Steady Navier—Stokes—Fourier system with nonlinear dependence of viscosity on temperature

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September 20, 2012

Steady Navier–Stokes–Fourier system

We study the steady Navier–Stokes–Fourier system

$$\operatorname{div}(\varrho \mathbf{u}) = 0$$

$$\operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - \operatorname{div} \mathbb{S} + \nabla p = \varrho \mathbf{f}$$

$$\operatorname{div}(\varrho E \mathbf{u}) = \varrho \mathbf{f} \cdot \mathbf{u} - \operatorname{div}(p\mathbf{u}) + \operatorname{div}(\mathbb{S}\mathbf{u}) - \operatorname{div} \mathbf{q}$$

Unknowns:

- $\varrho(\mathbf{x}) \geq 0$ … density
- $\mathbf{u}(\mathbf{x})$ … velocity
- $\vartheta(\mathbf{x}) > 0$ … temperature (appearing implicitly)

Conditions

- $\mathbb{S}(\vartheta, \nabla \mathbf{u}) = \mu(\vartheta)\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T - \frac{2}{3}\mathrm{div}\,\mathbf{u}\mathbb{I}\right] + \xi(\vartheta)\mathrm{div}\,\mathbf{u}\mathbb{I}$
- $p(\varrho, \vartheta) = (\gamma - 1)\varrho e(\varrho, \vartheta)$ ... generalized law for monoatomic gas
- $E(\varrho, \mathbf{u}, \vartheta) = \frac{1}{2}|\mathbf{u}|^2 + e(\varrho, \vartheta)$ ... specific total energy
- $\mathbf{q}(\vartheta) = -\kappa(\vartheta)\nabla\vartheta$ ... Fourier's law

- $p(\varrho, \vartheta) \sim \varrho^\gamma + \varrho\vartheta$
- $\kappa(\vartheta) \sim 1 + \vartheta^m$
- $\mu(\vartheta), \xi(\vartheta) \sim (1 + \vartheta)^\alpha$

Additional conditions

We prescribe total mass of the gas:

$$\int_\Omega \varrho \mathrm{d}\mathbf{x} = M > 0.$$

Boundary conditions:

- $\mathbf{u} \cdot \mathbf{n} = \mathbf{0}$, $[\mathbb{S}\mathbf{n}] \times \mathbf{n} = \mathbf{0}$
- $-\mathbf{q}(\vartheta) \cdot \mathbf{n} + L(\vartheta)(\vartheta - \Theta_0) = 0,$

with $0 < c_1 < L(\vartheta) < c_2$

Previous works I

Isentropic flows, steady case:

- P. L. Lions: effective viscous flux identity, $\gamma > \frac{5}{3}$
- E. Feireisl: oscillation defect measure
- S. Novo, A. Novotný: adapted this method to steady case, $\gamma > 1$, some a–priori estimates for ϱ required
- J. Frehse, S. Goj, M. Steinhauer and P. I. Plotnikow, J. Sokolowski: independently achieved improved estimates for ϱ
- J. Březina, A. Novotný: first rigorous proof of existence of weak solutions for some $\gamma < \frac{5}{3}$, periodic BC
- J. Frehse, M. Steinhauer, W. Weigant: $\gamma > \frac{4}{3}$ with Dirichlet BC (also slip BC)
- S. Jiang, C. Zhou: $\gamma > 1$, space periodic and Dirichlet BC
- D. Jesslé, A. Novotný: $\gamma > 1$, slip BC

Previous works II

Heat conducting flows, steady case:

- P. L. Lions: additional assumption for ϱ to be bdd in L^p for large p
- P. Mucha, M. Pokorný: viscosity does not depend on ϑ, slip and Dirichlet BC, $\gamma > \frac{7}{3}$
- A. Novotný, M. Pokorný: viscosity depend on ϑ, Dirichlet BC, VES $\gamma > \frac{\sqrt{41}+3}{8}$, WS $\gamma > \frac{4}{3}$
- D. Jesslé, A. Novotný, M. Pokorný: slip BC, VES $\gamma > 1$, WS $\gamma > \frac{5}{4}$
- O. K., Š. Nečasová, M. Pokorný: NSF coupled with radiation transport eq., $\alpha < 1$, VES $\gamma > \frac{3}{2}$, WS: $\gamma > \frac{5}{3}$ - only Bogovskii estimates used

Weak solutions I

The triple $(\varrho, \mathbf{u}, \vartheta)$ is called a weak solution to the mentioned system, if

- $\varrho \geq 0$ a.e. in Ω, $\varrho \in L^{\gamma \frac{3p}{4p-3}}(\Omega)$, $\int_\Omega \varrho \mathrm{d}\mathbf{x} = M$,
- $\mathbf{u} \in W^{1,p}(\Omega)$ for some $p \in (1,2]$,
- $\vartheta > 0$ a.e. in Ω, $\vartheta \in W^{1,r}(\Omega) \cap L^{3m}(\Omega)$,

moreover $\varrho |\mathbf{u}|^2 \in L^{\frac{3p}{4p-3}}(\Omega)$, $\varrho \mathbf{u} \vartheta \in L^1(\Omega)$, $\mathbb{S}(\vartheta, \nabla \mathbf{u})\mathbf{u} \in L^1(\Omega)$, $\vartheta^m \nabla \vartheta \in L^1(\Omega)$ and

$$\int_\Omega \varrho \mathbf{u} \cdot \nabla \psi \, \mathrm{d}\mathbf{x} = 0 \quad \forall \psi \in C^\infty(\overline{\Omega}), \tag{1}$$

$$\begin{aligned} \int_\Omega \left(-\varrho(\mathbf{u} \otimes \mathbf{u}) : \nabla \boldsymbol{\varphi} - p(\varrho, \vartheta) \mathrm{div}\, \boldsymbol{\varphi} + \mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \boldsymbol{\varphi} \right) \mathrm{d}\mathbf{x} \\ = \int_\Omega \varrho \mathbf{f} \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathbf{x} \quad \forall \boldsymbol{\varphi} \in C^\infty(\overline{\Omega}), \boldsymbol{\varphi} \cdot \mathbf{n} = 0 \text{ at } \partial\Omega \end{aligned} \tag{2}$$

and

$$\int_\Omega -\left(\frac{1}{2}\varrho\,|\mathbf{u}|^2 + \rho e(\varrho,\vartheta)\right)\mathbf{u}\cdot\nabla\psi\,\mathrm{d}\mathbf{x} = \int_\Omega \left(\varrho\mathbf{f}\cdot\mathbf{u}\psi + p(\varrho,\vartheta)\mathbf{u}\cdot\nabla\psi\right)\,\mathrm{d}\mathbf{x}$$
$$-\int_\Omega \left((\mathbb{S}(\vartheta,\nabla\mathbf{u})\mathbf{u})\cdot\nabla\psi + \kappa(\vartheta)\nabla\vartheta\cdot\nabla\psi\right)\mathrm{d}\mathbf{x}$$
$$-\int_{\partial\Omega} L(\vartheta)(\vartheta-\Theta_0)\psi\,\mathrm{d}S \quad \forall\psi\in C^\infty(\overline{\Omega}), \tag{3}$$

Variational entropy solutions I

The triple $(\varrho, \mathbf{u}, \vartheta)$ is called a variational entropy solution to the mentioned system, if

- $\varrho \geq 0$ a.e. in Ω, $\varrho \in L^\gamma(\Omega)$, $\int_\Omega \varrho \mathrm{d}\mathbf{x} = M$,
- $\mathbf{u} \in W^{1,p}(\Omega)$ for some $p \in (1, 2]$,
- $\vartheta > 0$ a.e. in Ω, $\vartheta \in W^{1,r}(\Omega) \cap L^{3m}(\Omega)$,

moreover $\varrho\,|\mathbf{u}|^2 \in L^1(\Omega)$, $\varrho\vartheta \in L^1(\Omega)$, $\vartheta^{-1}\mathbb{S}(\vartheta, \nabla\mathbf{u})\mathbf{u} \in L^1(\Omega)$, $\vartheta^m \frac{\nabla\vartheta}{\vartheta} \in L^1(\Omega)$, $\vartheta^{-1} \in L^1(\partial\Omega)$,
the equalities (1) and (2) are satisfied in the same sense as in previous definition...

and instead of (3) we have the entropy inequality

$$\int_\Omega \Big(\frac{\mathbb{S}(\vartheta,\nabla\mathbf{u}):\nabla\mathbf{u}}{\vartheta} + \kappa(\vartheta)\frac{|\nabla\vartheta|^2}{\vartheta^2}\Big)\psi\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega}\frac{L(\vartheta)}{\vartheta}\Theta_0\psi\,\mathrm{d}S \leq \int_\Omega \Big(\kappa(\vartheta)\frac{\nabla\vartheta:\nabla\psi}{\vartheta} - \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla\psi\Big)\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} L(\vartheta)\psi\,\mathrm{d}S \qquad (4)$$

for all nonnegative $\psi \in C^\infty(\overline{\Omega})$ together with the global total energy balance

$$\int_{\partial\Omega} L(\vartheta)(\vartheta - \Theta_0)\,\mathrm{d}S = \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}\mathbf{x}. \qquad (5)$$

Main Theorem I

Theorem

Let $\Omega \in C^2$ be a bounded domain in $\mathbb{R}^3$, $\mathbf{f} \in L^\infty(\Omega)$, $\Theta_0 \geq K_0 > 0$ a.e. at $\partial\Omega$, $\Theta_0 \in L^1(\partial\Omega)$, $M > 0$. Moreover, let

$$\alpha \in (0,1], \quad \gamma > 1 \quad \text{and} \quad m > F(\alpha,\gamma) \tag{6}$$

Then there exists a variational entropy solution to our system. Moreover, the pair $(\varrho, \mathbf{u})$ is a renormalized solution to the continuity equation.

Main Theorem II

Theorem

If additionally

$$\gamma > g(\alpha) \quad \text{and} \quad m > G(\alpha, \gamma) \tag{7}$$

then this solution is a weak solution.

Remark

The lowest possible γ for weak solution is $\frac{5}{4} + \varepsilon$ for $\alpha \in (\frac{1}{3}, 1)$ and $1 + \frac{1}{3(1+\alpha)}$ for $\alpha \in (0, \frac{1}{3})$.

Outline of proof

- 4-parameter $(N, \eta, \varepsilon, \delta)$ approximative system: existence, a–priori estimates
- Limit passages with N and η
- Recovering a–priori estimates, especially estimates for density
- Limit passages with ε and δ
- Strong convergence of the sequence of densities

Why is $\alpha < 1$ worse than $\alpha = 1$? I

At levels ε and δ we use the entropy inequality to read a–priori estimates. The key term is

$$\frac{\mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u}}{\vartheta} \sim \vartheta^{\alpha-1} |\nabla \mathbf{u}|^2 .$$

- $\alpha = 1$: use only Korn's inequality to read L^2 estimates of $\nabla \mathbf{u}$
- $\alpha < 1$: apart from Korn's inequality we need to use the following procedure

Denote $p = \frac{6m}{3m+1-\alpha}$. Then

$$\|\mathbf{u}\|_{1,p} \leq \|\vartheta\|_{3m}^{\frac{1-\alpha}{2}} \left\| \vartheta^{\alpha-1} |\nabla \mathbf{u}|^2 \right\|_1^{\frac{1}{2}} \leq C \|\vartheta\|_{3m}^{\frac{1-\alpha}{2}}$$

Why is $\alpha < 1$ worse than $\alpha = 1$? II

Using entropy inequality and global total energy balance

$$\|\vartheta\|_{3m} \le C\left(\|\vartheta\|_{1,\partial\Omega} + \left\|\nabla \vartheta^{\frac{m}{2}}\right\|_2^{\frac{2}{m}} \right) \le$$
$$\le C(1 + \int_\Omega |\varrho \mathbf{u} \cdot \mathbf{f}| \,\mathrm{d}\mathbf{x}) \le \|\mathbf{u}\|_{1,p} \|\varrho\|_{\frac{3p}{4p-3}} \le$$
$$\|\vartheta\|_{3m}^{\frac{1-\alpha}{2}} \|\varrho\|_{\frac{3p}{4p-3}}$$

and thus

$$\|\vartheta\|_{3m} \le C \|\varrho\|_{\frac{3p}{4p-3}}^{\frac{2}{1+\alpha}}$$
$$\|\mathbf{u}\|_{1,p} \le C \|\varrho\|_{\frac{3p}{4p-3}}^{\frac{1-\alpha}{1+\alpha}}$$

or similarly

$$\|\vartheta\|_{3m} \le C(1 + \|\varrho \mathbf{u}\|_1)$$
$$\|\mathbf{u}\|_{1,p} \le C(1 + \|\varrho \mathbf{u}\|_1^{\frac{1-\alpha}{2}})$$

- Introduce $\mathcal{A} = \int_\Omega \left(\varrho_\delta^a |\mathbf{u}_\delta|^p + \varrho_\delta^b |\mathbf{u}_\delta|^{2b+p} \right) \mathrm{d}\mathbf{x}$ with $1 \leq a \leq \gamma$ and $0 < b < 1$.
- Use Bogovskii test function in ME to obtain $\|\varrho\|_{s\gamma} \leq C\mathcal{A}^\omega$ for some ω and $s > 1$.
- Obtain local pressure estimates using special test functions in ME
  - $\varphi(\mathbf{x}) = \eta(\mathbf{x}) \frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|^A}$ for fixed $\mathbf{x}_0$ away from boundary $\partial\Omega$, η cut-off function
  - near boundary $\partial\Omega$: more complicated structure of $\varphi(\mathbf{x})$ (boundary conditions!) but similar behaviour

This yields:

$$\int_\Omega \frac{p(\varrho_\delta, \vartheta_\delta) + \delta(\varrho_\delta^\beta + \varrho_\delta^2)}{|\mathbf{x}-\mathbf{x}_0|^A} \mathrm{d}\mathbf{x} + (1-A) \int_\Omega \frac{\varrho_\delta |\mathbf{u}_\delta|^2}{|\mathbf{x}-\mathbf{x}_0|^A} \mathrm{d}\mathbf{x}$$
$$\leq C(1 + \delta \|\varrho_\delta\|_\beta^\beta + \|p(\varrho_\delta, \vartheta_\delta)\|_1 + (1 + \|\vartheta_\delta\|_{3m}^\alpha) \|\mathbf{u}_\delta\|_{1,p} + \left\| \varrho_\delta |\mathbf{u}_\delta|^2 \right\|_1)$$

Goal of this work

Our ultimate goal is the inequality of the following type

$$\mathcal{A} = \left\| (\varrho_\delta^a + (\varrho_\delta \left|\mathbf{u}_\delta\right|^2)^b)\mathbf{u}^p \right\|_{L^1(\Omega)} \leq$$
$$\leq C \left(\sup_{\mathbf{x}_0 \in \overline{\Omega}} \int_\Omega \frac{(\varrho_\delta^a + (\varrho_\delta \left|\mathbf{u}_\delta\right|^2)^b)^{\frac{1}{p-1}}(\mathbf{x})}{\left|\mathbf{x}-\mathbf{x}_0\right|^{\frac{3-p}{p-1}}} \mathrm{d}\mathbf{x} \right)^{p-1} \|\mathbf{u}_\delta\|_{W^{1,p}(\Omega)}^p . \tag{8}$$

so we can use the inequality from previous slide and estimating all terms on the right hand side derive

$$\mathcal{A} \leq C\mathcal{A}^z$$

for $z < 1$. This then yields $\|\varrho_\delta\|_{s\gamma}, \|\vartheta_\delta\|_{3m}, \|\mathbf{u}_\delta\|_{1,p} < C$.

Bessel potentials I

For any $\alpha \in \mathbb{R}$ define Bessel kernel

$$G_\alpha(\mathbf{x}) := \mathcal{F}^{-1}((1+|\xi|^2)^{-\frac{\alpha}{2}}) \tag{9}$$

Properties:

- real
- radially symmetric
- decreasing
- positive
- exponential decay at ∞
- $G_\alpha(\mathbf{x}) \leq C(\alpha, N)\,|\mathbf{x}|^{\alpha-N}$ as $|\mathbf{x}| \to 0$ for $\alpha \in (0, N)$

Bessel potentials II

Define Bessel potential space

$$L^{\alpha,p}(\mathbb{R}^N) := \left\{ \varphi = G_\alpha * f, f \in L^p(\mathbb{R}^N) \right\} \tag{10}$$

equipped with the norm

$$\|G_\alpha * f\|_{L^{\alpha,p}(\mathbb{R}^N)} := \|f\|_{L^p(\mathbb{R}^N)} . \tag{11}$$

Theorem

For $\alpha \in N$ and $1 < p < \infty$, $W^{\alpha,p}(\mathbb{R}^N) = L^{\alpha,p}(\mathbb{R}^N)$ with equivalence of norms. In particular, for all $\varphi \in W^{\alpha,p}(\mathbb{R}^N)$ there exists a unique $f \in L^p(\mathbb{R}^N)$ such that $\varphi = G_\alpha * f$ and there exists a constant $A > 0$ such that

$$A^{-1} \|\varphi\|_{L^{\alpha,p}(\mathbb{R}^N)} \le \|\varphi\|_{W^{\alpha,p}(\mathbb{R}^N)} \le A \|\varphi\|_{L^{\alpha,p}(\mathbb{R}^N)} . \tag{12}$$

Bessel potentials III

Key theorem:

Theorem

Let G be radially decreasing convolution kernel and let $\mu \in \mathcal{M}^+(\mathbb{R}^N)$. Then for $1 < p \leq q < \infty$ the following statements are equivalent:

1) There is a constant A_1 such that $\left(\int_{\mathbb{R}^N} |G * f|^q \, \mathrm{d}\mu\right)^{\frac{1}{q}} \leq A_1 \|f\|_{L^p}$ for all $f \in L^p(\mathbb{R}^N)$.

2) There is a constant A_2 such that $\|G * \mu_K\|_{L^{p'}} \leq A_2 \mu(K)^{\frac{1}{q'}}$ for all compact sets K.

Moreover the constants A_1, A_2 are comparable, in fact we can choose $A_1 = A_2$.

Bessel potentials IV

- for components of our velocity field $u_\delta^i \in W^{1,p}(\Omega)$ we find unique $f^i \in L^p(\Omega)$ such that $E(u_\delta^i) = G_1 * f^i$ (E is extension operator)
- $N = 3$, $p = q \in (1,2)$
- $\mathrm{d}\mu(\mathbf{x}) = (\varrho_\delta^a + (\varrho_\delta |\mathbf{u}_\delta|^2)^b)(\mathbf{x})\mathrm{d}\mathbf{x}$, $\varrho_\delta = 0$ outside Ω
- $G = G_1$ and $f = f^i$ defined above

First we check that statement 2) holds. Denote $h = (\varrho_\delta^a + (\varrho_\delta |\mathbf{u}_\delta|^2)^b)$ and then

$$\int_{\mathbb{R}^3} |G_1 * h|_K|^{p'} \,\mathrm{d}\mathbf{x} \le C(\Omega) \left\| ((G_1 * G_1^{p'-1}) * h^{p'-1}|_K) \right\|_{L^\infty(\mathbb{R}^3)} \|h\|_{L^1(K)} .$$

Therefore we denote

$$A_2 := C \left\| ((G_1 * G_1^{p'-1}) * h^{p'-1}) \right\|_{L^\infty(\Omega)}^{\frac{1}{p'}} .$$

Bessel potentials VI

This means that also statement 1) of the key theorem holds, i.e.

$$\left\| (\varrho_\delta^a + (\varrho_\delta |\mathbf{u}_\delta|^2)^b)\mathbf{u}_\delta^p \right\|_{L^1(\Omega)} \leq C A_2^p \|\mathbf{u}_\delta\|_{W^{1,p}(\Omega)}^p .$$

Now we have to investigate behaviour of

$$A_2 := C \left\| ((G_1 * G_1^{p'-1}) * h^{p'-1}) \right\|_{L^\infty(\Omega)}^{\frac{1}{p'}},$$

more precisely we are interested in studying properties of the convolution kernel $(G_1 * G_1^{p'-1})$.
If $\alpha = 1$, then $p = 2$, $p' - 1 = 1$ and the kernel is just $G_1 * G_1 = G_2 \leq \frac{C}{|\mathbf{x}|}$.
The case $\alpha < 1$ is more complicated.

Decay properties of Fourier transform I

Unfortunately, $G_1^{p'-1}$ is not equal to any Bessel potential G_α. We have to proceed in other way - derive decay properties of Fourier transform.

Lemma

Let $f(\mathbf{x}) \in C^3(\mathbb{R}^3 \setminus \{\mathbf{0}\})$ *such that for some* $\alpha \in (0,3)$

$$\sup_{\mathbf{x}\in\mathbb{R}^3\setminus\{\mathbf{0}\}} |\mathbf{x}|^{\alpha+n} D^n f(\mathbf{x}) < \infty, \quad n = 0, 1, \dots, 4 - \lceil \alpha \rceil .$$

Then

$$\sup_{\boldsymbol{\xi}\in\mathbb{R}^3\setminus\{\mathbf{0}\}} |\boldsymbol{\xi}|^{3-\alpha} \widehat{f}(\boldsymbol{\xi}) < \infty.$$

This works also other way round.

Lemma

Let $\widehat{f}(\xi) \in C^3(\mathbb{R}^3 \setminus \{\mathbf{0}\})$ *such that for some* $\beta \in (0,3)$

$$\sup_{\xi \in \mathbb{R}^3 \setminus \{\mathbf{0}\}} |\xi|^{\beta+n} D^n \widehat{f}(\xi) < \infty, \quad n = 0, 1, ..., 4 - \lceil \beta \rceil .$$

Then

$$\sup_{\mathbf{x} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}} |\mathbf{x}|^{3-\beta} f(\mathbf{x}) < \infty.$$

Repeatedly using these Lemmas we are able to show

$$(G_1 * G_1^r)(\mathbf{x}) \leq C\,|\mathbf{x}|^{1-2r} \tag{13}$$

and thus

$$\mathcal{A} = \left\| (\varrho_\delta^a + (\varrho_\delta\,|\mathbf{u}_\delta|^2)^b)\mathbf{u}^p \right\|_{L^1(\Omega)} \leq CA_2^p\,\|\mathbf{u}_\delta\|^p_{W^{1,p}(\Omega)} \leq$$
$$\leq C\left(\sup_{\mathbf{x}_0\in\overline{\Omega}}\int_\Omega \frac{(\varrho_\delta^a + (\varrho_\delta\,|\mathbf{u}_\delta|^2)^b)^{\frac{1}{p-1}}(\mathbf{x})}{|\mathbf{x}-\mathbf{x}_0|^{\frac{3-p}{p-1}}}\mathrm{d}\mathbf{x}\right)^{p-1} \|\mathbf{u}_\delta\|^p_{W^{1,p}(\Omega)}\,. \tag{14}$$

which is the desired estimate which enables us to close the set of inequalities.

Last steps in the proof

What remains is to derive the set of conditions, under which we really get

$$\mathcal{A} \leq C\mathcal{A}^z$$

with $z < 1$, this then leads to conditions on α, γ and m in the theorem.
Final step is to prove strong convergence of sequence of densities, which is somehow classical using the oscillation defect measure and is basically the same as in the case $p = 2$.

Thank you

Thank you for attention.