A posteriori error estimates and adaptivity for Stokes flow with implicit constitutive laws

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Outline

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Stokes flow with implicit constitutive laws

Stokes flow

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\begin{aligned}
-\nabla\cdot\mathbb{s} + \nabla p &= \mathbf{f} && \text{in } \Omega,\\
\nabla\cdot\mathbf{u} &= 0 && \text{in } \Omega,\\
\mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega,\\
g(\mathbb{s}, \mathbb{d}(\mathbf{u})) &= \mathbb{O} && \text{in } \Omega
\end{aligned}
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Notation

- $\mathbf{u}$ velocity
- p pressure
- $\mathbb{s}$ shear stress
- $\mathbb{d}(\mathbf{u}) := \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^t)$ symmetric velocity gradient
- $\mathbf{f}$ volume forces
- μ viscosity
- τ_* yield stress

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Examples of nonlinear implicit constitutive laws

Bingham fluid

- $|\mathbb{s}| \leq \tau_* \Leftrightarrow \mathbb{d}(\mathbf{u}) = \mathbb{o}$,
  $|\mathbb{s}| > \tau_* \Leftrightarrow \mathbb{s} = \tau_* \frac{\mathbb{d}(\mathbf{u})}{|\mathbb{d}(\mathbf{u})|} + 2\mu\mathbb{d}(\mathbf{u})$ clas.
- $g(\mathbb{s}, \mathbb{d}(\mathbf{u})) := \mathbb{d}(\mathbf{u}) - \frac{(|\mathbb{s}|-\tau_*)^+}{2\mu(\tau_*+(|\mathbb{s}|-\tau_*)^+)}\mathbb{s}$ $\mathbb{d}(\mathbf{u}) = f(\mathbb{s})$

Herschel–Bulkley fluid, $r \in (1, \infty)$

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Power law fluid, $r \in (1, \infty)$

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Analysis of implicit constitutive laws

- Rajagopal (2003), introduction
- Bulíček, Gwiazda, Málek, Rajagopal, & Świerczewska-Gwiazda (2012), rigorous analysis

Numerical schemes for implicit constitutive laws

- Diening, Kreuzer, & Süli (2013), convergence of FEs
- Hron, Málek, Stebel, & Touška (2015), 3-field discretization

A posteriori error estimates for nonlinear Stokes flow

- Padra (1997), power and Carreau laws, nonconforming discretizations, residual energy norm estimators
- Fuchs & Repin (2006, 2009), Bingham, power law, Powell–Eyring, functional estimators
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Functional setting

Function spaces

- $\mathbf{V} := [W_0^{1,r}(\Omega)]^d$
- $Q := L_0^s(\Omega) := \{q \in L^s(\Omega);\ (q,1) = 0\};\ \frac{1}{r} + \frac{1}{s} = 1$
- $\mathbb{T} := \mathbb{t} \in [L^s(\Omega)]^{d\times d}_{\mathrm{sym}};\ \mathrm{tr}\,\mathbb{t} = 0$ a.e. in Ω

Corresponding inf–sup conditions

$$\inf_{q\in Q} \sup_{\mathbf{v}\in\mathbf{V}} \frac{(q, \nabla\cdot\mathbf{v})}{\|\nabla\mathbf{v}\|_r \|q\|_s} = \beta > 0,$$

$$\inf_{\mathbf{v}\in\mathbf{V}} \sup_{\mathbb{t}\in\mathbb{T}} \frac{(\mathbb{t}, \nabla\mathbf{v})}{\|\nabla\mathbf{v}\|_r \|\mathbb{t}\|_s} = \gamma > 0.$$

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Outline

Weak solution, approximate solution, and residuals

Weak solution

$(\mathbf{u}, p, \mathbb{s}) \in \mathbf{V} \times Q \times \mathbb{T}$ such that

$$
\begin{aligned}
(\mathbb{s}, \nabla \mathbf{v}) - (\nabla{\cdot}\mathbf{v}, p) &= (\mathbf{f}, \mathbf{v}) && \forall \mathbf{v} \in \mathbf{V}, && \text{linear}\\
(\nabla{\cdot}\mathbf{u}, q) &= 0 && \forall q \in Q, && \text{linear}\\
(g(\mathbb{s}, \mathbb{d}(\mathbf{u})), \mathbb{t}) &= 0 && \forall \mathbb{t} \in \mathbb{T}. && \text{nonlinear}
\end{aligned}
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Approximate solution

$$(\mathbf{u}_h, p_h, \mathbb{s}_h) \in \mathbf{V} \times Q \times \mathbb{T}$$

Residuals

$$\mathcal{R}_1(\mathbb{s}_h, p_h) \in \mathbf{V}', \mathcal{R}_2(\mathbf{u}_h) \in Q', \mathcal{R}_3(\mathbb{s}_h, \mathbf{u}_h) \in \mathbb{T}'$$

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\begin{aligned}
\langle \mathcal{R}_1(\mathbb{s}_h, p_h), \mathbf{v}\rangle_{\mathbf{V}',\mathbf{V}} &:= (\mathbf{f}, \mathbf{v}) - (\mathbb{s}_h, \nabla \mathbf{v}) + (\nabla{\cdot}\mathbf{v}, p_h), && \mathbf{v} \in \mathbf{V},\\
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Residuals

Dual norms of the residuals

$$\|\mathcal{R}_1(\mathbb{s}_h, p_h)\|_{\mathbf{V}'} := \sup_{\mathbf{v}\in\mathbf{V};\ \mu_*\|\nabla\mathbf{v}\|_r=1} \langle \mathcal{R}_1(\mathbb{s}_h, p_h), \mathbf{v}\rangle_{\mathbf{V}',\mathbf{V}},$$

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$$= \underbrace{\beta^{-1}\|\nabla\cdot(\mathbf{u}-\mathbf{u}_h)\|_r}_{\text{strong error norm}} = \underbrace{\beta^{-1}\|\nabla\cdot\mathbf{u}_h\|_r}_{\text{computable estimate}},$$

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$$= \underbrace{\gamma^{-1}\|^{\delta}(\mathbb{g}(\mathbb{s},\mathbb{d}(\mathbf{u}))-\mathbb{g}(\mathbb{s}_h,\mathbb{d}(\mathbf{u}_h)))\|_r}_{\text{strong error norm}} = \underbrace{\gamma^{-1}\|^{\delta}\mathbb{g}(\mathbb{s}_h,\mathbb{d}(\mathbf{u}_h))\|_r}_{\text{computable estimate}}$$

Weak solution definition

- $\mathbf{u}_h = \mathbf{u}$, $p_h = p$, and $\mathbb{s}_h = \mathbb{s}$ if and only if $\|\mathcal{R}_1(\mathbb{s}_h, p_h)\|_{\mathbf{V}'} = 0$, $\|\mathcal{R}_2(\mathbf{u}_h)\|_{Q'} = 0$, and $\|\mathcal{R}_3(\mathbb{s}_h, \mathbf{u}_h)\|_{\mathbb{T}'} = 0$

Outline

Outline

A posteriori error estimate

Theorem (A guaranteed a posteriori error estimate)

Let

- $(\mathbf{u}_h, p_h, \mathbb{s}_h) \in \mathbf{V} \times Q \times \mathbb{T}$ *be arbitrary,*
- *Assumption A hold.*

Then there holds

$$\|\mathcal{R}_1(\mathbb{s}_h, p_h)\|_{\mathbf{V}'}^{s} + \|\mathcal{R}_2(\mathbf{u}_h)\|_{Q'}^{r} + \|\mathcal{R}_3(\mathbb{s}_h, \mathbf{u}_h)\|_{\mathbb{T}'}^{r}$$
$$\leq \sum_{K\in\mathcal{T}_h} (\eta_{\mathrm{O},K} + \eta_{\mathrm{F},K})^s + \sum_{K\in\mathcal{T}_h} (\eta_{\mathrm{D},K})^r + \sum_{K\in\mathcal{T}_h} (\eta_{\mathrm{I},K})^r.$$

Assumption A (Cauchy stress reconstruction)

There exists a stress reconstruction $\mathbb{w}_h \in \mathbb{H}^s(\mathrm{div}, \Omega)$ *such that*

$$-\nabla\cdot\mathbb{w}_h = \mathbf{f}_h, \qquad (\mathbf{f}_h, \mathbf{e}_m)_K = (\mathbf{f}, \mathbf{e}_m)_K \quad \forall 1 \leq m \leq d, \forall K \in \mathcal{T}_h.$$

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Outline

Finite elements, regularization

Finite element approximation

Find $(\mathbf{u}_h, p_h, \mathbb{s}_h) \in \mathbf{V}_h \subset \mathbf{V} \times Q_h \subset Q \times \mathbb{T}_h \subset \mathbb{T}$ such that

$$\begin{aligned} (\mathbb{s}_h, \nabla \mathbf{v}_h) - (\nabla \cdot \mathbf{v}_h, p_h) &= (\mathbf{f}, \mathbf{v}_h) && \forall \mathbf{v}_h \in \mathbf{V}_h, \\ (\nabla \cdot \mathbf{u}_h, q_h) &= 0 && \forall q_h \in Q_h, \\ (g(\mathbb{s}_h, \mathbb{d}(\mathbf{u}_h)), \mathbb{t}_h) &= 0 && \forall \mathbb{t}_h \in \mathbb{T}_h. \end{aligned}$$

Regularization of $g(\cdot,\cdot)$ by $g^\varepsilon(\cdot,\cdot)$, $\varepsilon > 0$

Find $(\mathbf{u}_h^\varepsilon, p_h^\varepsilon, \mathbb{s}_h^\varepsilon) \in \mathbf{V}_h \times Q_h \times \mathbb{T}_h$ such that

$$\begin{aligned} (\mathbb{s}_h^\varepsilon, \nabla \mathbf{v}_h) - (\nabla \cdot \mathbf{v}_h, p_h^\varepsilon) &= (\mathbf{f}, \mathbf{v}_h) && \forall \mathbf{v}_h \in \mathbf{V}_h, \\ (\nabla \cdot \mathbf{u}_h^\varepsilon, q_h) &= 0 && \forall q_h \in Q_h, \\ (g^\varepsilon(\mathbb{s}_h^\varepsilon, \mathbb{d}(\mathbf{u}_h^\varepsilon)), \mathbb{t}_h) &= 0 && \forall \mathbb{t}_h \in \mathbb{T}_h. \end{aligned}$$

Example (power law):

$$g^\varepsilon(\mathbb{s}, \mathbb{d}(\mathbf{u})) := \mathbb{d}(\mathbf{u}) - (2\mu)^{-\frac{1}{r-1}} (|\mathbb{s}|^2 + \varepsilon^2)^{\frac{2-r}{2(r-1)}} \mathbb{s}$$

Finite elements, regularization

Finite element approximation

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Linearization

Linearization of $g^{\varepsilon}(\cdot,\cdot)$ by $g^{\varepsilon,k-1}(\cdot,\cdot)$, $k \geq 1$

Find $(\mathbf{u}_h^{\varepsilon,k}, p_h^{\varepsilon,k}, \mathbb{s}_h^{\varepsilon,k}) \in \mathbf{V}_h \times Q_h \times \mathbb{T}_h$ such that

$$
\begin{aligned}
(\mathbb{s}_h^{\varepsilon,k}, \nabla \mathbf{v}_h) - (\nabla{\cdot}\mathbf{v}_h, p_h^{\varepsilon,k}) &= (\mathbf{f}, \mathbf{v}_h) && \forall \mathbf{v}_h \in \mathbf{V}_h,\\
(\nabla{\cdot}\mathbf{u}_h^{\varepsilon,k}, q_h) &= 0 && \forall q_h \in Q_h,\\
(g^{\varepsilon,k-1}(\mathbb{s}_h^{\varepsilon,k}, \mathbb{d}(\mathbf{u}_h^{\varepsilon,k})), \mathbb{t}_h) &= 0 && \forall \mathbb{t}_h \in \mathbb{T}_h.
\end{aligned}
$$

Example (power law, fixed point):

$$g^{\varepsilon,k-1}(\mathbb{s}, \mathbb{d}(\mathbf{u})) := \mathbb{d}(\mathbf{u}) - (2\mu)^{-\frac{1}{r-1}} (|\mathbb{s}_h^{\varepsilon,k-1}|^2 + \varepsilon^2)^{\frac{2-r}{2(r-1)}} \mathbb{s}$$

Example (power law, Newton):

$$
\begin{aligned}
g^{\varepsilon,k-1}(\mathbb{s}, \mathbb{d}(\mathbf{u})) := {}& \mathbb{d}(\mathbf{u}) - (2\mu)^{-\frac{1}{r-1}} (|\mathbb{s}_h^{\varepsilon,k-1}|^2 + \varepsilon^2)^{\frac{2-r}{2(r-1)}} \mathbb{s}\\
& - (2\mu)^{-\frac{1}{r-1}} \frac{2-r}{(r-1)} (|\mathbb{s}_h^{\varepsilon,k-1}|^2 + \varepsilon^2)^{\frac{4-3r}{2(r-1)}}\\
& (\mathbb{s}_h^{\varepsilon,k-1} \otimes \mathbb{s}_h^{\varepsilon,k-1})(\mathbb{s} - \mathbb{s}_h^{\varepsilon,k-1})
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$$

Linearization

Linearization of $g^{\varepsilon}(\cdot,\cdot)$ by $g^{\varepsilon,k-1}(\cdot,\cdot)$, $k \geq 1$

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(\mathbb{s}_h^{\varepsilon,k}, \nabla \mathbf{v}_h) - (\nabla\cdot \mathbf{v}_h, p_h^{\varepsilon,k}) &= (\mathbf{f}, \mathbf{v}_h) && \forall \mathbf{v}_h \in \mathbf{V}_h,\\
(\nabla\cdot \mathbf{u}_h^{\varepsilon,k}, q_h) &= 0 && \forall q_h \in Q_h,\\
(g^{\varepsilon,k-1}(\mathbb{s}_h^{\varepsilon,k}, \mathbb{d}(\mathbf{u}_h^{\varepsilon,k})), \mathbb{t}_h) &= 0 && \forall \mathbb{t}_h \in \mathbb{T}_h.
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$$

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$$
\begin{aligned}
g^{\varepsilon,k-1}(\mathbb{s}, \mathbb{d}(\mathbf{u})) := \mathbb{d}(\mathbf{u}) &- (2\mu)^{-\frac{1}{r-1}} (|\mathbb{s}_h^{\varepsilon,k-1}|^2 + \varepsilon^2)^{\frac{2-r}{2(r-1)}} \mathbb{s}\\
&- (2\mu)^{-\frac{1}{r-1}} \frac{2-r}{(r-1)} (|\mathbb{s}_h^{\varepsilon,k-1}|^2 + \varepsilon^2)^{\frac{4-3r}{2(r-1)}}\\
&(\mathbb{s}_h^{\varepsilon,k-1} \otimes \mathbb{s}_h^{\varepsilon,k-1})(\mathbb{s} - \mathbb{s}_h^{\varepsilon,k-1})
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$$

Linearization

Linearization of $g^{\varepsilon}(\cdot,\cdot)$ by $g^{\varepsilon,k-1}(\cdot,\cdot)$, $k \geq 1$

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\end{aligned}$$

Algebraic solver

Algebraic system solution, $i \geq 1$

Basis functions $\mathbf{v}_j$, q_j, $\mathbb{t}_j$ of $\mathbf{V}_h$, Q_h, $\mathbb{T}_h$, respectively. Algebraic residual vectors $R_{1,j}^{\varepsilon,k,i}$, $R_{2,j}^{\varepsilon,k,i}$, $R_{3,j}^{\varepsilon,k,i}$. Find $(\mathbf{u}_h^{\varepsilon,k,i}, p_h^{\varepsilon,k,i}, \mathbb{s}_h^{\varepsilon,k,i}) \in \mathbf{V}_h \times Q_h \times \mathbb{T}_h$ such that

$$\begin{aligned}
(\mathbb{s}_h^{\varepsilon,k,i}, \nabla \mathbf{v}_j) - (\nabla\cdot \mathbf{v}_j, p_h^{\varepsilon,k,i}) &= (\mathbf{f}, \mathbf{v}_j) - R_{1,j}^{\varepsilon,k,i} && \forall \mathbf{v}_j \in \mathbf{V}_h,\\
(\nabla\cdot \mathbf{u}_h^{\varepsilon,k,i}, q_j) &= -R_{2,j}^{\varepsilon,k,i} && \forall q_j \in Q_h,\\
(g^{\varepsilon,k-1}(\mathbb{s}_h^{\varepsilon,k,i}, \mathbb{d}(\mathbf{u}_h^{\varepsilon,k,i})), \mathbb{t}_j) &= -R_{3,j}^{\varepsilon,k,i} && \forall \mathbb{t}_j \in \mathbb{T}_h.
\end{aligned}$$

Outline

Estimate distinguishing different error components

Theorem (Estimate distinguishing different error components)

Let

- *regularization* ε, *lin. step* k, *and alg. step* i *be given,*
- $(\mathbf{u}_h^{\varepsilon,k,i}, p_h^{\varepsilon,k,i}, \mathbb{s}_h^{\varepsilon,k,i}) \in \mathbf{V}_h \times Q_h \times \mathbb{T}_h$ *be arbitrary,*
- *Assumption B hold.*

Then

$$\|\mathcal{R}_1(\mathbb{s}_h^{\varepsilon,k,i}, p_h^{\varepsilon,k,i})\|_{\mathbf{V}'}^{s} + \|\mathcal{R}_2(\mathbf{u}_h^{\varepsilon,k,i})\|_{Q'}^{r} + \|\mathcal{R}_3(\mathbb{s}_h^{\varepsilon,k,i}, \mathbf{u}_h^{\varepsilon,k,i})\|_{\mathbb{T}'}^{r}$$
$$\leq 4^{\max\{r,s\}-1}\left(\eta_{\mathrm{disc}}^{\varepsilon,k,i} + \eta_{\mathrm{reg}}^{\varepsilon,k,i} + \eta_{\mathrm{lin}}^{\varepsilon,k,i} + \eta_{\mathrm{alg}}^{\varepsilon,k,i} + \eta_{\mathrm{rem}}^{\varepsilon,k,i} + \eta_{\mathrm{osc}}^{\varepsilon,k,i}\right).$$

Assumption B (Stress reconstruction for each ε, k, and i)

For each ε, k, *and* i, *there exists a Cauchy stress reconstruct.* $\mathbb{w}_h^{\varepsilon,k,i} \in \mathbb{H}^s(\mathrm{div},\Omega)$ *& an algebraic remainder* $\rho^{\varepsilon,k,i} \in [L^s(\Omega)]^d$ *s.t.*

$$-\nabla\cdot\mathbb{w}_h^{\varepsilon,k,i} = \mathbf{f}_h - \rho^{\varepsilon,k,i}.$$

Moreover, $\mathbb{w}_h^{\varepsilon,k,i} = \mathbb{d}_h^{\varepsilon,k,i} + \mathbb{a}_h^{\varepsilon,k,i}$, $\mathbb{d}_h^{\varepsilon,k,i}, \mathbb{a}_h^{\varepsilon,k,i} \in [L^s(\Omega)]^{d\times d}$, *such that* $\|\mathbb{a}_h^{\varepsilon,k,i}\|_s \to 0$ *and* $\|\rho_h^{\varepsilon,k,i}\|_s \to 0$ *as the lin. solver converges.*

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Different error components

Discretization error

$$\eta_{\mathrm{disc}}^{\varepsilon,k,i} := \sum_{K\in\mathcal{T}_h} \{ \|\mu_*^{-1}(\mathbb{s}_h^{\varepsilon,k,i} - p_h^{\varepsilon,k,i}\mathbb{I} - \mathbb{d}_h^{\varepsilon,k,i})\|_{s,K}^s + \|\beta^{-1}(I - \Pi_{Q_h'})\nabla\cdot\mathbf{u}_h^{\varepsilon,k,i}\|_{r,K}^r + \|\gamma^{-1}(I - \Pi_{\mathbb{T}_h'})\ {}^\delta\mathbb{g}^{\varepsilon,k-1}(\mathbb{s}_h^{\varepsilon,k,i}, \mathbb{d}(\mathbf{u}_h^{\varepsilon,k,i}))\|_{r,K}^r \}$$

Regularization error

$$\eta_{\mathrm{reg}}^{\varepsilon,k,i} := \sum_{K\in\mathcal{T}_h} \|\gamma^{-1}\ {}^\delta(\mathbb{g}(\mathbb{s}_h^{\varepsilon,k,i}, \mathbb{d}(\mathbf{u}_h^{\varepsilon,k,i})) - \mathbb{g}^\varepsilon(\mathbb{s}_h^{\varepsilon,k,i}, \mathbb{d}(\mathbf{u}_h^{\varepsilon,k,i})))\|_{r,K}^r$$

Linearization error

$$\eta_{\mathrm{lin}}^{\varepsilon,k,i} := \sum_{K\in\mathcal{T}_h} \|\gamma^{-1}\ {}^\delta(\mathbb{g}^\varepsilon(\mathbb{s}_h^{\varepsilon,k,i}, \mathbb{d}(\mathbf{u}_h^{\varepsilon,k,i})) - \mathbb{g}^{\varepsilon,k-1}(\mathbb{s}_h^{\varepsilon,k,i}, \mathbb{d}(\mathbf{u}_h^{\varepsilon,k,i})))\|_{r,K}^r$$

Different error components

Discretization error

$$\eta_{\rm disc}^{\varepsilon,k,i} := \sum_{K\in\mathcal{T}_h} \{\|\mu_*^{-1}(\mathbb{s}_h^{\varepsilon,k,i} - p_h^{\varepsilon,k,i}\mathbb{I} - \mathbb{d}_h^{\varepsilon,k,i})\|_{s,K}^s + \|\beta^{-1}(I - \Pi_{Q_h'})\nabla\cdot\mathbf{u}_h^{\varepsilon,k,i}\|_{r,K}^r + \|\gamma^{-1}(I - \Pi_{\mathbb{T}_h'})\ {}^{\delta}\mathbb{g}^{\varepsilon,k-1}(\mathbb{s}_h^{\varepsilon,k,i}, \mathbb{d}(\mathbf{u}_h^{\varepsilon,k,i}))\|_{r,K}^r\}$$

Regularization error

$$\eta_{\rm reg}^{\varepsilon,k,i} := \sum_{K\in\mathcal{T}_h} \|\gamma^{-1}\ {}^{\delta}(\mathbb{g}(\mathbb{s}_h^{\varepsilon,k,i}, \mathbb{d}(\mathbf{u}_h^{\varepsilon,k,i})) - \mathbb{g}^{\varepsilon}(\mathbb{s}_h^{\varepsilon,k,i}, \mathbb{d}(\mathbf{u}_h^{\varepsilon,k,i})))\|_{r,K}^r$$

Linearization error

$$\eta_{\rm lin}^{\varepsilon,k,i} := \sum_{K\in\mathcal{T}_h} \|\gamma^{-1}\ {}^{\delta}(\mathbb{g}^{\varepsilon}(\mathbb{s}_h^{\varepsilon,k,i}, \mathbb{d}(\mathbf{u}_h^{\varepsilon,k,i})) - \mathbb{g}^{\varepsilon,k-1}(\mathbb{s}_h^{\varepsilon,k,i}, \mathbb{d}(\mathbf{u}_h^{\varepsilon,k,i})))\|_{r,K}^r$$

Different error components

Discretization error

$$\eta_{\mathrm{disc}}^{\varepsilon,k,i} := \sum_{K\in\mathcal{T}_h} \Big\{ \|\mu_*^{-1}(\mathbb{s}_h^{\varepsilon,k,i} - p_h^{\varepsilon,k,i}\mathbb{I} - \mathbb{d}_h^{\varepsilon,k,i})\|_{s,K}^{s} + \|\beta^{-1}(I - \Pi_{Q_h'})\nabla\cdot\mathbf{u}_h^{\varepsilon,k,i}\|_{r,K}^{r} + \|\gamma^{-1}(I - \Pi_{\mathbb{T}_h'})\ ^{\delta}\mathbb{g}^{\varepsilon,k-1}(\mathbb{s}_h^{\varepsilon,k,i}, \mathbb{d}(\mathbf{u}_h^{\varepsilon,k,i}))\|_{r,K}^{r} \Big\}$$

Regularization error

$$\eta_{\mathrm{reg}}^{\varepsilon,k,i} := \sum_{K\in\mathcal{T}_h} \|\gamma^{-1}\ ^{\delta}(\mathbb{g}(\mathbb{s}_h^{\varepsilon,k,i}, \mathbb{d}(\mathbf{u}_h^{\varepsilon,k,i})) - \mathbb{g}^{\varepsilon}(\mathbb{s}_h^{\varepsilon,k,i}, \mathbb{d}(\mathbf{u}_h^{\varepsilon,k,i})))\|_{r,K}^{r}$$

Linearization error

$$\eta_{\mathrm{lin}}^{\varepsilon,k,i} := \sum_{K\in\mathcal{T}_h} \|\gamma^{-1}\ ^{\delta}(\mathbb{g}^{\varepsilon}(\mathbb{s}_h^{\varepsilon,k,i}, \mathbb{d}(\mathbf{u}_h^{\varepsilon,k,i})) - \mathbb{g}^{\varepsilon,k-1}(\mathbb{s}_h^{\varepsilon,k,i}, \mathbb{d}(\mathbf{u}_h^{\varepsilon,k,i})))\|_{r,K}^{r}$$

Different error components

Algebraic error

$$\eta_{\mathrm{alg}}^{\varepsilon,k,i} := \sum_{K\in\mathcal{T}_h} \{\|\mu_*^{-1}\mathbb{a}_h^{\varepsilon,k,i}\|_{s,K}^s + \|\beta^{-1}\Pi_{Q_h'}\nabla\cdot\mathbf{u}_h^{\varepsilon,k,i}\|_{r,K}^r + \|\gamma^{-1}\Pi_{\mathbb{T}_h'}{}^{\delta}\mathbb{g}^{\varepsilon,k-1}(\mathbb{s}_h^{\varepsilon,k,i},\mathbb{d}(\mathbf{u}_h^{\varepsilon,k,i}))\|_{r,K}^r\}$$

Algebraic remainder error

$$\eta_{\mathrm{rem}}^{\varepsilon,k,i} := \sum_{K\in\mathcal{T}_h} (C_{\mathrm{F},r,d}h_\Omega\mu_*^{-1}\|\rho_h^{\varepsilon,k,i}\|_{s,K})^s$$

Data oscillation error

$$\eta_{\mathrm{osc}}^{\varepsilon,k,i} := \sum_{K\in\mathcal{T}_h} (C_{\mathrm{P},r,d}h_K\mu_*^{-1}\|\mathbf{f}-\mathbf{f}_h\|_{s,K})^s$$

Different error components

Algebraic error

$$\eta_{\mathrm{alg}}^{\varepsilon,k,i} := \sum_{K\in\mathcal{T}_h} \Big\{ \|\mu_*^{-1} \mathbb{a}_h^{\varepsilon,k,i}\|_{s,K}^{s} + \|\beta^{-1} \Pi_{Q_h'} \nabla{\cdot}\mathbf{u}_h^{\varepsilon,k,i}\|_{r,K}^{r} + \|\gamma^{-1} \Pi_{\mathbb{T}_h'}{}^{\delta}\mathbb{g}^{\varepsilon,k-1}(\mathbb{s}_h^{\varepsilon,k,i}, \mathbb{d}(\mathbf{u}_h^{\varepsilon,k,i}))\|_{r,K}^{r} \Big\}$$

Algebraic remainder error

$$\eta_{\mathrm{rem}}^{\varepsilon,k,i} := \sum_{K\in\mathcal{T}_h} \left(C_{\mathrm{F},r,d}\, h_\Omega \mu_*^{-1} \|\boldsymbol{\rho}_h^{\varepsilon,k,i}\|_{s,K}\right)^{s}$$

Data oscillation error

$$\eta_{\mathrm{osc}}^{\varepsilon,k,i} := \sum_{K\in\mathcal{T}_h} \left(C_{\mathrm{P},r,d}\, h_K \mu_*^{-1} \|\mathbf{f} - \mathbf{f}_h\|_{s,K}\right)^{s}$$

Different error components

Algebraic error

$$\eta_{\mathrm{alg}}^{\varepsilon,k,i} := \sum_{K\in\mathcal{T}_h} \Big\{ \|\mu_*^{-1} \mathbb{a}_h^{\varepsilon,k,i}\|_{s,K}^{s} + \|\beta^{-1} \Pi_{Q_h'} \nabla\cdot \mathbf{u}_h^{\varepsilon,k,i}\|_{r,K}^{r} + \|\gamma^{-1} \Pi_{\mathbb{T}_h'}\, {}^{\delta}\mathbb{g}^{\varepsilon,k-1}(\mathbb{s}_h^{\varepsilon,k,i}, \mathbb{d}(\mathbf{u}_h^{\varepsilon,k,i}))\|_{r,K}^{r} \Big\}$$

Algebraic remainder error

$$\eta_{\mathrm{rem}}^{\varepsilon,k,i} := \sum_{K\in\mathcal{T}_h} (C_{\mathrm{F},r,d} h_{\Omega} \mu_*^{-1} \|\boldsymbol{\rho}_h^{\varepsilon,k,i}\|_{s,K})^{s}$$

Data oscillation error

$$\eta_{\mathrm{osc}}^{\varepsilon,k,i} := \sum_{K\in\mathcal{T}_h} (C_{\mathrm{P},r,d} h_K \mu_*^{-1} \|\mathbf{f} - \mathbf{f}_h\|_{s,K})^{s}$$

Outline

Construction of the ingredients

Construction of $\mathbb{d}_h^{\varepsilon,k,i}$, $\mathbb{a}_h^{\varepsilon,k,i}$, and $\boldsymbol{\rho}^{\varepsilon,k,i}$

- construction of $\mathbb{d}_h^{\varepsilon,k,i}$; after ν additional algebraic solver steps: $\mathbb{d}_h^{\varepsilon,k,i+\nu}$
- set $\mathbb{a}_h^{\varepsilon,k,i} := \mathbb{d}_h^{\varepsilon,k,i+\nu} - \mathbb{d}_h^{\varepsilon,k,i}$
- set $\boldsymbol{\rho}^{\varepsilon,k,i}$ as the lifting of $R_1^{\varepsilon,k,i+\nu}$ to pw polynomials
- choose ν adaptively by a stopping criterion

Construction of $\mathbb{d}_h^{\varepsilon,k,i}$, $\rho^{\varepsilon,k,i}$ verifying Assumption B

Local Neumann(/Dirichlet) MFE problem on each patch $\mathcal{T}_{\mathbf{a}}$

Find $\mathbb{d}_{\mathbf{a}}^{\varepsilon,k,i} \in \mathbb{RTN}_l^{\mathrm{N},0}(\mathcal{T}_{\mathbf{a}})$ and $\mathbf{q}_{\mathbf{a}}^{\varepsilon,k,i} \in [\mathcal{P}_l^*(\mathcal{T}_{\mathbf{a}})]^d$ such that

$$(\mu_*^{-1}\mathbb{d}_{\mathbf{a}}^{\varepsilon,k,i}, \mathbb{v}_h)_{\mathcal{T}_{\mathbf{a}}} + (\mathbf{q}_{\mathbf{a}}^{\varepsilon,k,i}, \nabla{\cdot}\mathbb{v}_h^{\varepsilon,k,i})_{\mathcal{T}_{\mathbf{a}}} = (\mu_*^{-1}\psi_{\mathbf{a}}\mathbb{s}_h^{\varepsilon,k,i}, \mathbb{v}_h)_{\mathcal{T}_{\mathbf{a}}},$$

$$-((\nabla{\cdot}\mathbb{d}_{\mathbf{a}}^{\varepsilon,k,i})_m, (\phi_h)_m)_{\mathcal{T}_{\mathbf{a}}} = (\mathbf{f}{\cdot}\psi_{\mathbf{a},m} + \nabla{\cdot}\psi_{\mathbf{a},m}p_h^{\varepsilon,k,i} - \mathbb{s}_h^{\varepsilon,k,i} : \nabla\psi_{\mathbf{a},m}, (\phi_h)_m)_{\mathcal{T}_{\mathbf{a}}} - (R_{1,\mathbf{a},m}^{\varepsilon,k,i}, (\phi_h)_m)_{\mathcal{T}_{\mathbf{a}}}|\mathcal{T}_{\mathbf{a}}|^{-1}, \qquad 1 \le m \le d,$$

for all $(\mathbb{v}_h, \phi_h) \in \mathbb{RTN}_l^{\mathrm{N},0}(\mathcal{T}_{\mathbf{a}}) \times [\mathcal{P}_l^*(\mathcal{T}_{\mathbf{a}})]^d$; $\psi_{\mathbf{a},m} = (0, \ldots, \underbrace{\psi_{\mathbf{a}}}_{m}, \ldots, 0)^t$. Neumann compatibility condition from the scheme.

Construction of $\mathbb{d}_h^{\varepsilon,k,i}$

Set

$$\mathbb{d}_h^{\varepsilon,k,i} := \sum_{\mathbf{a}\in\mathcal{V}_h} \mathbb{d}_{\mathbf{a}}^{\varepsilon,k,i}.$$

Construction of $\rho^{\varepsilon,k,i}$

Set

$$(\rho^{\varepsilon,k,i-\nu}|_K)_m := \sum_{\mathbf{a}\in\mathcal{V}_K} |\mathcal{T}_{\mathbf{a}}|^{-1} R_{1,\mathbf{a},m}^{\varepsilon,k,i}, \qquad 1 \le m \le d$$

Construction of $\mathbb{d}_h^{\varepsilon,k,i}$, $\rho^{\varepsilon,k,i}$ verifying Assumption B

Local Neumann(/Dirichlet) MFE problem on each patch $\mathcal{T}_{\mathbf{a}}$

Find $\mathbb{d}_{\mathbf{a}}^{\varepsilon,k,i} \in \mathbb{RTN}_l^{\mathrm{N},0}(\mathcal{T}_{\mathbf{a}})$ and $\mathbf{q}_{\mathbf{a}}^{\varepsilon,k,i} \in [\mathcal{P}_l^*(\mathcal{T}_{\mathbf{a}})]^d$ such that

$$(\mu_*^{-1}\mathbb{d}_{\mathbf{a}}^{\varepsilon,k,i}, \mathbb{v}_h)_{\mathcal{T}_{\mathbf{a}}} + (\mathbf{q}_{\mathbf{a}}^{\varepsilon,k,i}, \nabla\cdot\mathbb{v}_h^{\varepsilon,k,i})_{\mathcal{T}_{\mathbf{a}}} = (\mu_*^{-1}\psi_{\mathbf{a}}\mathbb{s}_h^{\varepsilon,k,i}, \mathbb{v}_h)_{\mathcal{T}_{\mathbf{a}}},$$

$$-((\nabla\cdot\mathbb{d}_{\mathbf{a}}^{\varepsilon,k,i})_m, (\phi_h)_m)_{\mathcal{T}_{\mathbf{a}}} = (\mathbf{f}\cdot\psi_{\mathbf{a},m} + \nabla\cdot\psi_{\mathbf{a},m}p_h^{\varepsilon,k,i} - \mathbb{s}_h^{\varepsilon,k,i} : \nabla\psi_{\mathbf{a},m}, (\phi_h)_m)_{\mathcal{T}_{\mathbf{a}}} - (R_{1,\mathbf{a},m}^{\varepsilon,k,i}, (\phi_h)_m)_{\mathcal{T}_{\mathbf{a}}}|\mathcal{T}_{\mathbf{a}}|^{-1}, \qquad 1 \le m \le d,$$

for all $(\mathbb{v}_h, \phi_h) \in \mathbb{RTN}_l^{\mathrm{N},0}(\mathcal{T}_{\mathbf{a}}) \times [\mathcal{P}_l^*(\mathcal{T}_{\mathbf{a}})]^d$; $\psi_{\mathbf{a},m} = (0, \ldots, \underbrace{\psi_{\mathbf{a}}}_{m}, \ldots, 0)^t$. Neumann compatibility condition from the scheme.

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Find $\mathbb{d}_{\mathbf{a}}^{\varepsilon,k,i} \in \mathbb{RTN}_l^{\mathrm{N},0}(\mathcal{T}_{\mathbf{a}})$ and $\mathbf{q}_{\mathbf{a}}^{\varepsilon,k,i} \in [\mathcal{P}_l^*(\mathcal{T}_{\mathbf{a}})]^d$ such that

$$(\mu_*^{-1}\mathbb{d}_{\mathbf{a}}^{\varepsilon,k,i}, \mathbb{v}_h)_{\mathcal{T}_{\mathbf{a}}} + (\mathbf{q}_{\mathbf{a}}^{\varepsilon,k,i}, \nabla\cdot\mathbb{v}_h^{\varepsilon,k,i})_{\mathcal{T}_{\mathbf{a}}} = (\mu_*^{-1}\psi_{\mathbf{a}}\mathbb{s}_h^{\varepsilon,k,i}, \mathbb{v}_h)_{\mathcal{T}_{\mathbf{a}}},$$

$$-((\nabla\cdot\mathbb{d}_{\mathbf{a}}^{\varepsilon,k,i})_m, (\phi_h)_m)_{\mathcal{T}_{\mathbf{a}}} = (\mathbf{f}\cdot\boldsymbol{\psi}_{\mathbf{a},m} + \nabla\cdot\boldsymbol{\psi}_{\mathbf{a},m} p_h^{\varepsilon,k,i} - \mathbb{s}_h^{\varepsilon,k,i} : \nabla\boldsymbol{\psi}_{\mathbf{a},m}, (\phi_h)_m)_{\mathcal{T}_{\mathbf{a}}} - (R_{1,\mathbf{a},m}^{\varepsilon,k,i}, (\phi_h)_m)_{\mathcal{T}_{\mathbf{a}}} |\mathcal{T}_{\mathbf{a}}|^{-1}, \qquad 1 \le m \le d,$$

for all $(\mathbb{v}_h, \phi_h) \in \mathbb{RTN}_l^{\mathrm{N},0}(\mathcal{T}_{\mathbf{a}}) \times [\mathcal{P}_l^*(\mathcal{T}_{\mathbf{a}})]^d$; $\boldsymbol{\psi}_{\mathbf{a},m} = (0, \ldots, \underbrace{\psi_{\mathbf{a}}}_{m}, \ldots, 0)^t$. Neumann compatibility condition from the scheme.

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Set

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Construction of $\rho^{\varepsilon,k,i}$

Set

$$(\boldsymbol{\rho}^{\varepsilon,k,i-\nu}|_K)_m := \sum_{\mathbf{a}\in\mathcal{V}_K} |\mathcal{T}_{\mathbf{a}}|^{-1} R_{1,\mathbf{a},m}^{\varepsilon,k,i}, \qquad 1 \le m \le d$$

Outline

Stopping criteria

Global stopping criteria

- stop whenever:

$$\eta_{\mathrm{rem}}^{\varepsilon,k,i} \le \gamma_{\mathrm{rem}} \max\{\eta_{\mathrm{disc}}^{\varepsilon,k,i}, \eta_{\mathrm{reg}}^{\varepsilon,k,i}, \eta_{\mathrm{lin}}^{\varepsilon,k,i}, \eta_{\mathrm{alg}}^{\varepsilon,k,i}\} \qquad \text{choice of } \nu,$$

$$\eta_{\mathrm{alg}}^{\varepsilon,k,i} \le \gamma_{\mathrm{alg}} \max\{\eta_{\mathrm{disc}}^{\varepsilon,k,i}, \eta_{\mathrm{reg}}^{\varepsilon,k,i}, \eta_{\mathrm{lin}}^{\varepsilon,k,i}\} \qquad \text{algebraic solver},$$

$$\eta_{\mathrm{lin}}^{\varepsilon,k,i} \le \gamma_{\mathrm{lin}} \max\{\eta_{\mathrm{disc}}^{\varepsilon,k,i}, \eta_{\mathrm{reg}}^{\varepsilon,k,i}\} \qquad \text{linearization solver},$$

$$\eta_{\mathrm{reg}}^{\varepsilon,k,i} \le \gamma_{\mathrm{reg}} \eta_{\mathrm{disc}}^{\varepsilon,k,i} \qquad \text{choice of } \varepsilon.$$

- $\gamma_{\mathrm{rem}}, \gamma_{\mathrm{alg}}, \gamma_{\mathrm{lin}}, \gamma_{\mathrm{reg}} \approx 0.1$

Local stopping criteria

- same as above, mesh element by mesh element

Stopping criteria

Global stopping criteria

- stop whenever:

$$\eta_{\mathrm{rem}}^{\varepsilon,k,i} \leq \gamma_{\mathrm{rem}} \max\left\{\eta_{\mathrm{disc}}^{\varepsilon,k,i}, \eta_{\mathrm{reg}}^{\varepsilon,k,i}, \eta_{\mathrm{lin}}^{\varepsilon,k,i}, \eta_{\mathrm{alg}}^{\varepsilon,k,i}\right\} \qquad \text{choice of } \nu,$$

$$\eta_{\mathrm{alg}}^{\varepsilon,k,i} \leq \gamma_{\mathrm{alg}} \max\left\{\eta_{\mathrm{disc}}^{\varepsilon,k,i}, \eta_{\mathrm{reg}}^{\varepsilon,k,i}, \eta_{\mathrm{lin}}^{\varepsilon,k,i}\right\} \qquad \text{algebraic solver},$$

$$\eta_{\mathrm{lin}}^{\varepsilon,k,i} \leq \gamma_{\mathrm{lin}} \max\left\{\eta_{\mathrm{disc}}^{\varepsilon,k,i}, \eta_{\mathrm{reg}}^{\varepsilon,k,i}\right\} \qquad \text{linearization solver},$$

$$\eta_{\mathrm{reg}}^{\varepsilon,k,i} \leq \gamma_{\mathrm{reg}} \eta_{\mathrm{disc}}^{\varepsilon,k,i} \qquad \text{choice of } \varepsilon.$$

- $\gamma_{\mathrm{rem}}, \gamma_{\mathrm{alg}}, \gamma_{\mathrm{lin}}, \gamma_{\mathrm{reg}} \approx 0.1$

Local stopping criteria

- same as above, mesh element by mesh element

Outline

Efficiency of the a posteriori error estimate

Theorem (Global efficiency of the a posteriori error estimate)

Let the mesh $\mathcal{T}_h$ be shape-regular, the global stopping criteria hold, $\gamma_{\rm lin}$ and $\gamma_{\rm reg}$ be small enough, and $\mathbf{f} = \mathbf{f}_h$. Then

$$\eta_{\rm disc}^{\varepsilon,k,i} + \eta_{\rm reg}^{\varepsilon,k,i} + \eta_{\rm lin}^{\varepsilon,k,i} + \eta_{\rm alg}^{\varepsilon,k,i} + \eta_{\rm rem}^{\varepsilon,k,i} + \eta_{\rm osc}^{\varepsilon,k,i}$$
$$\lesssim \|\mathcal{R}_1(\mathbb{s}_h^{\varepsilon,k,i}, p_h^{\varepsilon,k,i})\|_{\mathbf{V}'}^{s} + \|\mathcal{R}_2(\mathbf{u}_h^{\varepsilon,k,i})\|_{Q'}^{r} + \|\mathcal{R}_3(\mathbb{s}_h^{\varepsilon,k,i}, \mathbf{u}_h^{\varepsilon,k,i})\|_{\mathbb{T}'}^{r}.$$

- robustness with respect to the nonlinearity thanks to the choice of the dual norms of the residuals as error measure
- local efficiency under local stopping criteria

Efficiency of the a posteriori error estimate

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$$\lesssim \|\mathcal{R}_1(\mathbb{s}_h^{\varepsilon,k,i}, p_h^{\varepsilon,k,i})\|_{\mathbf{V}'}^{s} + \|\mathcal{R}_2(\mathbf{u}_h^{\varepsilon,k,i})\|_{Q'}^{r} + \|\mathcal{R}_3(\mathbb{s}_h^{\varepsilon,k,i}, \mathbf{u}_h^{\varepsilon,k,i})\|_{\mathbb{T}'}^{r}.$$

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$$\eta_{\rm disc}^{\varepsilon,k,i} + \eta_{\rm reg}^{\varepsilon,k,i} + \eta_{\rm lin}^{\varepsilon,k,i} + \eta_{\rm alg}^{\varepsilon,k,i} + \eta_{\rm rem}^{\varepsilon,k,i} + \eta_{\rm osc}^{\varepsilon,k,i}$$
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Outline

1. Introduction
2. Velocity–pressure–stress formulation
   - Weak solution, approximate solution, residuals
   - Dual norms of the residuals
3. A posteriori error estimates
   - Basic estimate
   - Finite element implementation
   - Estimate distinguishing different error components
   - Construction of the ingredients
   - Stopping criteria
   - Efficiency and robustness
4. Localization of the dual norms of residuals
5. Link dual norms of the residuals – energy-type errors
6. 1st numerical results
7. Outlook and references

Localization of the dual norms of residuals

Localization of $\mathcal{R}_2(\mathbf{u}_h)$ and $\mathcal{R}_3(\mathbf{u}_h)$ (reminder)

$$\|\mathcal{R}_2(\mathbf{u}_h)\|_{Q'} = \beta^{-1}\left\{\sum_{K\in\mathcal{T}_h}\|\nabla\cdot(\mathbf{u}-\mathbf{u}_h)\|_{r,K}^r\right\}^{\frac{1}{r}}$$

$$\|\mathcal{R}_3(\mathbb{s}_h,\mathbf{u}_h)\|_{\mathbb{T}'} = \gamma^{-1}\left\{\sum_{K\in\mathcal{T}_h}\|^{\delta}(\mathrm{g}(\mathbb{s},\mathbb{d}(\mathbf{u}))-\mathrm{g}(\mathbb{s}_h,\mathbb{d}(\mathbf{u}_h)))\|_{r,K}^r\right\}^{\frac{1}{r}}$$

Theorem (Localization of the equilibrium residual $\mathcal{R}_1$)

Let $\langle\mathcal{R}_1(\mathbb{s}_h,p_h),\psi_{\mathbf{a},m}\rangle_{\mathbf{V}',\mathbf{V}}=0$ for all $1\le m\le d$, $\forall\mathbf{a}\in\mathcal{V}_h^{\mathrm{int}}$. Then

$$\|\mathcal{R}_1(\mathbb{s}_h,p_h)\|_{\mathbf{V}'}\le(d+1)^{\frac{1}{r}}C_{\mathrm{cont,PF},d}\left\{\sum_{\mathbf{a}\in\mathcal{V}_h}\|\mathcal{R}_1(\mathbb{s}_h,p_h)\|_{\mathbf{V}(\omega_{\mathbf{a}})'}^s\right\}^{\frac{1}{s}},$$

$$\left\{\sum_{\mathbf{a}\in\mathcal{V}_h}\|\mathcal{R}_1(\mathbb{s}_h,p_h)\|_{\mathbf{V}(\omega_{\mathbf{a}})'}^s\right\}^{\frac{1}{s}}\le(d+1)^{\frac{1}{s}}\|\mathcal{R}_1(\mathbb{s}_h,p_h)\|_{\mathbf{V}'}.$$

Localization of the dual norms of residuals

Localization of $\mathcal{R}_2(\mathbf{u}_h)$ and $\mathcal{R}_3(\mathbf{u}_h)$ (reminder)

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Theorem (Localization of the equilibrium residual $\mathcal{R}_1$)

Let $\langle\mathcal{R}_1(\mathbb{s}_h,p_h),\psi_{\mathbf{a},m}\rangle_{\mathbf{V}',\mathbf{V}} = 0$ *for all* $1\leq m\leq d$, $\forall\mathbf{a}\in\mathcal{V}_h^{\mathrm{int}}$. *Then*

$$\|\mathcal{R}_1(\mathbb{s}_h,p_h)\|_{\mathbf{V}'} \leq (d+1)^{\frac{1}{r}}C_{\mathrm{cont,PF},d}\left\{\sum_{\mathbf{a}\in\mathcal{V}_h}\|\mathcal{R}_1(\mathbb{s}_h,p_h)\|_{\mathbf{V}(\omega_{\mathbf{a}})'}^s\right\}^{\frac{1}{s}},$$

$$\left\{\sum_{\mathbf{a}\in\mathcal{V}_h}\|\mathcal{R}_1(\mathbb{s}_h,p_h)\|_{\mathbf{V}(\omega_{\mathbf{a}})'}^s\right\}^{\frac{1}{s}} \leq (d+1)^{\frac{1}{s}}\|\mathcal{R}_1(\mathbb{s}_h,p_h)\|_{\mathbf{V}'}.$$

Outline

Link dual norm of the residual – error

Perturbation nonlinearities

$\mathbb{s} \in \partial(\mu|\mathbb{d}(\mathbf{u})|^2 + \pi(\mathbb{d}(\mathbf{u})))$, π convex

- $\pi(\mathbb{d}(\mathbf{u})) = \tau_*|\mathbb{d}(\mathbf{u})|$: Bingham
- $\pi(\mathbb{d}(\mathbf{u})) = k|\mathbb{d}(\mathbf{u})|^r$, $k > 0$, $r \in (1, \infty)$: modified power law

Theorem (Energy error estimate for perturbation nonlinearities)

Let $\mathbf{V}_0 := \{\mathbf{v} \in \mathbf{V};\ \nabla\cdot\mathbf{v} = 0\}$ *and set* $\mu_* = 1$. *Then, for perturbation nonlinearities,*

$$\|\mu^{1/2}\mathbb{d}(\mathbf{u}-\mathbf{u}_h)\|^2 + \underbrace{(\pi'(\mathbb{d}(\mathbf{u})) - \pi'(\mathbb{d}(\mathbf{u}_h)), \mathbb{d}(\mathbf{u}-\mathbf{u}_h))}_{\geq 0}$$

$$\leq \frac{1}{2}\|\mathcal{R}_1(\mathbb{s}(\mathbb{d}(\mathbf{u}_h)), p_h)\|^2_{\mathbf{V}_0'}$$

Link dual norm of the residual – error

Perturbation nonlinearities

$\mathbb{s} \in \partial(\mu|\mathbb{d}(\mathbf{u})|^2 + \pi(\mathbb{d}(\mathbf{u})))$, π convex

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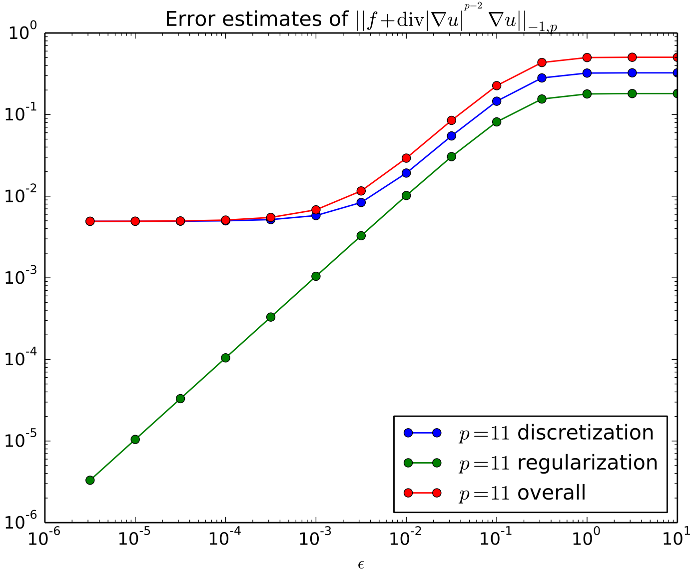
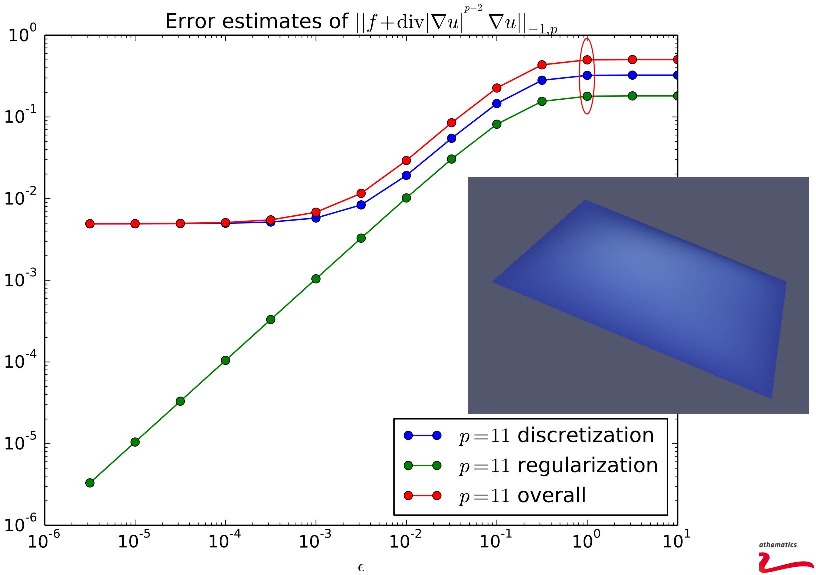
$$\leq \frac{1}{2}\|\mathcal{R}_1(\mathbb{s}(\mathbb{d}(\mathbf{u}_h)), p_h)\|^2_{\mathbf{V}_0'}$$

Outline

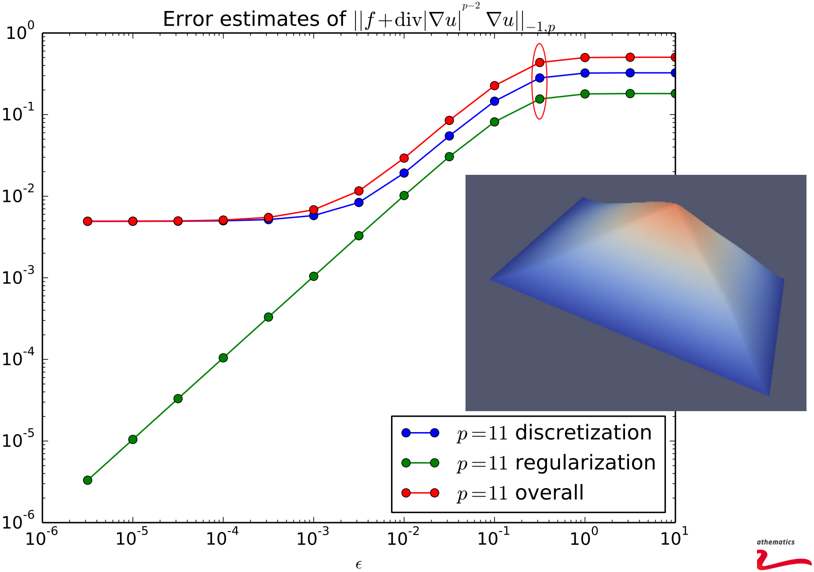
p-Laplace regularization test case



p-Laplace regularization test case



p-Laplace regularization test case



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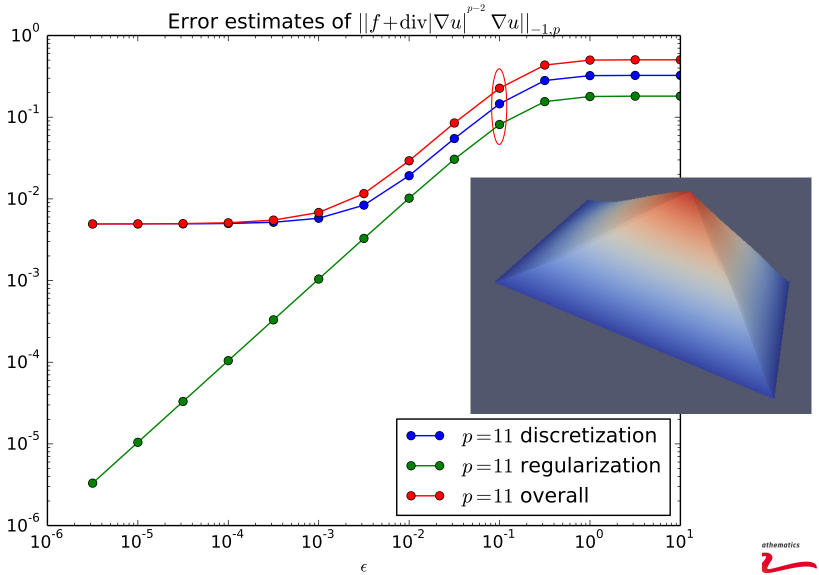

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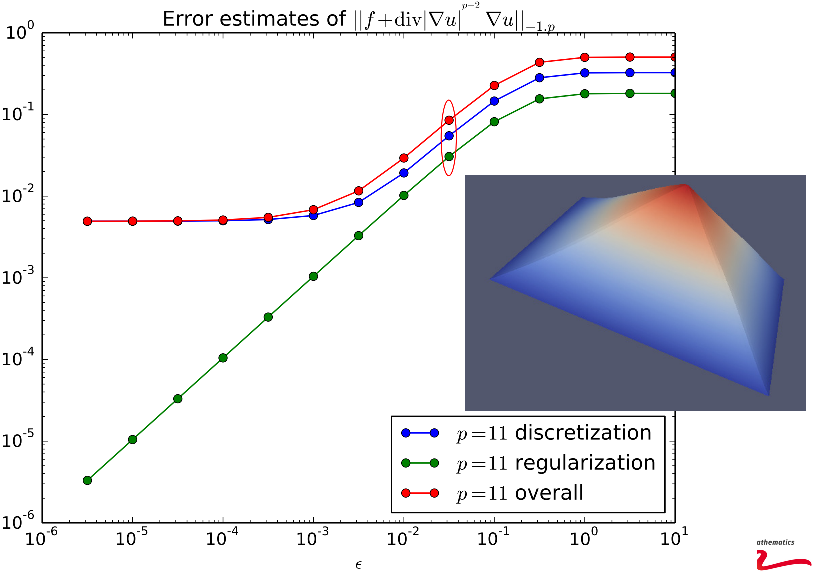
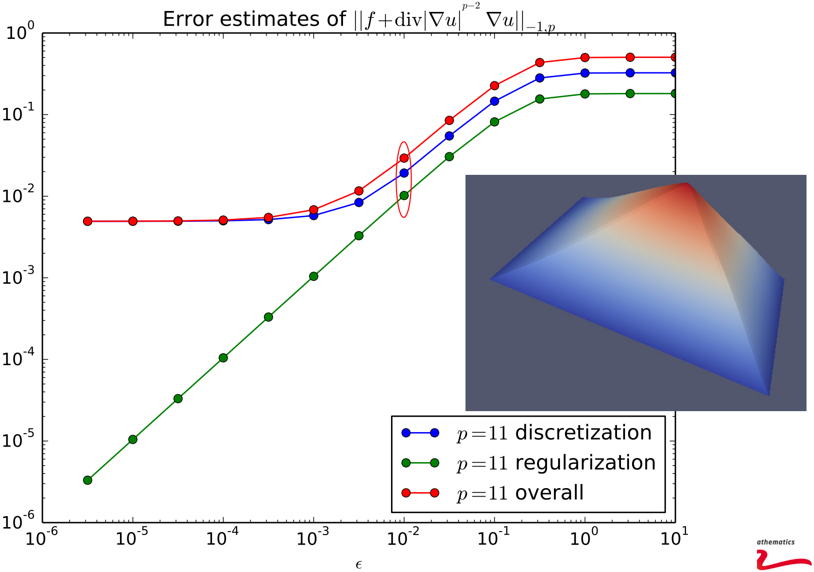


p-Laplace regularization test case



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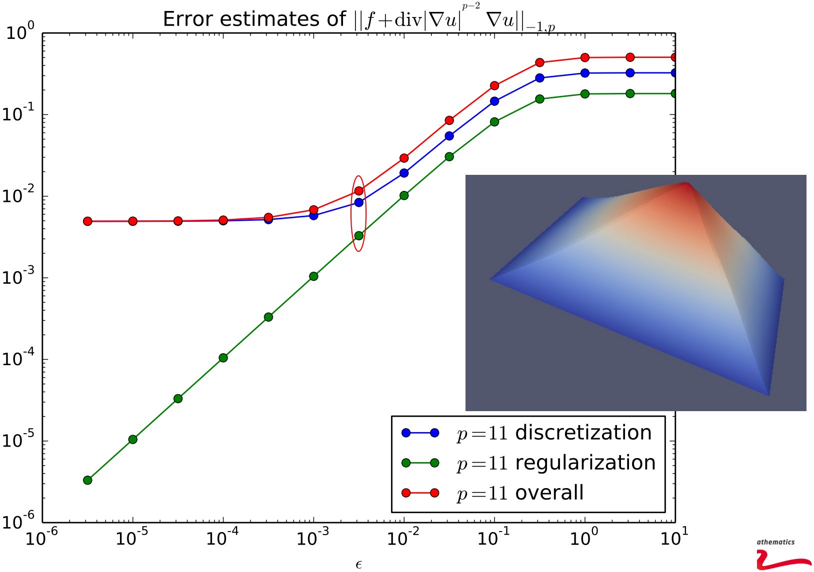
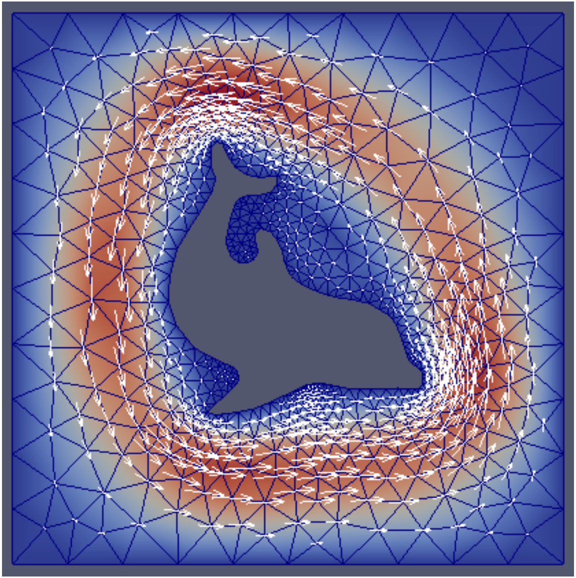


Power law flow around a dolphin with mesh adaptivity



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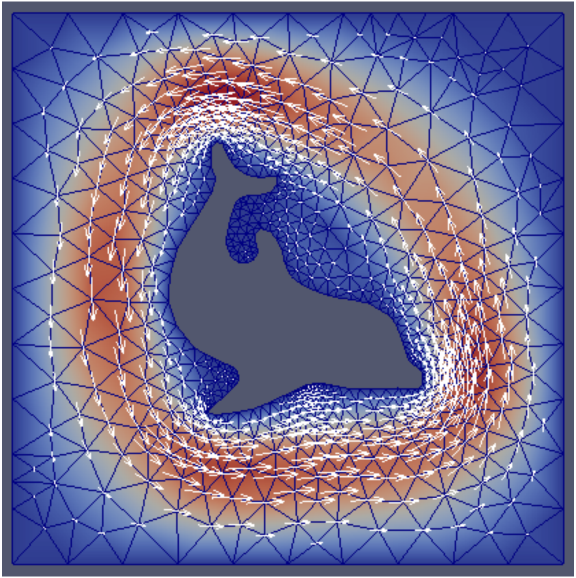

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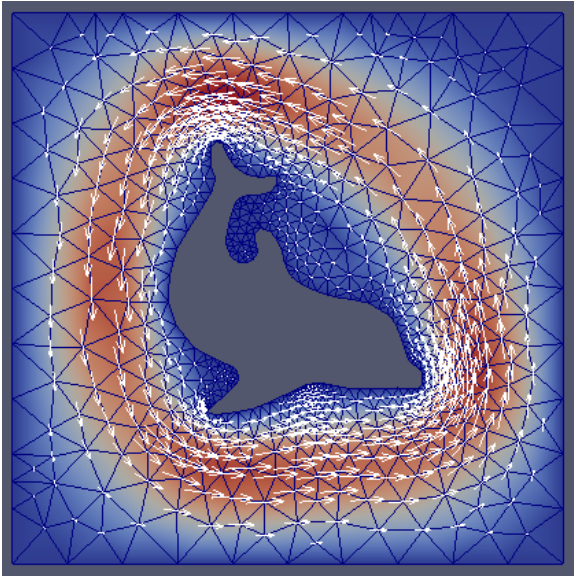

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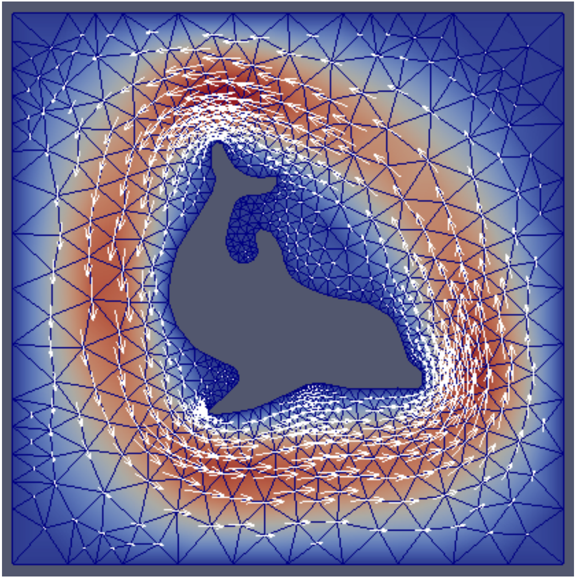

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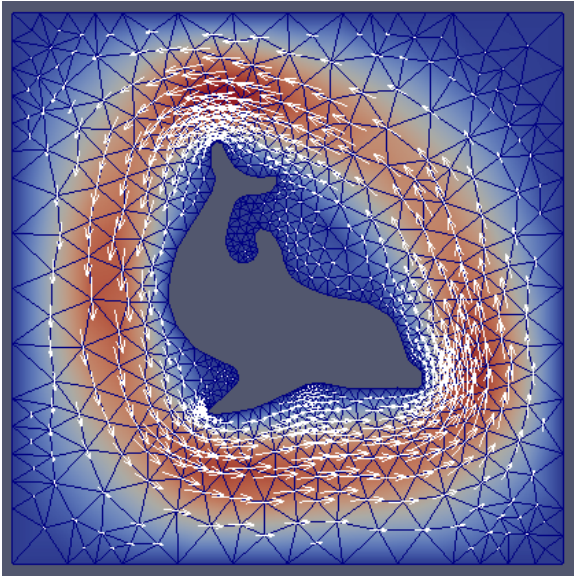

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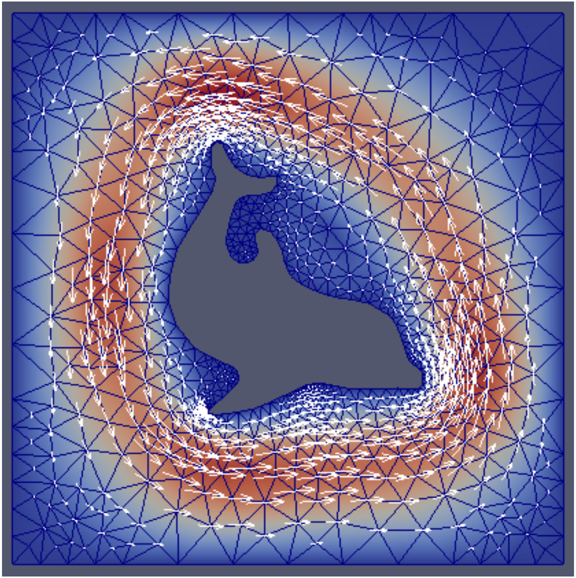

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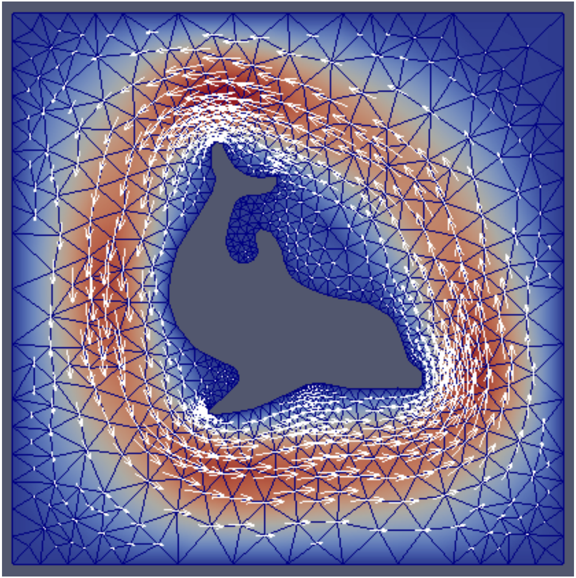

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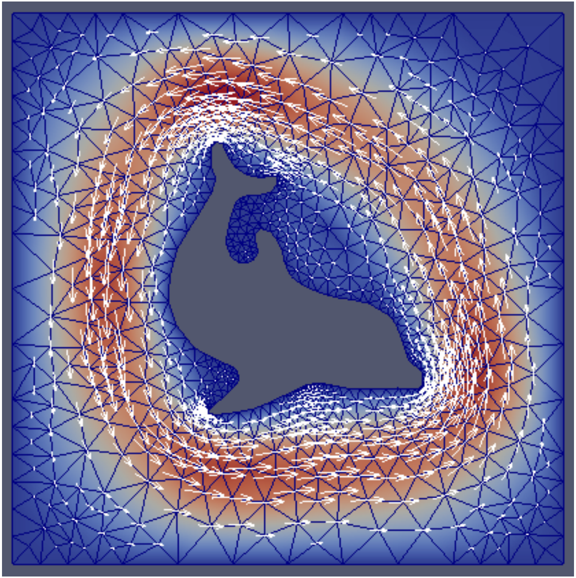

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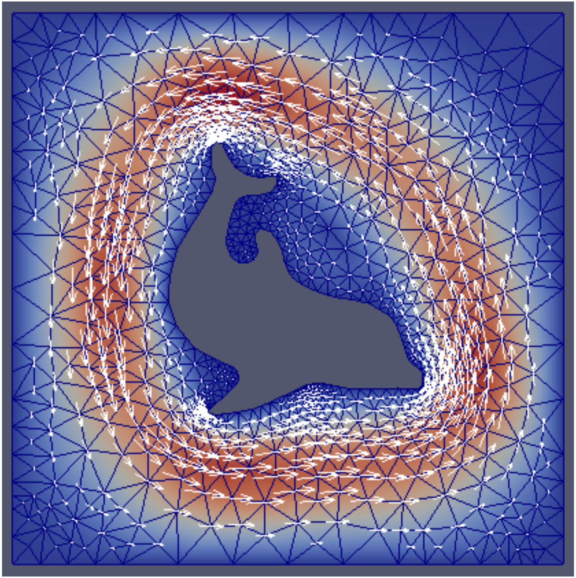

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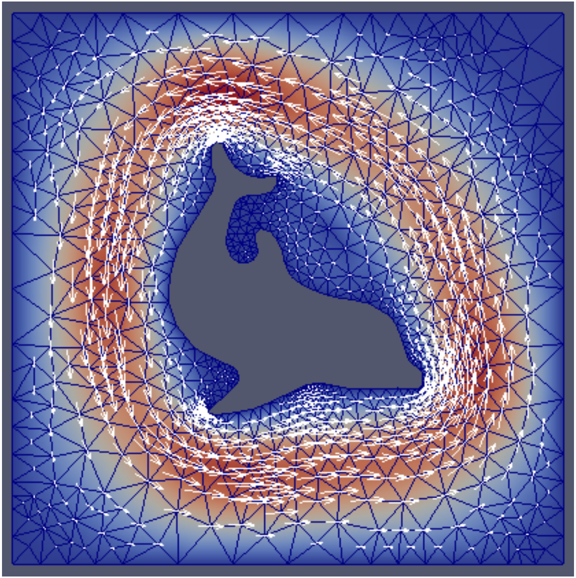

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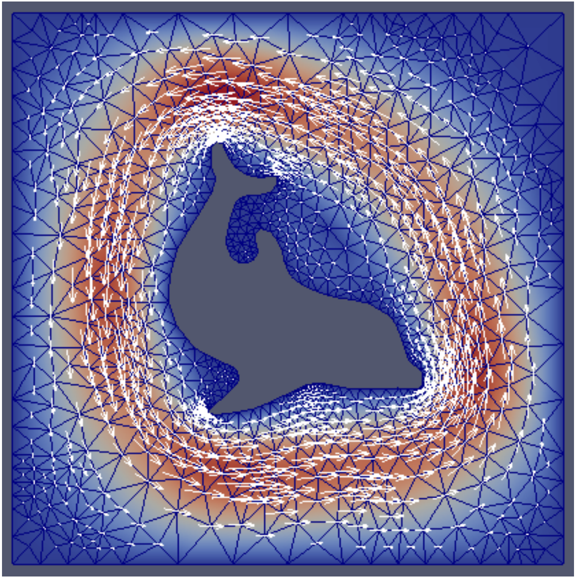

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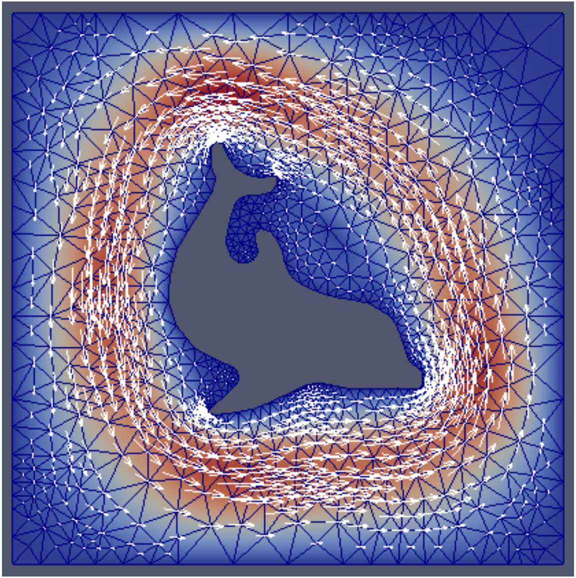

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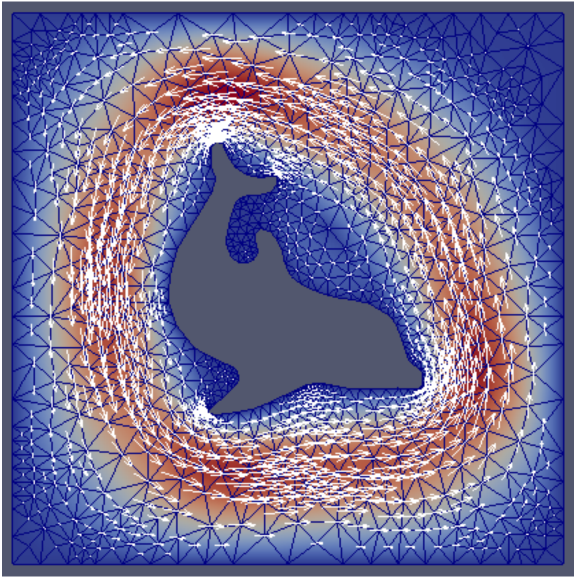

Outline

Outlook and references

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- simplified evaluation of the estimators
- guaranteed upper bound and robustness in quasi-norms
- generalizations (Navier–Stokes, time-dependent, ...)

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Thank you for your attention!

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