examples of nonlinear incompressible fluids (by graphs and formulas), paying a special attention to symmetric role of $\mathbb{S}$ and $\mathbb{D}$

Incompressible fluids with pressure dependent viscosity

$$\boxed{\mathbb{S} = 2\mu_* \exp(\alpha_* p)\mathbb{D}}$$ - Barus fluids (1893) $\qquad p = m := \frac{1}{3}\operatorname{Tr}\mathbb{T}$

$$\mathbb{T} = \mathfrak{f}(\mathbb{D}) \implies \mathbb{T} = \Phi\mathbb{I} + \hat{\mathfrak{f}}(\mathbb{D})$$

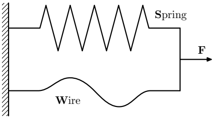
$\implies$ the viscosity can depend only on $\operatorname{Tr}\mathbb{D}^2$ and $\operatorname{Tr}\mathbb{D}^3$!!!

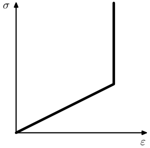
$\mathbb{D} = \mathfrak{f}(\mathbb{T}) \implies$ the viscosity can depend $\boxed{\operatorname{Tr}\mathbb{T}}$, $\operatorname{Tr}\mathbb{T}^2$ and $\operatorname{Tr}\mathbb{T}^3$

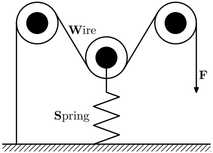
Solids with bounded linearized strain

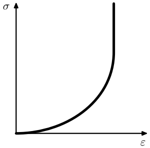
Is it possible to have models with a nonlinear relation between $\mathbb{T}$ and $\varepsilon(\mathbf{u})$?
Why?

- behavior of brittle materials near the crack tip
- concentrated loads resulting at small strain deformation
- to capture response analogous to mechanical analogs:









K. R. Rajagopal: Elasticity of Elasticity, *Z. Angew. Math. Physics*, Vol. 58, pp. 309–317 (2007)

Solids with bounded linearized strain

Is it possible to have models with a nonlinear relation between $\mathbb{T}$ and $\varepsilon(\mathbf{u})$?

$\mathbb{T} = \mathfrak{f}(\mathbb{B}) \implies$ linear Hooke's elastic solid !!!

$\mathbb{B} = \mathfrak{f}(\mathbb{T})$ and $|\nabla \mathbf{u}| = \delta \ll 1 \implies \varepsilon(\mathbf{u}) = \hat{\mathfrak{f}}(\mathbb{T})$

Rajagopal's "strain-limiting" model

$$\varepsilon(\mathbf{u}) = \beta \left(1 - \exp \frac{-\lambda \operatorname{Tr} \mathbb{T}}{(1 + |\mathbb{T}|^{b})^{1/b}}\right) \mathbb{I} + \frac{\mathbb{T}}{2\mu\,(1 + \kappa |\mathbb{T}|^{a})^{1/a}}$$

a, b, β, λ, μ, κ are positive constants

Linearization gives Hooke's linear elasticity model

A. D. Freed: *Soft Solids: A Primer to the Theoretical mechanics of Materials* (2012)

Compressible fluids with bounded $\operatorname{div} \mathbf{v}$

Implicit relations

$$g(m + p(\varrho), \operatorname{div} \mathbf{v}) = 0 \qquad \mathfrak{g}(\mathbb{T}_\delta, \mathbb{D}_\delta) = \mathbb{O}$$

includes

$$\operatorname{div} \mathbf{v} = \frac{1}{b} \frac{m + p(\rho)}{(1 + |m + p(\rho)|^a)^{1/a}} \qquad a > 0, b > 0$$
$$\mathfrak{g}(\mathbb{T}_\delta, \mathbb{D}_\delta) = \mathbb{O}$$

which implies

$$|\operatorname{div} \mathbf{v}| < \frac{1}{b}$$

Consequently,

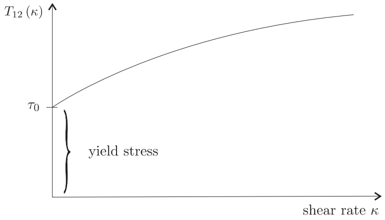
$$0 < \rho_* \le \rho(0, \cdot) \le \rho^* \implies 0 < \rho_* \exp(-\frac{t}{b}) \le \rho(t, \cdot) \le \rho^* \exp(-\frac{t}{b})$$

E. Feireisl, X. Liao, J. Málek: *Global weak solutions to a class of non-Newtonian compressible fluids* online first in Mathematical Methods in the Applied Sciences (2015)

Bingham fluids

Standard description

$$|\mathbb{S}| \le \tau_* \Leftrightarrow \mathbb{D} = \mathbb{O}$$
$$|\mathbb{S}| > \tau_* \Leftrightarrow \mathbb{S} = \tau_* \frac{\mathbb{D}}{|\mathbb{D}|} + 2\nu_* \mathbb{D}$$



is equivalent to

$$\boxed{\mathbb{D} = \frac{1}{2\nu_*} \frac{(|\mathbb{S}| - \tau_*)^+}{|\mathbb{S}|} \mathbb{S}}$$

$$x^+ = \max\{x, 0\},\ \nu_* > 0,\ \tau_* \ge 0$$

$$\mathfrak{f}(\mathbb{S}, \mathbb{D}) = \mathbb{D} - \frac{1}{2\nu_*} \frac{(|\mathbb{S}| - \tau_*)^+}{|\mathbb{S}|} \mathbb{S}$$

Stick-slip boundary conditions

Threshold slip condition usually written as

$$\mathbf{s} := (\mathbb{S}\mathbf{n})_\tau$$

$$|\mathbf{s}| \leq \sigma_* \Leftrightarrow \mathbf{v}_\tau = \mathbf{0}$$
$$|\mathbf{s}| > \sigma_* \Leftrightarrow \mathbf{s} = \sigma_* \frac{\mathbf{v}_\tau}{|\mathbf{v}_\tau|} + \gamma_* \mathbf{v}_\tau$$

can be equivalently rewritten as

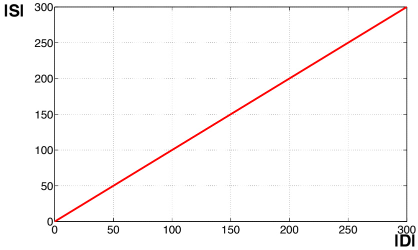
$$\gamma_* \mathbf{v}_\tau = \frac{(|\mathbf{s}| - \sigma_*)^+}{|\mathbf{s}|}\mathbf{s}$$

$x^+ = \max\{x, 0\}$, $\gamma_* > 0$, $\sigma_* \geq 0$

$$\mathbf{h}(\mathbf{s}, \mathbf{v}_\tau) = \gamma_* \mathbf{v}_\tau - \frac{(|\mathbf{s}| - \sigma_*)^+}{|\mathbf{s}|}\mathbf{s}$$

Examples of constitutive relations

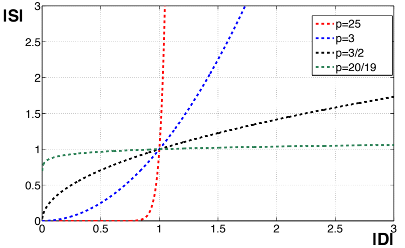
$$\mathbb{S} = \mathbb{D}$$



$$\mathbb{S} = |\mathbb{D}|^{p-2}\mathbb{D}$$

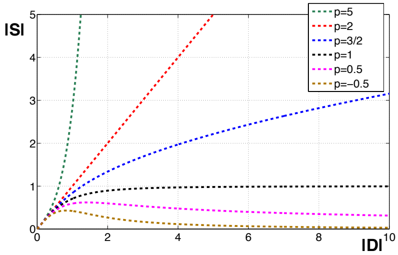
or

$$\mathbb{D} = |\mathbb{S}|^{\frac{2-p}{p-1}}\mathbb{S}$$

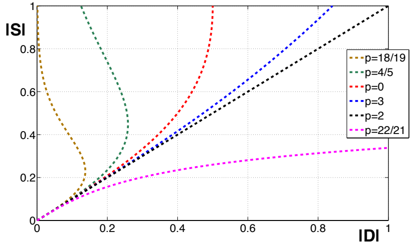


Examples of constitutive relations

$$\mathbb{S} = (1 + |\mathbb{D}|^2)^{\frac{p-2}{2}} \mathbb{D}$$

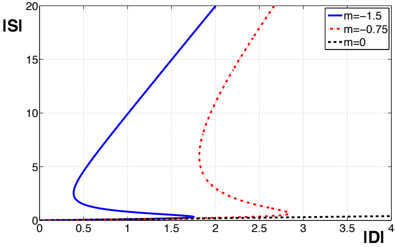


$$\mathbb{D} = (1 + |\mathbb{S}|^2)^{\frac{2-p}{2(p-1)}} \mathbb{S}$$



Examples of constitutive relations

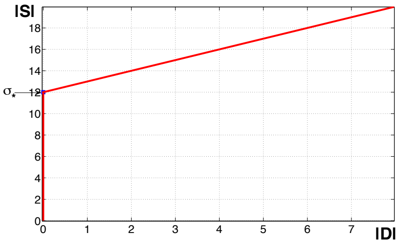
$$\mathbb{D} = \mathbb{S} + (1 + |\mathbb{S}|^2)^{\frac{2-p}{2(p-1)}}\mathbb{S}$$



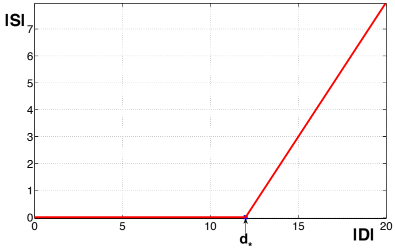
G. Tierra

Examples of constitutive relations

$$\mathbb{D} = \frac{(|\mathbb{S}| - \sigma_*)^+}{|\mathbb{S}|}\mathbb{S}$$

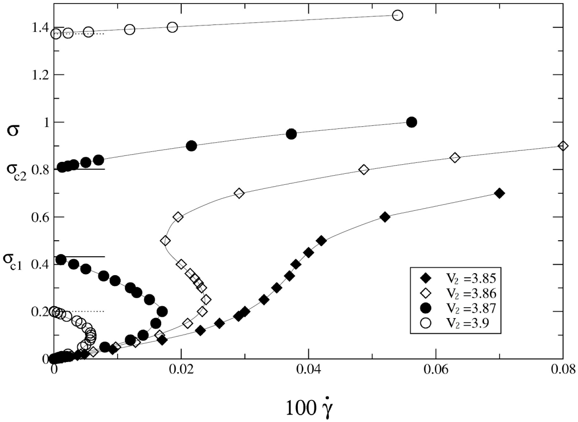


$$\mathbb{S} = \frac{(|\mathbb{D}| - d_*)^+}{|\mathbb{D}|}\mathbb{D}$$

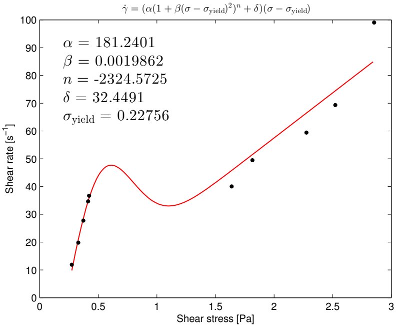


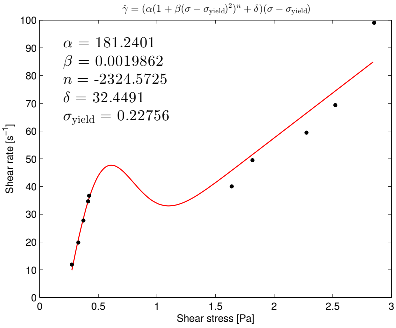
Experimental data for colloidal suspensions

Can one describe such non-monotone response of fluid-like materials?



C. B. Holmes, M. E. Cates, M. Fuchs, P. Sollich: Glass transitions and shear thickening suspension rheology, *J. Rheology*, Vol. 49, pp. 237–269 (2005)



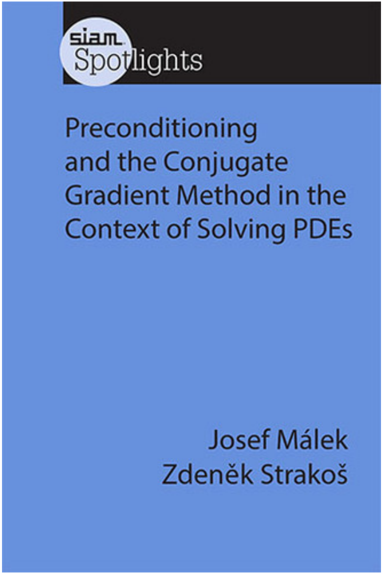


(Data fitted by Adam Janečka, Tereza Perlácová and Vít Pruša.)

T. Perlácová, V. Pruša: Tensorial implicit constitutive relations in mechanics of incompressible non-Newtonian fluids , *J. Non-Newton. Fluid Mech.*, Vol. 216, pp. 13–21 (2015)

Commercial break

- link PDE via functional analysis description with the finite dimensional computation
- preconditioning viewed as a natural object of functional analysis
- conjugate gradient method viewed as a model reduction of the original infinite dimensional problem to n-dimensional algebraic problem matching the first $2n$ moments
- un-preconditioned conjugate gradient method is understood as an oxymoron



PROBLEM

$$\left.\begin{aligned} \operatorname{div}\mathbf{v} &= 0 \\ \frac{\partial \mathbf{v}}{\partial t} + \operatorname{div}(\mathbf{v}\otimes\mathbf{v}) - \operatorname{div}\mathbb{S} &= -\nabla p + \mathbf{b} \\ \mathfrak{f}(\mathbb{S},\mathbb{D}) &= \mathbb{O} \end{aligned}\right\} \text{ in } Q_T$$

$$\left.\begin{aligned} \mathbf{v}\cdot\mathbf{n} &= 0 \\ \mathbf{h}(\mathbf{s},\mathbf{v}_\tau) &= \mathbf{0} \end{aligned}\right\} \text{ on } \Sigma_T$$

$$\mathbf{v}(0,\cdot) = \mathbf{v}_0 \qquad \text{in } \Omega$$

DATA

- $\Omega \subset \mathbb{R}^d$ bounded open connected set with $\partial\Omega \in \mathcal{C}^{1,1}(d=3)$
- $T > 0$ and $Q_T := (0,T)\times\Omega$, $\Sigma_T := (0,T)\times\partial\Omega$
- $\mathbf{v}_0$, $\mathbf{b}$
- $\mathfrak{f}$ and $\mathbf{h}$ - constitutive functions in Q_T and on Σ_T

Main questions addressed

UNKNOWNS triplet $(\mathbf{v}, p, \mathbb{S})$ defined on Q_T and $\mathbf{s}$ defined on Σ_T

AIM

- To define object useful for computation and show its existence
- To establish large data existence of solution for any set of data (Ω, T, $\mathbf{v}_0$, $\mathbf{b}$) and for robust class of constitutive equations described by $\mathfrak{f}$ and $\mathbf{h}$
- To develop theory towards the class of models described through

$$\mathbb{D} = \alpha(\mathrm{Tr}\,\mathbb{T}, \mathrm{Tr}\,\mathbb{S}^2, \mathrm{tr}\,\mathbb{D}^2)\mathbb{S}$$

- To develop a theory with integrable p

Implicit formulation - maximal monotone r-graph setting

$$\boxed{(\mathbb{S},\mathbb{D}) \in \mathcal{A} \quad \Longleftrightarrow \quad \mathfrak{f}(\mathbb{S},\mathbb{D}) = \mathbb{O}}$$

Assumptions ($\mathcal{A}$ is a maximal monotone r-graph with $r = 2$):

(A1) $(\mathbb{O},\mathbb{O}) \in \mathcal{A}$

(A2) **Monotone graph:** For any $(\mathbb{S}^1,\mathbb{D}^1),(\mathbb{S}^2,\mathbb{D}^2) \in \mathcal{A}$

$$(\mathbb{S}^1 - \mathbb{S}^2)\cdot(\mathbb{D}^1 - \mathbb{D}^2) \geq 0$$

(A3) **Maximal monotone graph:** Let $(\mathbb{S},\mathbb{D}) \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^{d\times d}_{\mathrm{sym}}$.

$$\text{If } (\mathbb{S}-\tilde{\mathbb{S}})\cdot(\mathbb{D}-\tilde{\mathbb{D}}) \geq 0 \quad \forall\,(\tilde{\mathbb{S}},\tilde{\mathbb{D}}) \in \mathcal{A} \text{ then } (\mathbb{S},\mathbb{D}) \in \mathcal{A}$$

(A4) **r-graph with $r = 2$:** There are $\alpha_* > 0$, $c_* \geq 0$ so that for any $(\mathbb{S},\mathbb{D}) \in \mathcal{A}$

$$\mathbb{S}\cdot\mathbb{D} \geq \alpha_*\left(|\mathbb{D}|^r + |\mathbb{S}|^{r'}\right) - c_*$$

Implicit formulation - maximal monotone 2-graph setting

$$\boxed{(\mathbf{s}, \mathbf{v}_\tau) \in \mathcal{B} \iff \mathbf{h}(\mathbf{s}, \mathbf{v}_\tau) = \mathbf{0}}$$

(B1) $\mathcal{B}$ **contains the origin.** $(\mathbf{0}, \mathbf{0}) \in \mathcal{B}$.

(B2) $\mathcal{B}$ **is a monotone graph.**

$$(\mathbf{s}^1 - \mathbf{s}^2) \cdot (\mathbf{v}_\tau^1 - \mathbf{v}_\tau^2) \geq 0 \qquad \text{for all } (\mathbf{s}^1, \mathbf{v}_\tau^1), (\mathbf{s}^2, \mathbf{v}_\tau^2) \in \mathcal{B}.$$

(B3) $\mathcal{B}$ **is a maximal monotone graph.** Let for some $(\mathbf{s}, \mathbf{u})$ holds:

$$\text{If} \quad (\bar{\mathbf{s}} - \mathbf{s}) \cdot (\bar{\mathbf{v}}_\tau - \mathbf{u}) \geq 0 \quad \text{for all } (\bar{\mathbf{s}}, \bar{\mathbf{v}}_\tau) \in \mathcal{B} \quad \text{then } (\mathbf{s}, \mathbf{u}) \in \mathcal{B}.$$

(B4) $\mathcal{B}$ **is a** 2**-graph.** There are $d_* > 0$ and $n_* \geq 0$ such that

$$\mathbf{s} \cdot \mathbf{v}_\tau \geq -c_* + d_*(|\mathbf{v}_\tau|^2 + |\mathbf{s}|^2) \quad \text{for all } (\mathbf{s}, \mathbf{v}_\tau) \in \mathcal{B}\,.$$

No-slip boundary condition is excluded by **(B4)**.

Theorem

Let $\Omega \subset \mathbb{R}^d$ be a $\mathcal{C}^{1,1}$ domain. Then for any $\mathbf{v}_0 \in L^2_{0,\mathrm{div}}$ there exists

$$\mathbf{v} \in L^\infty(0,T; L^2(\Omega)^d) \cap L^2(0,T; W^{1,2}_{\mathbf{n},\mathrm{div}})$$
$$\mathbb{S} \in L^2(Q_T)^{d\times d}_{sym}, \quad \mathbf{s} \in L^2(\Sigma_T)^d$$
$$p_1 \in L^2(Q_T), \quad p_2 \in L^{\frac{d+2}{d+1}}(0,T; W^{1,\frac{d+2}{d+1}}(\Omega))$$

solving for almost all time $t \in (0,T)$ and for all $\mathbf{w} \in W^{1,\infty}_{\mathbf{n}}$

$$\langle \mathbf{v}', \mathbf{w}\rangle - \int_\Omega (\mathbf{v}\otimes\mathbf{v})\cdot\nabla\mathbf{w} + \int_\Omega \mathbb{S}\cdot\mathbb{D}(\mathbf{w}) + \int_{\partial\Omega} \mathbf{s}\cdot\mathbf{w} = \int_\Omega (p_1+p_2)\,\mathrm{div}\,\mathbf{w}$$

and fulfilling

$$\mathfrak{f}(\mathbb{S}, \mathbb{D}(\mathbf{v})) = \mathbb{O} \ \ a.e. \ in \ Q_T \quad and \quad \mathbf{h}(\mathbf{s}, \mathbf{v}_\tau) = \mathbf{0} \ \ a.e. \ in \ \Sigma_T$$

M. Bulíček, J. Málek: On unsteady internal flows of Bingham fluids subject to threshold slip on the impermeable boundary, *accepted for publication* in "Recent Developments of Mathematical Fluid Mechanics", series: Advances in Mathematical Fluid Mechanics, Birkhauser-Verlag (2014), Preprint MORE/2013/06.

Implicit formulation - maximal monotone ψ-graph or r-graph setting

$$\boxed{(\mathbb{S},\mathbb{D}) \in \mathcal{A} \quad \Longleftrightarrow \quad \mathfrak{f}(\mathbb{S},\mathbb{D}) = \mathbb{O}}$$

Assumptions ($\mathcal{A}$ is a maximal monotone ψ-graph):

(A1) $(\mathbb{O},\mathbb{O}) \in \mathcal{A}$

(A2) **Monotone graph:** For any $(\mathbb{S}^1,\mathbb{D}^1),(\mathbb{S}^2,\mathbb{D}^2) \in \mathcal{A}$

$$(\mathbb{S}^1 - \mathbb{S}^2)\cdot(\mathbb{D}^1 - \mathbb{D}^2) \geq 0$$

(A3) **Maximal monotone graph:** Let $(\mathbb{S},\mathbb{D}) \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^{d\times d}_{\mathrm{sym}}$.

$$\text{If } (\mathbb{S} - \tilde{\mathbb{T}}_\delta)\cdot(\mathbb{D} - \tilde{\mathbb{D}}) \geq 0 \quad \forall\,(\tilde{\mathbb{T}}_\delta,\tilde{\mathbb{D}}) \in \mathcal{A} \text{ then } (\mathbb{S},\mathbb{D}) \in \mathcal{A}$$

(A4) ψ**-graph:** There are $\alpha_* > 0, c_* \geq 0$ so that for any $(\mathbb{S},\mathbb{D}) \in \mathcal{A}$

$$\mathbb{S}\cdot\mathbb{D} \geq \alpha_*\left(\psi(|\mathbb{D}|) + \psi^*(|\mathbb{S}|)\right) - c_*$$

Main result

Theorem

Let $\Omega \subset \mathbb{R}^d$ and $\mathcal{A}$ satisfy the assumptions **(A1)–(A4)** *with ψ fulfilling*

$$\boxed{c_1 s^r - c_2 \le \psi(s) \le c_3 s^{\tilde{r}} + c_4 \quad \textit{with } r > \frac{2d}{d+2}}$$

Then for any $\Omega \in \mathcal{C}^{1,1}$ and $T \in (0, \infty)$ and for arbitrary

$$\mathbf{v}_0 \in L^2_{0,\mathrm{div}}, \quad \mathbf{b} \in L^2(0, T; L^2(\Omega)^d) \ \textit{and} \ \sigma_* \ge 0, \gamma_* > 0$$

there exists weak solution to Problem.

M. Bulíček, P. Gwiazda, J. Málek, A. Świerczewska-Gwiazda: On Unsteady Flows of Implicitly Constituted Incompressible Fluids, *SIAM J. Math. Anal.*, Vol. 44, No. 4, pp. 2756–2801 (2012)

L. Diening, Ch. Kreuzer, E. Süli: Finite element approximation of steady flows of incompressible fluid with implicit power-law-like rheology, *SIAM J. Numer. Anal.*, Vol. 51, pp. 984-1015 (2013)

Comments to the result

Novel tools:

- Structural assumptions **(A1)–(A4)** on $\mathfrak{f}(\mathbb{S}, \mathbb{D}) = \mathbb{O}$
- Convergence lemma (its local character) $\Longrightarrow$ strict monotonicity not needed
- Understanding the interplay between the chosen boundary conditions and *global* integrability of p
- Lipschitz approximations of Sobolev-Orlicz and Bochner functions

D. Breit, L. Diening, S. Schwarzacher: Solenoidal Lipschitz truncation for parabolic PDE's *Mathematical Models and Methods in Applied Sciences (M3AS)*, Vol. 23, No. 14, pp. 2671-2700 (2013)

Generalized Minty's method - Convergence lemma

Lemma

Let $U \subset Q_T$ be arbitrary (measurable) and $r \in (1, \infty)$. Assume that

- *$\mathcal{A}$ is a maximal monotone graph (satisfying* **(A2)–(A3)***)*
- *$\{\mathbb{S}^n\}_{n=1}^{\infty}$ and $\{\mathbb{D}^n\}_{n=1}^{\infty}$ satisfy*

$$
\begin{aligned}
(\mathbb{S}^n, \mathbb{D}^n) &\in \mathcal{A} && \textit{for a.a. } (t,x) \in U \\
\mathbb{D}^n &\rightharpoonup \mathbb{D} && \textit{weakly in } L^r(U)^{d\times d} \\
\mathbb{S}^n &\rightharpoonup \mathbb{S} && \textit{weakly in } L^{r'}(U)^{d\times d}
\end{aligned}
$$

$$
\limsup_{n\to\infty} \int_U \mathbb{S}^n \cdot \mathbb{D}^n \, dx\, dt \le \int_U \mathbb{S} \cdot \mathbb{D} \, dx\, dt.
$$

Then

$$
(\mathbb{S}, \mathbb{D}) \in \mathcal{A} \quad \textit{almost everywhere in } U.
$$

- *Local* version
- Last assumption suggests to use energy (entropy) inequality

Generalized Darcy-Forchheimer's equations

To find $(\mathbf{m}, \mathbf{v}, p)$:

$$\boxed{\begin{aligned} \nabla p + \mathbf{m} &= \mathbf{f} && \text{in } \Omega \\ \operatorname{div} \mathbf{v} &= 0 && \text{in } \Omega \\ \mathbf{h}(\mathbf{m}, \mathbf{v}, p) &= \mathbf{0} && \text{in } \Omega \\ (\mathbf{v} - \mathbf{v}_0) \cdot \mathbf{n} &= 0 && \text{on } \Gamma_1 \\ p - p_0 &= 0 && \text{on } \Gamma_2 \end{aligned}}$$

To find $(\mathbf{m}, \mathbf{v}, p)$:

$$\boxed{\begin{aligned} \nabla p + \mathbf{m} &= \mathbf{f} && \text{in } \Omega \\ \operatorname{div} \mathbf{v} &= 0 && \text{in } \Omega \\ \mathbf{h}(\mathbf{m}, \mathbf{v}, p) &= \mathbf{0} && \text{in } \Omega \\ (\mathbf{v} - \mathbf{v}_0) \cdot \mathbf{n} &= 0 && \text{on } \Gamma_1 \\ p - p_0 &= 0 && \text{on } \Gamma_2 \end{aligned}}$$

- Example: $\boxed{\mathbf{m} = \alpha_0 \exp(\alpha_1 p)|\mathbf{v}|\mathbf{v}}$

Theorem

Let $\Omega \subset \mathbb{R}^d$ be Lipschitz and the implicit relation (parametrized by p) generate maximal monotone r-graph with $r \in (1, \infty)$. Then there is a weak solution to the problem.

Evenmore, if $\mathbf{f} = \nabla g$ and the graph is strictly monotone at the origin, $\mathbf{v}_0 \cdot \mathbf{n} = 0$ on Γ_1, then the pressure is bounded.

M. Bulíček, J. Málek, J. Žabenský: A generalization of the Darcy-Forchheimer equation involving an implicit, pressure-dependent relation between the drag force and the velocity *J. Math. Anal. Appl.*, Vol. 424, No. 1, pp. 785-801 (2015)

Generalized evolutionary Brinkman's-Darcy equations

$$
\begin{aligned}
\partial_t \mathbf{v} - \operatorname{div}(2\nu(p, |\mathbb{D}(\mathbf{v})|^2)\mathbb{D}(\mathbf{v})) + \nabla p + \alpha(p, |\mathbf{v}|, |\mathbb{D}(\mathbf{v})|^2)\mathbf{v} &= \mathbf{f} && \text{in } \Omega \\
\operatorname{div} \mathbf{v} &= 0 && \text{in } \Omega \\
(\mathbf{v} - \mathbf{v}_0) \cdot \mathbf{n} &= 0 && \text{on } \Gamma_1 \\
p - p_0 &= 0 && \text{on } \Gamma_2
\end{aligned}
$$

M. Bulíček and J. Žabenský

Limiting strain model

Rajagopal's limiting strain model

$$\varepsilon(\mathbf{u}) = \beta \left(1 - \exp \frac{-\lambda \operatorname{Tr} \mathbb{T}}{(1 + |\mathbb{T}|^b)^{1/b}}\right) \mathbb{I} + \frac{\mathbb{T}}{2\mu\,(1 + \kappa|\mathbb{T}|^{a})^{1/a}}$$

a, b, β, λ, μ, κ are positive constants

Simplification

$$\varepsilon(\mathbf{u}) = \frac{\mathbb{T}}{(1 + |\mathbb{T}|^{a})^{1/a}} \implies \boxed{\varepsilon(\mathbf{u}) \in L^\infty \qquad \mathbb{T} \in L^1}$$

Problem formulation

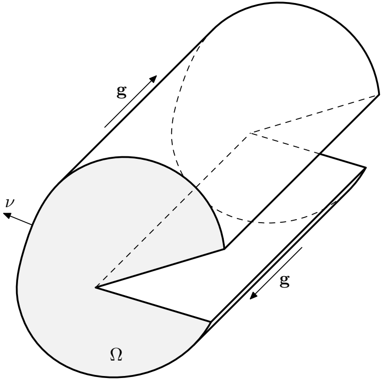
$$-\operatorname{div} \mathbb{T} = \mathbf{0}, \quad \varepsilon(\mathbf{u}) = \frac{\mathbb{T}}{(1 + |\mathbb{T}|^{a})^{1/a}} \quad \text{in } \mathcal{B}$$

$$\mathbb{T}\nu = \mathbf{g} \quad \text{on } \partial\mathcal{B}$$

Compatibility condition - $\mathcal{B}$ simply connected open bounded subset

$$\varepsilon = \frac{\nabla\mathbf{u} + (\nabla\mathbf{u})^T}{2} \iff \mathrm{curl}\mathrm{curl}\varepsilon = \mathbf{0}$$

Anti-Plane Stress Problem - Crack III Mode



$\mathcal{B} = \Omega \times \mathbb{R}$ and $\Omega \subset \mathbb{R}^2$ is simply connected open set

$$\mathbf{u} = \mathbf{u}(x_1, x_2) = (0, 0, u(x_1, x_2)), \quad \mathbf{g} = (0, 0, g)$$

$$\mathbb{T} = \begin{pmatrix} 0 & 0 & T_{13}(x_1, x_2) \\ 0 & 0 & T_{23}(x_1, x_2) \\ T_{13}(x_1, x_2) & T_{23}(x_1, x_2) & 0 \end{pmatrix}.$$

Introducing for $i = 1, 2$ the notation $D_i := \frac{\partial}{\partial x_i}$ we say that $U : \Omega \to \mathbb{R}$ solves Problem $\mathcal{P}$

$$-D_i\left(\frac{D_i U}{(1+|\nabla U|^a)^{\frac{1}{a}}}\right) = 0 \qquad \text{in } \Omega$$
$$U = U_0 \quad \text{on } \partial\Omega$$

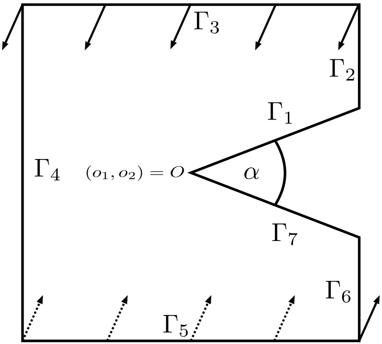
Theorem

Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain with $C^{0,1}$ boundary consisting of two parts: one is 'convex' and the other is composed of finite number of flat parts such that U_0 is piecewisely constant there. Let further $\tilde{U}_0 \in W^{1,\infty}(\Omega)$ be such that $\tilde{U}_0|_{\partial\Omega} = U_0$. Let $a \in (0, 2)$. Then there is a unique weak solution $U \in W^{2,2}_{loc}(\Omega)$ to Problem $\mathcal{P}$ satisfying

$$U - \tilde{U}_0 \in W^{1,1}_0(\Omega)\,,$$
$$\left(\frac{\nabla U}{(1+|\nabla U|^a)^{\frac{1}{a}}}, \nabla\phi\right) = 0 \quad \textit{for all } \phi \in W^{1,1}_0(\Omega)\,.$$

M. Bulíček, J. Málek, K. R. Rajagopal, J. Walton: *Existence of solutions for the anti-plane stress for a new class of "strain-limiting" elastic bodies* online in Calc. Var. (2015)

Boundary conditions



$$U[\Gamma_1] = a_0 \quad , \tag{1a}$$

$$U[\Gamma_2] = a_0 \quad , \tag{1b}$$

$$U[\Gamma_3] = a_0 + F(\frac{h}{2} - x_1) \quad , \tag{1c}$$

$$U[\Gamma_4] = a_0 + Fh \quad , \tag{1d}$$

$$U[\Gamma_5] = a_0 - F(x_1 - \frac{h}{2}) \quad , \tag{1e}$$

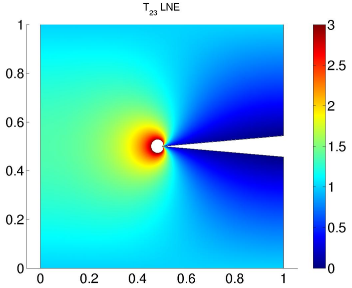
$$U[\Gamma_6] = a_0 \quad , \tag{1f}$$

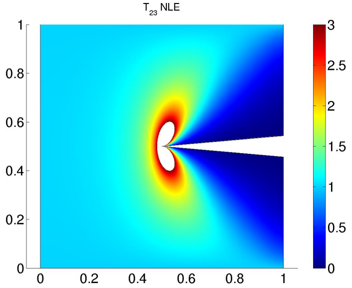
$$U[\Gamma_7] = a_0 \quad . \tag{1g}$$

For the computations we set $F = 1$, $a_0 = 0$.

Numerical results - Stress distribution

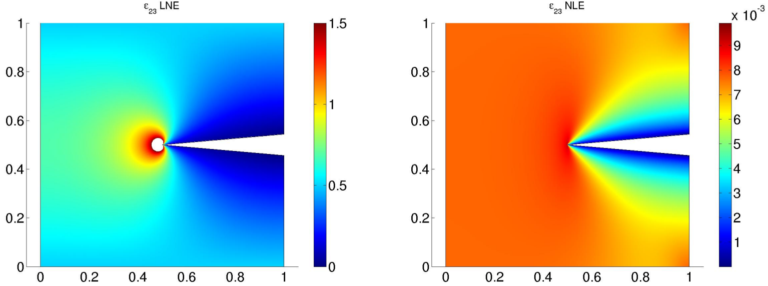
Comparison of T_{23} for linear (left) and nonlinear (right) model.





Numerical results - Strain distribution

Comparison of $\varepsilon(\mathbf{u})_{23}$ for linear (left) and nonlinear (right) model.



V. Kulvait, J. Málek, K. R. Rajagopal: Anti-plane stress state of a plate with a V-notch for a new class of elastic solids *Int. J. Fract* **179** (2013) 59–73.

L. Beck, M. Bulíček, E. Süli

Conlusion

- implicit constitutive theory is a very useful framework from modeller point of view
- generates new classes of systems of nonlinear PDEs of the first order (mixed formulation)
- gives alternative way how to study problems, different qualitative feature of involved quantities
- aim is to specify the concept of solution suitable for computer simulations
- show in what precise sense this object exists
- general data and three-dimensional deformations/flows