



Mixed formulations, numerical solutions and computations for visco-plastic flows using implicit material relations

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joint work with J. Málek, J. Stebel, K. Touška, K. Tůma, P. Minakowski

- balance equations

$$\varrho \frac{\partial \mathbf{v}}{\partial t} + \varrho [\nabla \mathbf{v}] \mathbf{v} = \operatorname{div} \mathbf{T} + \varrho \mathbf{f}$$
$$\operatorname{div} \mathbf{v} = 0$$

- constitutive equations

$$\mathcal{G}(\mathbf{T}^\delta, \mathbf{D}) = \mathbf{0}$$

- boundary conditions

$$\mathbf{v} \cdot \mathbf{n} = 0$$
$$\mathbf{g}((\mathbf{T}\mathbf{n})_t, \mathbf{v}_t) = \mathbf{0}$$

$$\mathbf{A}^\delta := \mathbf{A} - \frac{\operatorname{tr} \mathbf{A}}{\operatorname{tr} \mathbf{I}} \mathbf{I}$$

$$\mathbf{b}_t := (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \mathbf{b}$$

general non-Newtonian simple viscous fluid, implicit constitutive law

$$\mathcal{G}(\mathbf{S}, \mathbf{D}, \dots) = 0 \quad \mathbf{T} = -p\mathbf{I} + \mathbf{S} \quad \mathbf{D} = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T)$$

- incompressible Newtonian fluid

$$\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D} \quad \Rightarrow \quad \mathcal{G}(\mathbf{S}, \mathbf{D}) = \mathbf{S} - 2\mu\mathbf{D}$$

- generalized Newtonian fluid

$$\mathbf{T} = -p\mathbf{I} + 2\mu(p, |\mathbf{D}|^2, \dots)\mathbf{D} \quad \Rightarrow \quad \mathcal{G}(\mathbf{S}, \mathbf{D}) = \mathbf{S} - 2\mu(p, |\mathbf{D}|^2, \dots)\mathbf{D}$$

- more generalized, including stress power-law models

$$\mathcal{G}(\mathbf{S}, \mathbf{D}) = \alpha(|\mathbf{S}|^2, |\mathbf{D}|^2)\mathbf{S} - \beta(|\mathbf{S}|^2, |\mathbf{D}|^2)\mathbf{D}$$



K. R. Rajagopal ['06], Málek ['08]



M. Bulíček, P. Gwiazda, J. Málek, A. Świerczewska-Gwiazda ['09, '12, '12]



L. Diening, C. Kreuzer, E. Süli ['13]

Stokes like system:

$$\begin{aligned}-\operatorname{div} \mathbf{S} + \nabla p &= \mathbf{f}, \\ \operatorname{div} \mathbf{v} &= 0, \\ \mathbf{S} &= \mathcal{A}(\mathbf{D}(\mathbf{v}))\end{aligned}$$

- primal formulation: eliminate $\mathbf{S}$, p and solve for $\mathbf{v}$ only
- dual formulation: eliminate $\mathbf{v}$ and solve for $\mathbf{T} = -p\mathbf{I} + \mathbf{S}$ only
- classical mixed formulation: find $(\mathbf{v}, p) \in \mathbb{V} \times \mathbb{P}$ such that

$$\int_{\Omega} \mathcal{A}(\mathbf{D}(\mathbf{v})) : \mathbf{D}(\boldsymbol{\varphi}) - p \operatorname{div} \boldsymbol{\varphi} + \xi \operatorname{div} \mathbf{v} = \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\varphi}, \quad \forall (\boldsymbol{\varphi}, \xi) \in \mathbb{V} \times \mathbb{P}$$

$$\begin{bmatrix} \tilde{\mathcal{A}} & -\operatorname{div}^T \\ \operatorname{div} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ p \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \end{bmatrix}$$

finite element formulation: find $(\mathbf{v}_h, p_h) \in \mathbb{V}_h \times \mathbb{P}_h$ such that

$$\int_{\Omega} \mathcal{A}(\mathbf{D}(\mathbf{v}_h)) : \mathbf{D}(\boldsymbol{\varphi}) - p_h \operatorname{div} \boldsymbol{\varphi} + \xi \operatorname{div} \mathbf{v}_h = \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\varphi}, \quad \forall (\boldsymbol{\varphi}, \xi) \in \mathbb{V}_h \times \mathbb{P}_h$$

or if we define $a(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \mathcal{A}(\mathbf{D}(\mathbf{u})) : \mathbf{D}(\mathbf{v})$, $b(p, \mathbf{u}) = \int_{\Omega} p \operatorname{div} \mathbf{u}$

$$\begin{aligned} a(\mathbf{v}_h, \boldsymbol{\varphi}) - b(p_h, \boldsymbol{\varphi}) &= (f, \boldsymbol{\varphi}) \\ b(\xi, \mathbf{v}_h) &= 0 \end{aligned}$$

let $\{\boldsymbol{\varphi}^i\}$ denote a basis for $\mathbb{V}_h$ and $\{\xi^i\}$ denote a basis for $\mathbb{P}_h$ then we look for

$$\mathbf{v}_h = \sum V_i \boldsymbol{\varphi}^i \qquad p_h = \sum P_i \xi^i$$

denoting $\mathbf{X} = (V, P)$ we can write the finite dimensional nonlinear system as

$$\mathcal{R}(\mathbf{X}) = \mathbf{0}$$

- solvability of the linear system $\Rightarrow$ inf-sup condition
 $(P_k/P_{k-1}, Q_k/Q_{k-1}, Q_k/P_{k-1}^{\text{disc}})$

$$\inf_{p_h \in \mathbb{P}_h} \sup_{\mathbf{v}_h \in \mathbb{V}_h} \frac{b(p_h, \mathbf{v}_h)}{\|\mathbf{v}_h\|_1 \|p_h\|_0} = \beta_h \geq \beta > 0$$

- equal order elements $\Rightarrow$ need for additional stabilization
- inf-sup stability $(P_k/P_{k-1}, Q_k/Q_{k-1}, Q_k/P_{k-1}^{\text{disc}})$

$$\inf_{p_h \in \mathbb{P}_h} \sup_{\mathbf{v}_h \in \mathbb{V}_h} \frac{b(p_h, \mathbf{v}_h)}{\|\mathbf{v}_h\|_1 \|p_h\|_0} = \beta_h \geq \beta > 0$$

- conforming vs. nonconforming: $\mathbf{v} \in \mathbf{H}^1, p \in L^2, \mathbf{T} \in \mathbf{H}(\text{div}, \text{sym})$
- discretely div-free solution: if $\text{div } \mathbf{v}_h \in \mathbb{P}_h$ (Scott, Vogelius)

vast existing literature for example: Babuška, Brezzi, Fortin, etc.

$\Rightarrow$ assures that the linear problem is solvable

Stokes like system:

$$\begin{aligned} -\operatorname{div} \mathbf{T} &= \mathbf{f}, \\ \operatorname{div} \mathbf{v} &= 0, \\ \mathbf{D}(\mathbf{v}) &= \mathcal{A}(\mathbf{T}^\delta) \end{aligned}$$

- dual mixed formulation: find $(\mathbf{T}, \mathbf{v}) \in \mathbb{S} \times \mathbb{V}$ such that

$$\int_{\Omega} \mathcal{A}(\mathbf{T}) : \boldsymbol{\chi} + \mathbf{v} \cdot \operatorname{div} \boldsymbol{\chi} - \operatorname{div} \mathbf{T} \cdot \boldsymbol{\varphi} = \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\varphi}, \quad \forall (\boldsymbol{\chi}, \boldsymbol{\varphi}) \in \mathbb{S} \times \mathbb{V}$$

$$\begin{bmatrix} \tilde{\mathcal{A}}^{-1} & \operatorname{div}^T \\ -\operatorname{div} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \end{bmatrix}$$

- inf-sup condition for velocity-stress

Stokes like system with general implicit constitutive law:

$$\begin{aligned}-\operatorname{div} \mathbf{T} &= \mathbf{f}, \\ \operatorname{div} \mathbf{v} &= 0, \\ \mathbf{D} &= \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T), \\ \mathcal{G}(\mathbf{T}^\delta, \mathbf{D}) &= 0\end{aligned}$$

- dual mixed formulation: find $(\mathbf{D}, \mathbf{v}, \mathbf{T}^\delta) \in \mathbb{D} \times \mathbb{V} \times \mathbb{S}$ such that

$$\int_{\Omega} \mathcal{G}(\mathbf{T}, \mathbf{D}) : \boldsymbol{\omega} - \operatorname{div} \mathbf{T} \cdot \boldsymbol{\varphi} + \mathbf{D} : \boldsymbol{\chi} + \mathbf{v} \cdot \operatorname{div} \boldsymbol{\chi} = \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\varphi}, \quad \forall (\boldsymbol{\omega}, \boldsymbol{\varphi}, \boldsymbol{\chi}) \in \mathbb{D} \times \mathbb{V} \times \mathbb{S}$$

$$\begin{bmatrix} \mathcal{G}_{\mathbf{D}} & 0 & \mathcal{G}_{\mathbf{T}} \\ 0 & 0 & -\operatorname{div} \\ I & \operatorname{div}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{D} \\ \mathbf{v} \\ \mathbf{T} \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

- classical inf-sup for velocity-pressure and velocity-stress
- double inf-sup for $\mathbf{D}$ -velocity-stress

Generalized nonsymmetric saddle point problem

Sufficient and necessary conditions for well-posedness

[Bernardi et al. 1988]:

$$\begin{bmatrix} A & B_1^\top \\ B_2 & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

- A restricted to $\ker B_2$ is isomorphism onto $(\ker B_1)^*$
- B_1 and B_2 have full rank

Generalized twofold saddle point problem

$$\begin{bmatrix} A & 0 & B_1^\top \\ 0 & 0 & C_2 \\ B_2 & C_1^\top & 0 \end{bmatrix} \begin{bmatrix} u \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} f \\ g_1 \\ g_2 \end{bmatrix}$$

Sufficient and necessary conditions for well-posedness:

- A restricted to $\ker B_2$ is isomorphism onto $(\ker B_1)^*$
- $B_1^\top$ and $B_2^\top$ restricted to $\ker C_2$, $\ker C_1$, respectively, have full rank
- C_1 and C_2 have full rank



J.S. Howell, H.J. Walkington [10]

Theorem

Let $\mathbb{S}_h, \mathbb{V}_h$ satisfy the following conditions:

- (i) $\{\mathbf{D}\varphi; \varphi \in \mathbb{V}_h\} \subset \mathbb{S}_h$;
- (ii) There exists $c > 0$ such that: $\sup_{\varphi \in \mathbb{V}_h} \frac{(\text{tr } \mathbf{T}, \text{div } \varphi)}{\|\varphi\|_{1,2}} \geq c \|\text{tr } \mathbf{T}\|_2 \quad \forall \mathbf{T} \in \mathbb{S}_h$.

Then the linearized problem has a unique solution $(\mathbf{T}_h, \mathbf{v}_h)$.

Theorem

Let $\mathbb{D}_h, \mathbb{V}_h, \mathbb{S}_h$ satisfy the following conditions:

- (i) $\{\mathbf{D}\varphi; \varphi \in \mathbb{V}_h\} \subset \mathbb{S}_h$;
- (ii) $\{\mathbf{T}^\delta; \mathbf{T} \in \mathbb{S}_h\} \subset \mathbb{D}_h$;
- (iii) There exists $c > 0$ such that: $\sup_{\varphi \in \mathbb{V}_h} \frac{(\text{tr } \mathbf{T}, \text{div } \varphi)}{\|\varphi\|_{1,2}} \geq c \|\text{tr } \mathbf{T}\|_2 \quad \forall \mathbf{T} \in \mathbb{S}_h$.

Then the linearized problem has a unique solution $(\mathbf{D}_h, \mathbf{v}_h, \mathbf{T}_h)$.



L. Diening, C. Kreuzer, E. Süli [13]



J. Hron, J. Málek, J. Stebel, K. Touška [in preparation]

A. Unknowns $(\mathbf{S}, \mathbf{v}, p)$:

$$\operatorname{div} \mathbf{S} - \nabla p = \mathbf{f}$$

$$\operatorname{div} \mathbf{v} = 0$$

$$\mathbf{D}(\mathbf{v}) = (1 + |\mathbf{S}|^2)^n \mathbf{S}.$$

B. Unknowns $(\mathbf{T}, \mathbf{v})$:

$$\operatorname{div} \mathbf{T} = \mathbf{f}$$

$$\mathbf{D}(\mathbf{v}) = (1 + |\mathbf{T}^\delta|^2)^n \mathbf{T}^\delta$$

C. Unknowns $(\mathbf{T}, \mathbf{v}, \mathbf{D})$:

$$\operatorname{div} \mathbf{T} = \mathbf{f}$$

$$\mathbf{D}(\mathbf{v}) = \mathbf{D}$$

$$\mathbf{D} = (1 + |\mathbf{T}^\delta|^2)^n \mathbf{T}^\delta$$

Finite element approximation

triangular mesh

A		B		C	
$\mathbf{S}$	p_1^{disc}	$\mathbf{T}$	p_1^{disc}	$\mathbf{T}$	p_1^{disc}
$\mathbf{v}$	p_2	$\mathbf{v}$	p_2	$\mathbf{v}$	p_2
p	p_1			$\mathbf{D}$	p_1^{disc}

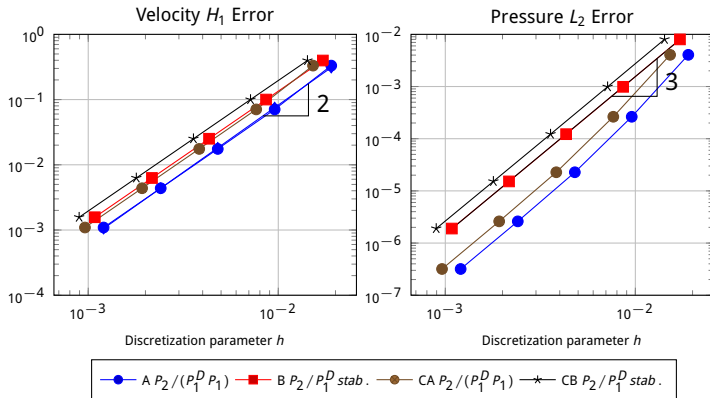
quadrilateral mesh

A		B		C	
$\mathbf{S}$	Q_2^{disc}	$\mathbf{T}$	Q_2^{disc}	$\mathbf{T}$	Q_2^{disc}
$\mathbf{v}$	Q_2	$\mathbf{v}$	Q_2	$\mathbf{v}$	Q_2
p	p_1^{disc}			$\mathbf{D}$	Q_2^{disc}

Observations:

- In the cases B and C it is necessary to stabilize jumps of $\operatorname{tr} \mathbf{T}$ across edges in order to satisfy the inf-sup condition for the pressure on simplex mesh.

Analytical solution derived in [Málek, Pruša, Rajagopal 2010].



Numerical simulations of stress power-law model

Flow around cylinder, curves for different values of the power n .

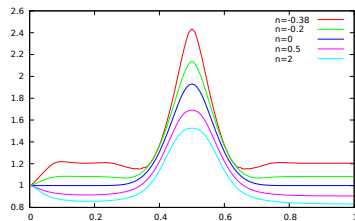
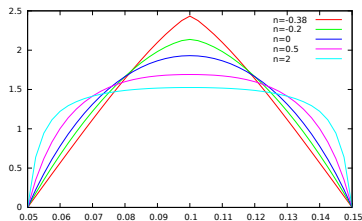


Figure: Velocity in the middle cross-section (left), along the channel (right).

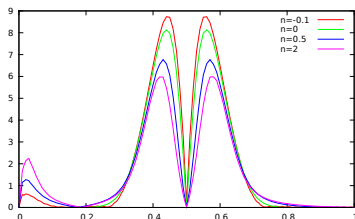
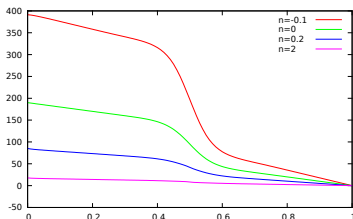


Figure: Pressure (left) and norm of $\mathbf{D}(\mathbf{v})$ (right) along the channel.

$$|\mathbf{T}^\delta| \leq \tau_* \quad \Leftrightarrow \quad \mathbf{D} = \mathbf{0},$$

$$|\mathbf{T}^\delta| > \tau_* \quad \Leftrightarrow \quad 2\mu\mathbf{D} = \left(1 - \frac{\tau_*}{|\mathbf{T}^\delta|}\right)\mathbf{T}^\delta;$$

alternatively,

$$|\mathbf{D}| = 0 \quad \Leftrightarrow \quad |\mathbf{T}^\delta| \leq \tau_*,$$

$$|\mathbf{D}| > 0 \quad \Leftrightarrow \quad \mathbf{T}^\delta = \left(2\mu + \frac{\tau_*}{|\mathbf{D}|}\right)\mathbf{D}.$$

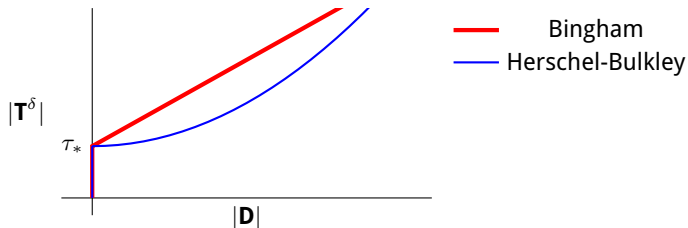


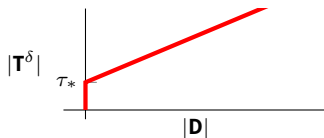
Figure: Yield stress response. Bingham and Herschel-Bulkley fluids.

$$\operatorname{div} \mathbf{T} = 0, \quad \operatorname{div} \mathbf{v} = 0, \quad \mathbf{D} = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T)$$

- common formulation

$$|\mathbf{D}| = 0 \Rightarrow |\mathbf{T}^\delta| \leq \tau^*$$

$$|\mathbf{D}| \neq 0 \Rightarrow \mathbf{T}^\delta = \tau^* \frac{\mathbf{D}}{|\mathbf{D}|} + 2\mu \mathbf{D}$$



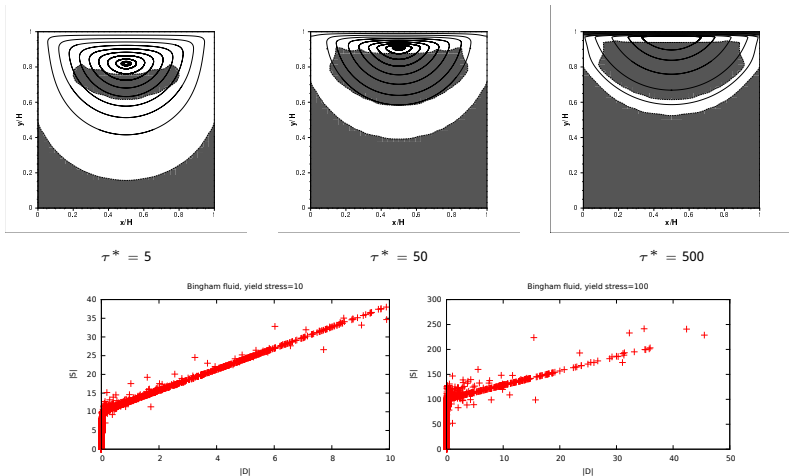
- implicit formulation

$$\mathcal{G}(\mathbf{T}^\delta, \mathbf{D}) = 2\mu\tau^*\mathbf{D} + (|\mathbf{T}^\delta| - \tau^*)^+(2\mu\mathbf{D} - \mathbf{T}^\delta) = 0 \quad (1)$$

$$\mathcal{G}(\mathbf{T}^\delta, \mathbf{D}) = 2\mu\tau^*\mathbf{D} + (2\mu|\mathbf{D}|)(2\mu\mathbf{D} - \mathbf{T}^\delta) = 0 \quad (2)$$

- regularization $|\mathbf{D}|_{\varepsilon_1} = \sqrt{\varepsilon_1^2 + |\mathbf{D}|^2}$, $(f)_{\varepsilon_2}^+ = \max(f, \varepsilon_2)$

Lid driven cavity with Bingham fluid



Satisfaction of the constitutive relation. Left: $\tau^* = 10$, right: $\tau^* = 100$.

- One can reach numerical solution with vanishing regularization in the $(\mathbf{D}, \mathbf{v}, \mathbf{T})$ formulation.

$$-\operatorname{div} \mathbf{T} = \mathbf{f}$$

$$\operatorname{div} \mathbf{v} = 0$$

$$\mathbf{T} + p\mathbf{I} - 2\mu\mathbf{D} = 0$$

$$\mathbf{v} \cdot \mathbf{n} = 0$$

$$\mathbf{g}((\mathbf{T}\mathbf{n})_t, \mathbf{v}_t) = 0$$

How to impose such boundary conditions:

- Lagrange multiplier method
- as Dirichlet condition
- as natural boundary condition $\rightarrow$ Nitsche penalty method

$$\mathbf{b}_t := (\mathbf{I} - \mathbf{n} \otimes \mathbf{n})\mathbf{b}$$

$$-\operatorname{div} \mathbf{T} = \mathbf{f} \quad \operatorname{div} \mathbf{v} = 0 \quad \mathbf{T} + p\mathbf{I} - 2\mu\mathbf{D} = 0$$

$$\mathbf{v} \cdot \mathbf{n} = 0 \quad \mathbf{g}((\mathbf{T}\mathbf{n})_t, \mathbf{v}_t) = 0$$

mixed formulation: find $(\mathbf{T}, \mathbf{v}, p) \in \mathbb{S} \times \mathbb{V} \times \mathbb{P}$ such that

$$\begin{aligned} & \int_{\Omega} (\mathbf{T} : \nabla \varphi - p \operatorname{tr} \boldsymbol{\sigma} + \xi \operatorname{div} \mathbf{v} + [\mathbf{T} + p\mathbf{I} - 2\mu\mathbf{D}(\mathbf{v})] : \boldsymbol{\sigma}) dx \\ & + \int_{\Gamma} (\mathbf{T}\mathbf{n} \cdot \boldsymbol{\varphi} + (\mathbf{v} \cdot \mathbf{n})(\boldsymbol{\sigma}\mathbf{n} \cdot \mathbf{n}) + \mathbf{g}((\mathbf{T}\mathbf{n})_t, \mathbf{v}_t) \cdot (\boldsymbol{\sigma}\mathbf{n})_t) ds \\ & + \int_{\Gamma} \left(\frac{\beta}{h} ((\mathbf{v} \cdot \mathbf{n})(\boldsymbol{\varphi} \cdot \mathbf{n}) + \mathbf{g}((\mathbf{T}\mathbf{n})_t, \mathbf{v}_t) \cdot \boldsymbol{\varphi}_t) \right) ds \\ & = \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\varphi}, \quad \forall (\boldsymbol{\sigma}, \boldsymbol{\varphi}, \xi) \in \mathbb{S} \times \mathbb{V} \times \mathbb{P} \end{aligned}$$



Nitsche, J. (1971). Über ein Variationsprinzip zur Lösung von Dirichlet Problemen bei Verwendung von Teilräumen, die kein Randbedingungen unterworfen sind.



Juntunen, M., & Stenberg, R. (2009). Nitsche's method for general boundary conditions.



Vergara, C. (2010). Nitsche's Method for Defective Boundary Value Problems in Incompressible Fluid-dynamics.

- Dirichlet condition at the top plate: $\mathbf{v} = (1, 0)$
- Threshold condition on the walls:

$$|(\mathbf{Tn})_t| \leq \tau^* \Rightarrow \mathbf{v} = 0$$

$$|(\mathbf{Tn})_t| > \tau^* \Rightarrow \mathbf{g}((\mathbf{Tn})_t, \mathbf{v}_t) = |(\mathbf{Tn})_t| \mathbf{v} + \gamma(|(\mathbf{Tn})_t| - \tau^*)(\mathbf{Tn})_t = 0$$

