Krylov Subspace-Based Model Order Reduction of RCL Circuit Equations

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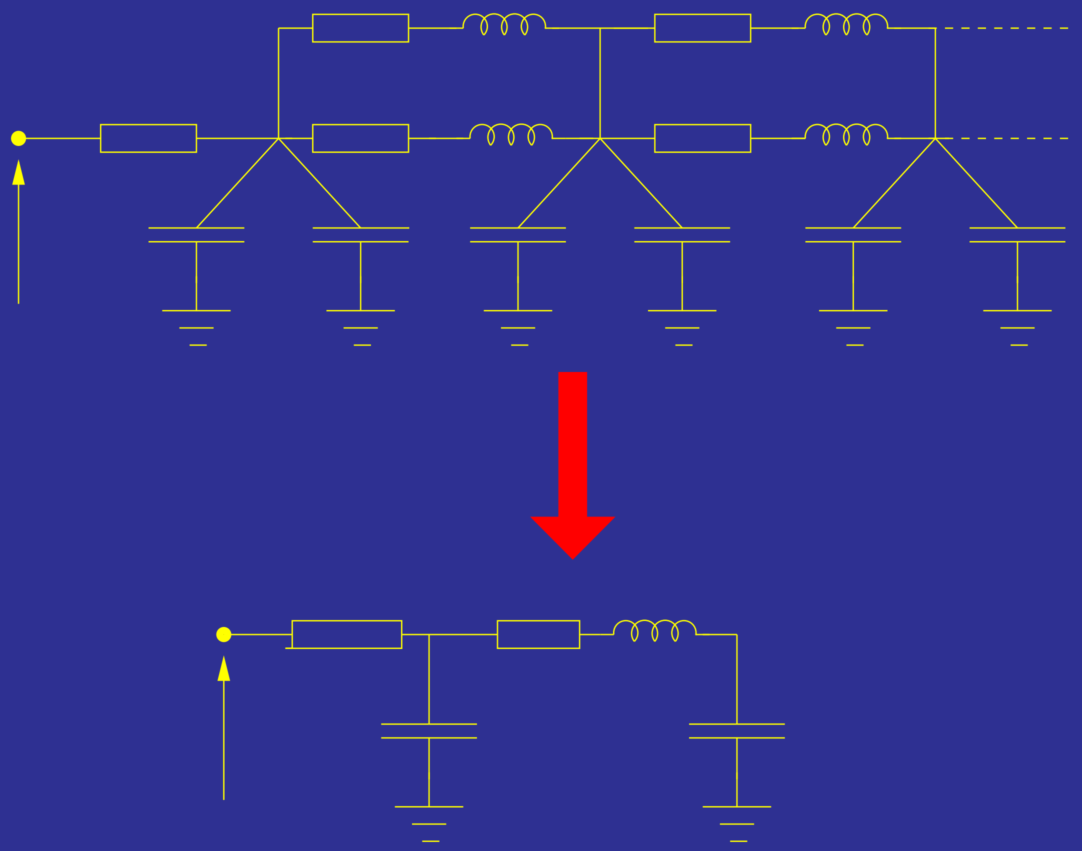
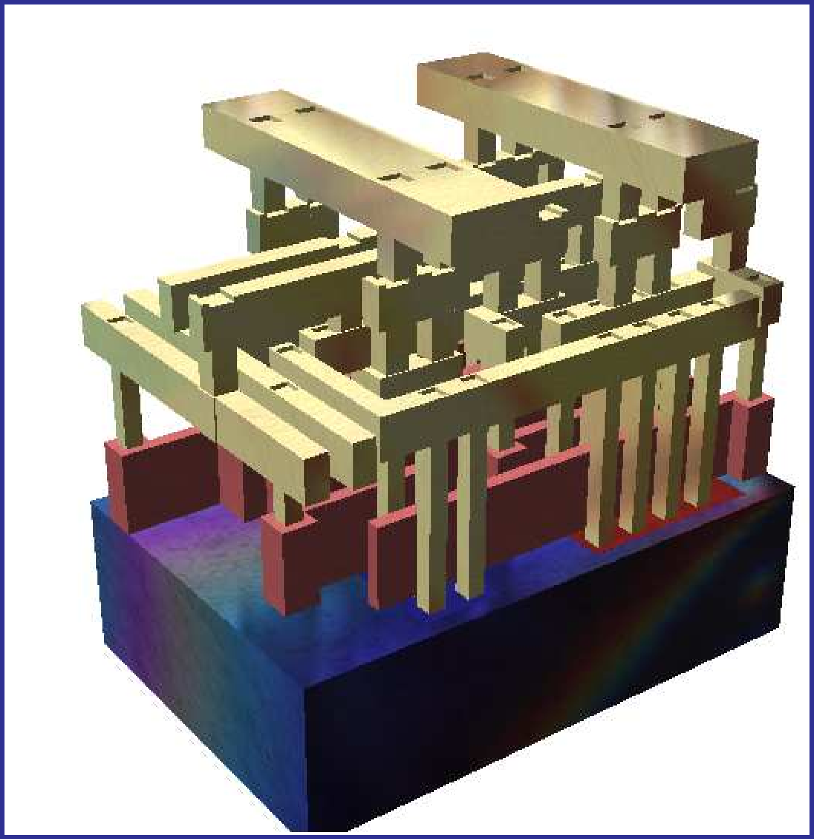
Problem: reduction of RCL networks

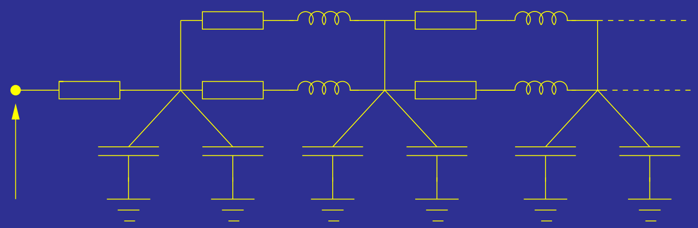


Modeling of chip's "wiring"



Efficient reduction of RCL networks

- RCL network as a linear dynamical system:



$$\Longleftrightarrow \quad \mathbf{H}(s) = \mathbf{B}^{\top}\Big(s\,\mathbf{C} + \mathbf{G}\Big)^{-1}\mathbf{B}$$

- Model order reduction (based on Krylov subspace methods):

$$\mathbf{H}(s) \;\approx\; \mathbf{H}_n(s) = \mathbf{B}_n^{\top}\Big(s\,\mathbf{C}_n + \mathbf{G}_n\Big)^{-1}\mathbf{B}_n$$

- Can we find an RCL network corresponding to $\mathbf{H}_n$?

In general, no!

- RCL networks (and $\mathbf{H}$) are passive and reciprocal
- Need to use model order reduction methods that guarantee passive and reciprocal $\mathbf{H}_n$
- Classical results from network synthesis:
  If $\mathbf{H}_n$ is passive, then there exists a corresponding physical electrical circuit, but not necessarily one with only R's, C's, and L's
  If $\mathbf{H}_n$ is also reciprocal, then 'fewer' non-RCL elements are needed

Some history

- **Elmore delay** of RC networks:
  Based on a model with a single R and a single C such that

$$\mathbf{H}(s) \approx \mathbf{H}_1(s) = \frac{a}{s+b}$$

  with matching of first two **moments** (= Taylor coefficients)

- **AWE** (Pillage and Rohrer, '90):

$$\mathbf{H}(s) \approx \mathbf{H}_n(s) = \frac{p_{n-1}(s)}{q_n(s)}$$

  with matching of first $2n$ moments

Some history

- **PVL, MPVL** (Feldmann and F., '94 and '95):
  Avoids numerical issues of AWE by computing Padé reduced-order models via the Lanczos process

- **PRIMA** (Odabasioglu, Celik, and Pileggi, '97):
  Passivity via explicit projection onto Krylov subspaces

- **Split congruence transformations** (Kerns and Yang, '97)

- **SPRIM** (F., '04 and '11)
  Passivity and reciprocity via explicit projection

Outline

- RCL network equations
- Projection onto Krylov subspaces
- PRIMA and SPRIM
- SPRIM revisited
- Open problems

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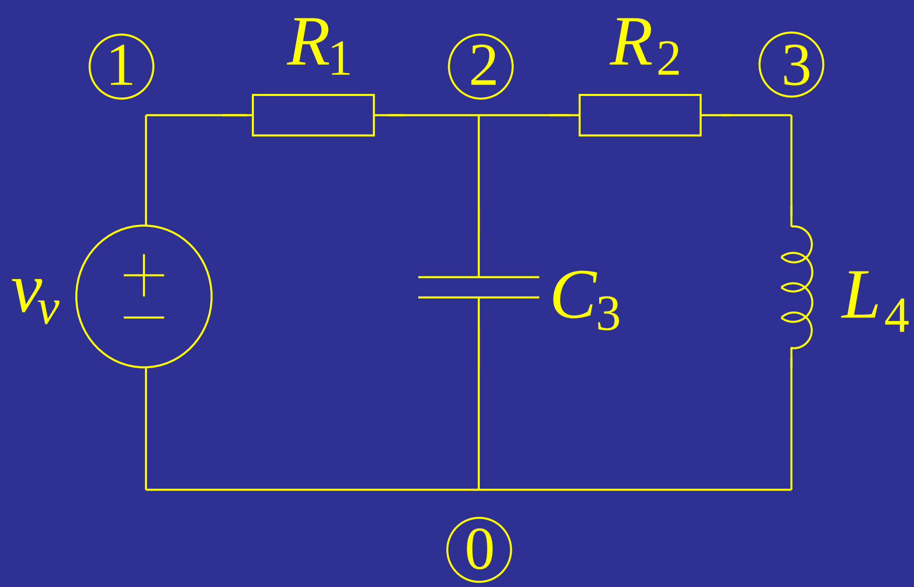
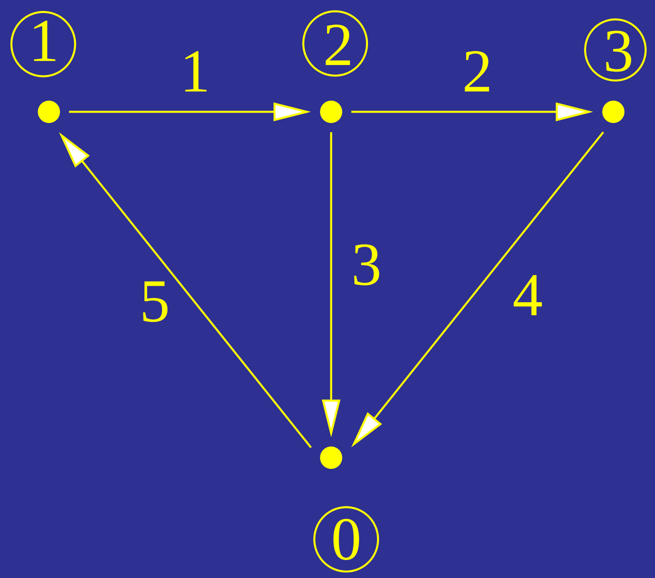
RCL networks as directed graphs



RCL network

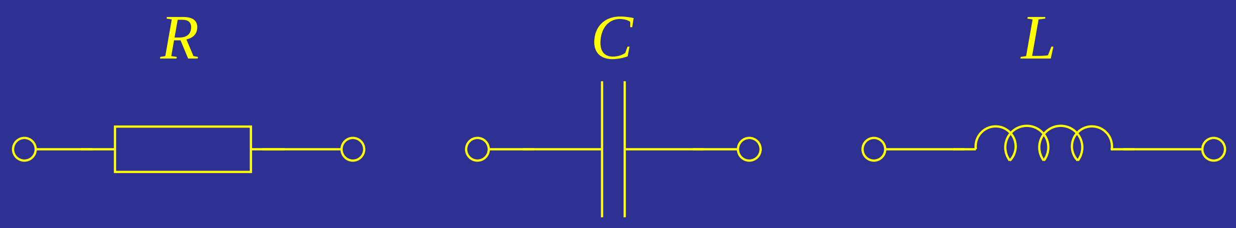


directed graph

Network topology $\Longleftrightarrow$ Graph incidence matrix $\mathcal{A}$

RCL network equations

- Kirchhoff's current laws: $\mathcal{A}\,\mathbf{i}_{\mathcal{E}} = \mathbf{0}$
- Kirchhoff's voltage laws: $\mathcal{A}^{\top}\mathbf{v} = \mathbf{v}_{\mathcal{E}}$
- Equations for R's, C's, and L's:

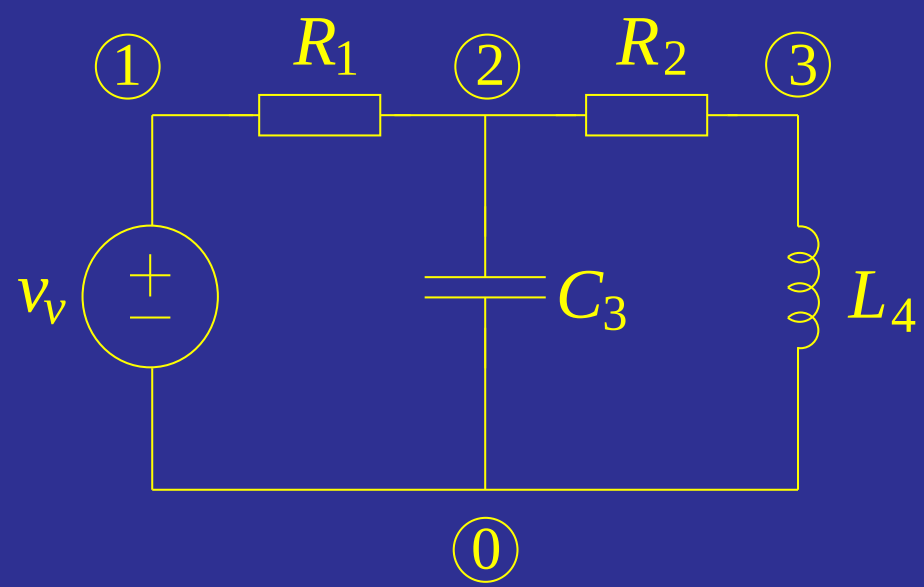


$$v_r = R\, i_r \qquad i_c = C \frac{d}{dt} v_c \qquad v_l = L \frac{d}{dt} i_l$$

Modified nodal analysis

- 'Easy' eliminations leave only $\mathbf{v}$, $\mathbf{i}_l$, and $\mathbf{i}_v$ as unknowns

Example



5 unknowns: v_1, v_2, v_3, $i_l = i_4$, and i_v

Example

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & C_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ i_l \\ i_v \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1} & -\frac{1}{R_1} & 0 & 0 & -1 \\ -\frac{1}{R_1} & \frac{1}{R_1} + \frac{1}{R_2} & -\frac{1}{R_2} & 0 & 0 \\ 0 & -\frac{1}{R_2} & \frac{1}{R_2} & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ i_l \\ i_v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} v_v$$

Input: $\mathbf{u}(t) = v_v(t)$

Output: $\mathbf{y}(t) = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ i_l \\ i_v \end{bmatrix} = -i_v(t)$

General RCL network equations

Linear time-invariant dynamical system:

$$\mathbf{C}\,\frac{d}{dt}\mathbf{x}(t) + \mathbf{G}\,\mathbf{x}(t) = \mathbf{B}\,\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{B}^{\top}\mathbf{x}(t)$$

where

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{v}(t) \\ \mathbf{i}_l(t) \\ \mathbf{i}_v(t) \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \mathcal{A}_c \mathcal{C} \mathcal{A}_c^{\top} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathcal{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \mathcal{A}_r \mathcal{R}^{-1} \mathcal{A}_r^{\top} & \mathcal{A}_l & \mathcal{A}_v \\ -\mathcal{A}_l^{\top} & \mathbf{0} & \mathbf{0} \\ -\mathcal{A}_v^{\top} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$\mathbf{u}(t) = \begin{bmatrix} \mathbf{v}_v(t) \\ -\mathbf{i}_i(t) \end{bmatrix}, \quad \mathbf{y}(t) = \begin{bmatrix} -\mathbf{i}_v(t) \\ \mathbf{v}_i(t) \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{0} & \mathcal{A}_i \\ \mathbf{0} & \mathbf{0} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}$$

Passivity and reciprocity

- $\mathcal{R}$ and $\mathcal{C}$ are diagonal with positive diagonal entries,

$$\mathcal{A}_r\mathcal{R}^{-1}\mathcal{A}_r^\top \succeq \mathbf{0} \quad \text{and} \quad \mathcal{A}_r\mathcal{C}\mathcal{A}_r^\top \succeq \mathbf{0},$$

  and $\mathcal{L} \succ \mathbf{0}$

- Passivity follows from

$$\mathbf{C} = \begin{bmatrix} \mathcal{A}_c\mathcal{C}\mathcal{A}_c^\top & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathcal{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \succeq \mathbf{0}, \qquad \frac{\mathbf{G}+\mathbf{G}^\top}{2} = \begin{bmatrix} \mathcal{A}_r\mathcal{R}^{-1}\mathcal{A}_r^\top & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \succeq \mathbf{0}$$

- Reciprocity follows from the zero structure of $\mathbf{C}$, $\mathbf{G}$, $\mathbf{B}$ and the symmetry of $\mathcal{A}_r\mathcal{C}\mathcal{A}_r^\top$, $\mathcal{A}_r\mathcal{R}^{-1}\mathcal{A}_r^\top$, $\mathcal{L}$

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- RCL network equations
- *Projection onto Krylov subspaces*
- PRIMA and SPRIM
- SPRIM revisited
- Open problems

General RCL network equations

- Linear time-invariant dynamical system of the form:

$$\mathbf{C}\,\frac{d}{dt}\mathbf{x}(t) + \mathbf{G}\,\mathbf{x}(t) = \mathbf{B}\,\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{B}^{\top}\mathbf{x}(t)$$

  where $\mathbf{C},\ \mathbf{G} \in \mathbb{R}^{N\times N}$ and $\mathbf{B} \in \mathbb{R}^{N\times m}$

- m is the total number of voltage and current sources

Reduced-order models

- Linear time-invariant dynamical system of the same form:

$$\begin{aligned} \mathbf{C}_n \frac{d}{dt}\mathbf{z}(t) + \mathbf{G}_n\,\mathbf{z}(t) &= \mathbf{B}_n\,\mathbf{u}(t) \\ \tilde{\mathbf{y}}(t) &= \mathbf{B}_n^\top\,\mathbf{z}(t) \end{aligned}$$

- But now:

$$\mathbf{C}_n,\ \mathbf{G}_n \in \mathbb{R}^{n\times n} \quad \text{and} \quad \mathbf{B}_n \in \mathbb{R}^{n\times m}$$

where $n \ll N$

Transfer functions

- Original system:

$$\mathbf{H}(s) = \mathbf{B}^{\mathsf{T}} \left(s\,\mathbf{C} + \mathbf{G} \right)^{-1} \mathbf{B}$$

- Reduced-order model:

$$\mathbf{H}_n(s) = \mathbf{B}_n^{\mathsf{T}} \left(s\,\mathbf{C}_n + \mathbf{G}_n \right)^{-1} \mathbf{B}_n$$

- 'Good' reduced-order model

  $\Longleftrightarrow$ 'Good' approximation $\mathbf{H}_n \approx \mathbf{H}$

Projection-based reduction

- Choose an $N \times n$ matrix

$\mathbf{V}_n =$  with Rank $\mathbf{V}_n = n$

and explicitly project the data matrices of

$$\mathbf{C}\,\frac{d}{dt}\mathbf{x}(t) + \mathbf{G}\,\mathbf{x}(t) = \mathbf{B}\,\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{B}^{\top}\mathbf{x}(t)$$

onto the subspace spanned by the columns of $\mathbf{V}_n$

Projection-based reduction

- Resulting reduced-order model:

$$\mathbf{C}_n \frac{d}{dt}\mathbf{z}(t) + \mathbf{G}_n\,\mathbf{z}(t) = \mathbf{B}_n\,\mathbf{u}(t)$$
$$\tilde{\mathbf{y}}(t) = \mathbf{B}_n^\top \mathbf{z}(t)$$

where

$$\mathbf{C}_n := \mathbf{V}_n^\top\,\mathbf{C}\,\mathbf{V}_n, \quad \mathbf{G}_n := \mathbf{V}_n^\top\,\mathbf{G}\,\mathbf{V}_n, \quad \mathbf{B}_n := \mathbf{V}_n^\top\,\mathbf{B}$$

- Preserves **passivity**:

$$\mathbf{C} \succeq \mathbf{0}, \quad \frac{\mathbf{G}+\mathbf{G}^\top}{2} \succeq \mathbf{0} \quad \Rightarrow \quad \mathbf{C}_n \succeq \mathbf{0}, \quad \frac{\mathbf{G}_n+\mathbf{G}_n^\top}{2} \succeq \mathbf{0}$$

Choice of projection matrix

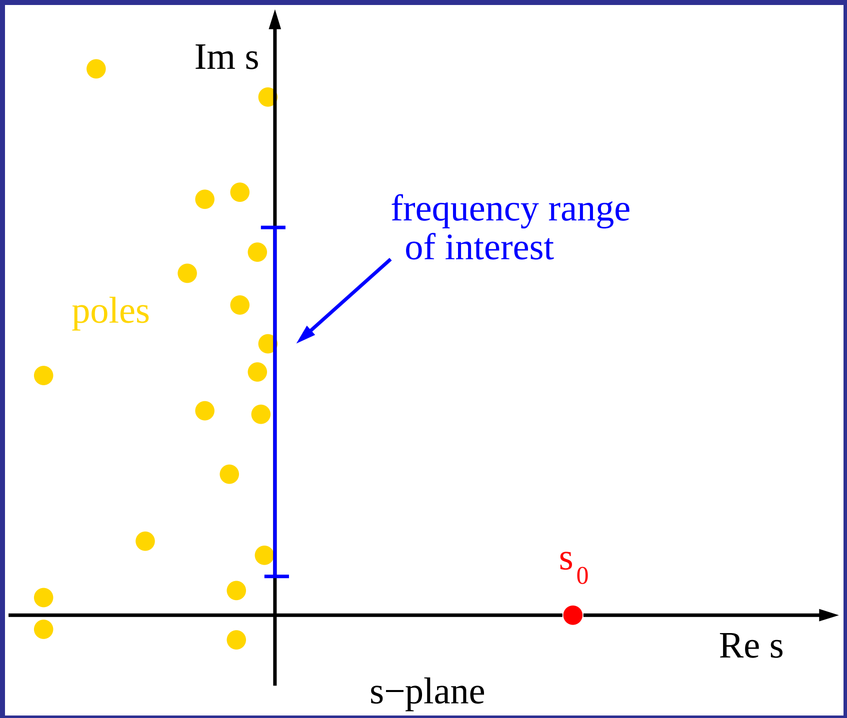
- Choose expansion point s_0 for transfer function and rewrite:

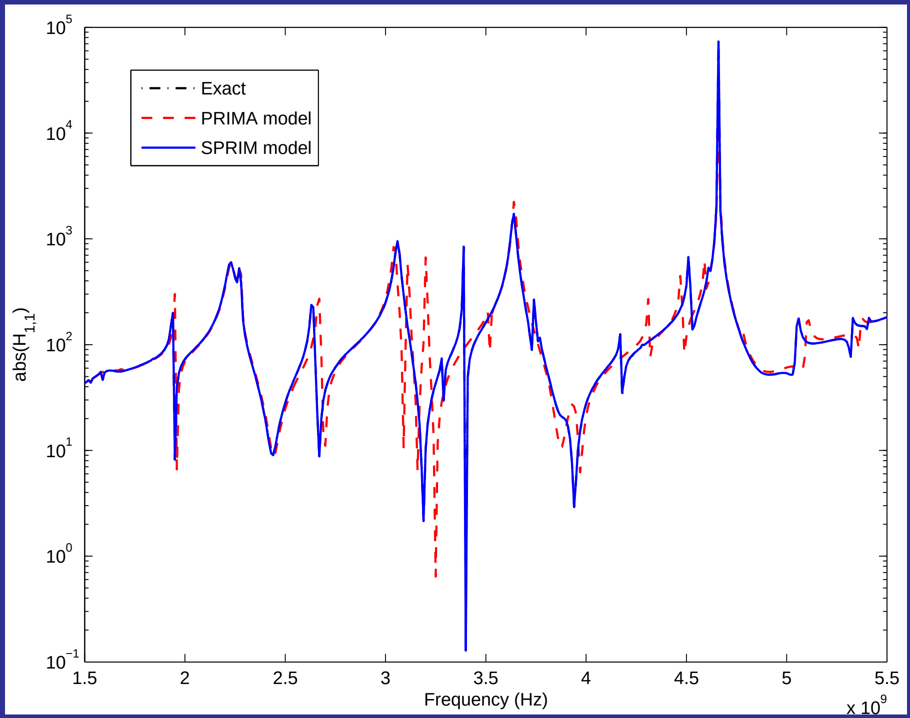
$$\begin{aligned}\mathbf{H}(s) &= \mathbf{B}^{\mathsf{T}} (s\,\mathbf{C} + \mathbf{G})^{-1} \mathbf{B} \\ &= \mathbf{B}^{\mathsf{T}} \Big(s_0\,\mathbf{C} + \mathbf{G} + (s - s_0)\,\mathbf{C} \Big)^{-1} \mathbf{B} \\ &= \mathbf{B}^{\mathsf{T}} \Big(\mathbf{I} + (s - s_0)\,\mathbf{A} \Big)^{-1} \mathbf{R}\end{aligned}$$

  where

$$\mathbf{A} := (s_0\,\mathbf{C} + \mathbf{G})^{-1}\,\mathbf{C} \quad \text{and} \quad \mathbf{R} := (s_0\,\mathbf{C} + \mathbf{G})^{-1}\,\mathbf{B}$$

Practical choice of expansion point





Projection + Krylov: Moment matching

- Recall: $\mathbf{A} = (s_0\,\mathbf{C} + \mathbf{G})^{-1}\,\mathbf{C}$ and $\mathbf{R} = (s_0\,\mathbf{C} + \mathbf{G})^{-1}\,\mathbf{B}$
- $\widehat{n}$-th **block Krylov subspace**:

$$\mathcal{K}_{\widehat{n}}(\mathbf{A}, \mathbf{R}) := \mathrm{colspan}_{\widehat{n}} \left[\, \mathbf{R} \quad \mathbf{A}\mathbf{R} \quad \mathbf{A}^2\mathbf{R} \quad \cdots \,\right]$$

- Choose the projection matrix $\mathbf{V}_n$ such that

$$\mathcal{K}_{\widehat{n}}(\mathbf{A}, \mathbf{R}) \subseteq \mathrm{Range}\,\mathbf{V}_n$$

- **Moment matching** about s_0:

$$\mathbf{H}_n(s) = \mathbf{H}(s) + \mathcal{O}\left((s - s_0)^{\tilde{q}}\right), \quad \text{where} \quad \tilde{q} \geq \lfloor \widehat{n}/m \rfloor$$

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PRIMA

- Projection onto n-th block Krylov subspace:

$$\text{Range}\, \mathbf{V}_n = \mathcal{K}_n(\mathbf{A}, \mathbf{R})$$

- Block structure of the data matrices:

$$\mathbf{C} = \begin{bmatrix} \mathcal{C}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathcal{C}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \mathcal{G}_1 & \mathcal{A}_l & \mathcal{A}_v \\ -\mathcal{A}_l^{\mathsf{T}} & \mathbf{0} & \mathbf{0} \\ -\mathcal{A}_v^{\mathsf{T}} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{0} & \mathcal{A}_i \\ \mathbf{0} & \mathbf{0} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}$$

- Reduced-order matrices:

$$\mathbf{C}_n = \begin{bmatrix} \blacksquare \end{bmatrix}, \quad \mathbf{G}_n = \begin{bmatrix} \blacksquare \end{bmatrix}, \quad \mathbf{B}_n = \begin{bmatrix} \blacksquare \end{bmatrix}$$

PRIMA

- Reduced-order models are passive
- Block structure of data matrices is not preserved
- Reduced-order models are not reciprocal

SPRIM

- Recall: moment matching if

$$\mathcal{K}_{\widehat{n}}(\mathbf{A},\mathbf{R}) \subseteq \mathsf{Range}\,\mathbf{V}_n$$

- Let $\widehat{\mathbf{V}}_{\widehat{n}}$ be any matrix such that

$$\mathsf{Range}\,\widehat{\mathbf{V}}_{\widehat{n}} = \mathcal{K}_{\widehat{n}}(\mathbf{A},\mathbf{R})$$

- Recall:

$$\mathbf{C} = \begin{bmatrix} \mathcal{C}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathcal{C}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \mathcal{G}_1 & \mathcal{A}_l & \mathcal{A}_v \\ -\mathcal{A}_l^{\mathsf{T}} & \mathbf{0} & \mathbf{0} \\ -\mathcal{A}_v^{\mathsf{T}} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{0} & \mathcal{A}_i \\ \mathbf{0} & \mathbf{0} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}$$

SPRIM

- Partition $\hat{\mathbf{V}}_{\hat{n}}$ accordingly:

$\hat{\mathbf{V}}_{\hat{n}} = \begin{bmatrix} \hat{\mathbf{V}}^{(1)} \\ \hat{\mathbf{V}}^{(2)} \\ \hat{\mathbf{V}}^{(3)} \end{bmatrix},$ $\quad \hat{\mathbf{V}}^{(1)} =$ , $\hat{\mathbf{V}}^{(2)} =$ , $\hat{\mathbf{V}}^{(3)} =$ 

- For $i = 1, 2, 3$:
  If Rank $\hat{\mathbf{V}}^{(i)} < \hat{n}$, replace $\hat{\mathbf{V}}^{(i)}$ by matrix of full column rank
- Usually:

$$\hat{\mathbf{V}}^{(3)} = \mathbf{I}$$

SPRIM

- Set

$$\mathbf{V}_n = \begin{bmatrix} \hat{\mathbf{V}}^{(1)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{V}}^{(2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \hat{\mathbf{V}}^{(3)} \end{bmatrix}$$

- Reduced-order matrices:

$$\mathbf{C}_n = \begin{bmatrix} \tilde{\mathcal{C}}_1 & 0 & 0 \\ 0 & \tilde{\mathcal{C}}_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{G}_n = \begin{bmatrix} \tilde{\mathcal{G}}_1 & \tilde{\mathcal{G}}_2 & \tilde{\mathcal{G}}_3 \\ -\tilde{\mathcal{G}}_2^T & 0 & 0 \\ -\tilde{\mathcal{G}}_3^T & 0 & 0 \end{bmatrix}, \quad \mathbf{B}_n = \begin{bmatrix} 0 & \tilde{\mathcal{B}}_1 \\ 0 & 0 \\ \tilde{\mathcal{B}}_2 & 0 \end{bmatrix}$$

- $\mathcal{K}_{\hat{n}}(\mathbf{A}, \mathbf{R}) = \text{Range}\, \hat{\mathbf{V}}_{\hat{n}} \subseteq \text{Range}\, \mathbf{V}_n \quad \Rightarrow$ moment matching!

SPRIM

- Block structure of data matrices is preserved
- Reduced-order models are passive and reciprocal
- Preservation of block structure implies that SPRIM matches twice as many moments as PRIMA
- PRIMA and SPRIM have the same computational costs
- SPRIM models are about twice as large as PRIMA models

Moment matching of SPRIM

- General theory of projection onto block Krylov subspaces: PRIMA and SPRIM produce reduced-order models with

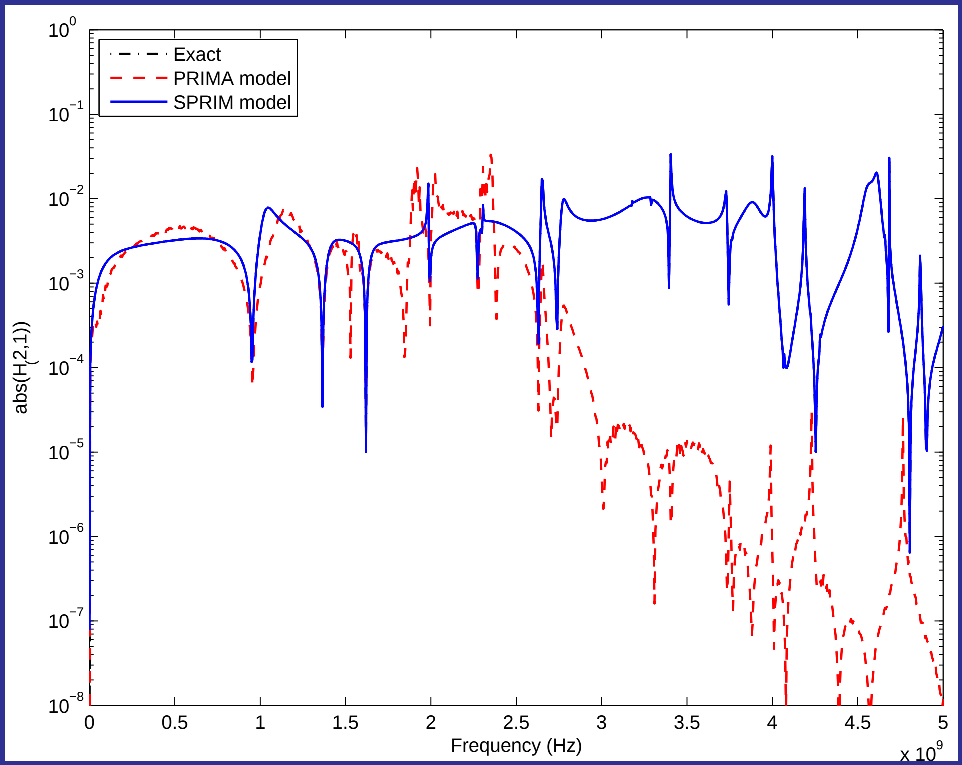
$$\mathbf{H}_n(s) = \mathbf{H}(s) + \mathcal{O}\left((s - s_0)^{\tilde{q}}\right), \quad \text{where} \quad \tilde{q} \geq \lfloor \hat{n}/m \rfloor$$

- **Theorem** (F., '08)

  The n-th SPRIM model satisfies

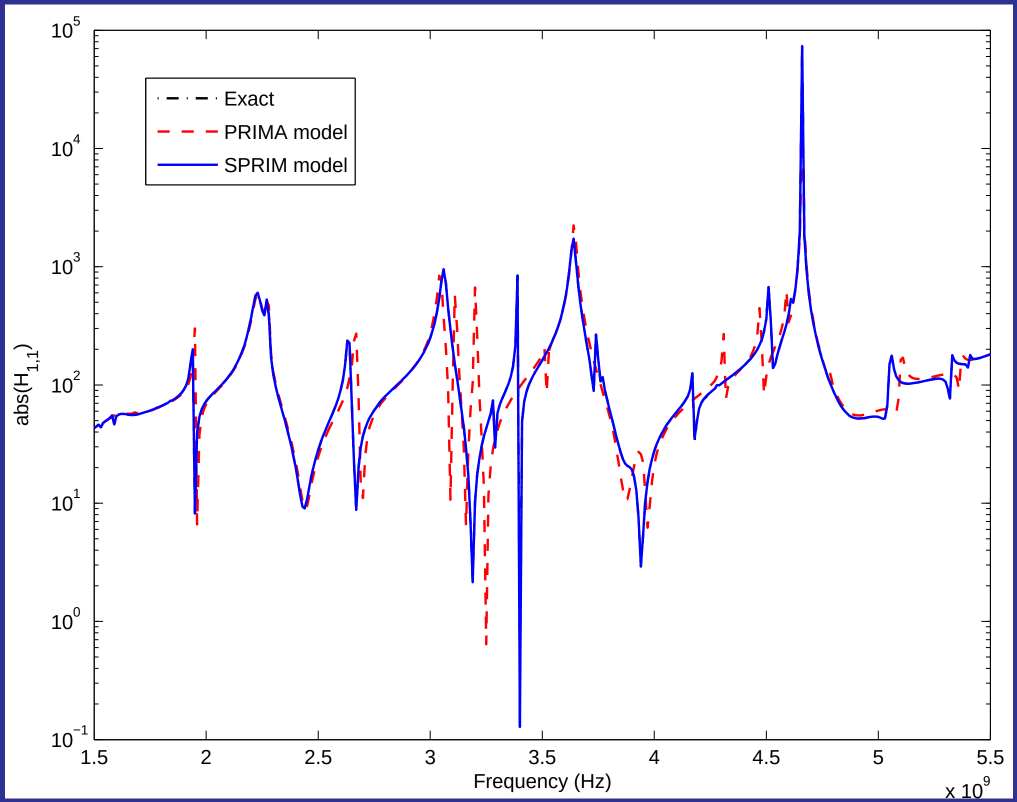
$$\mathbf{H}_n(s) = \mathbf{H}(s) + \mathcal{O}\left((s - s_0)^{\tilde{q}}\right), \quad \text{where} \quad \tilde{q} \geq 2 \lfloor \hat{n}/m \rfloor$$

An RCL network with mostly C's and L's



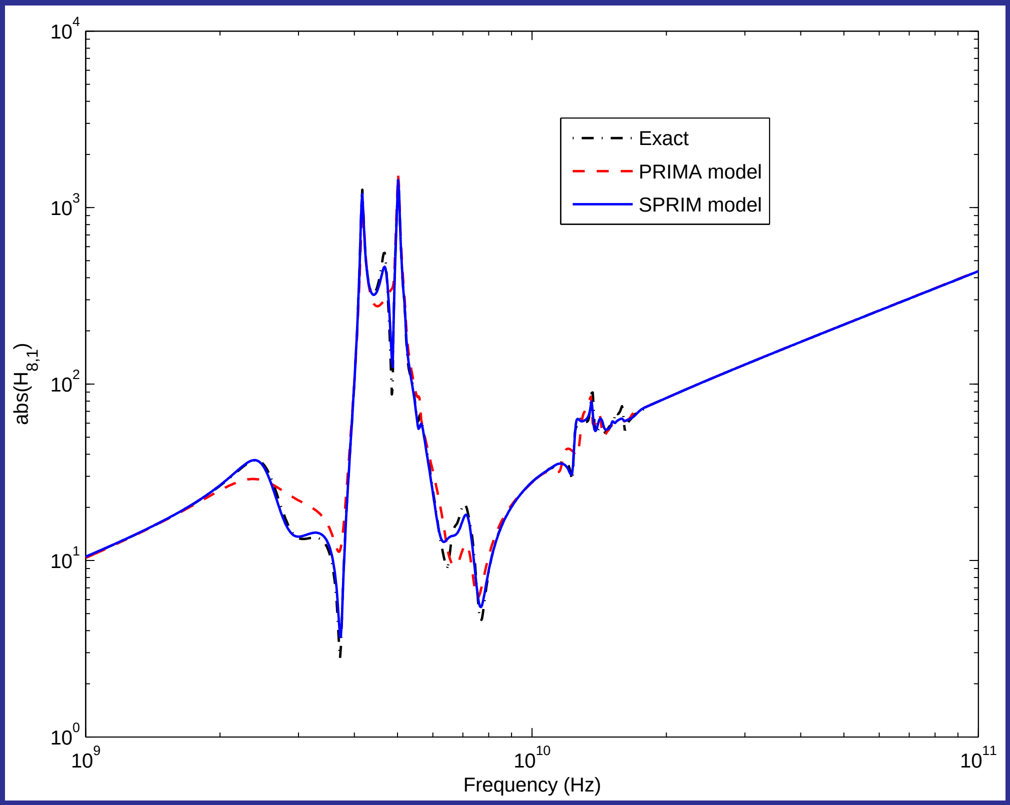
Exact and models corresponding to
block Krylov subspace of dimension $\widehat{n} = 120$

An RCL network with mostly C's and L's



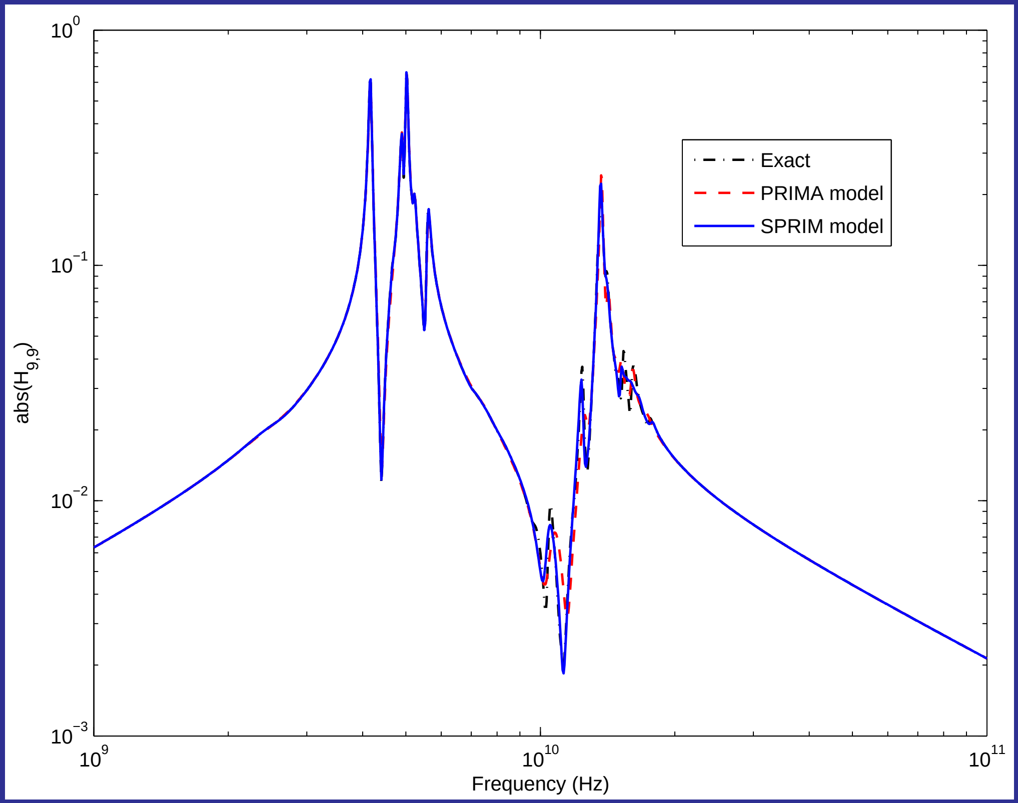
Exact and models corresponding to $\hat{n} = 90$

A package example



Exact and models corresponding to $\widehat{n} = 128$

A package example



Exact and models corresponding to $\hat{n} = 128$

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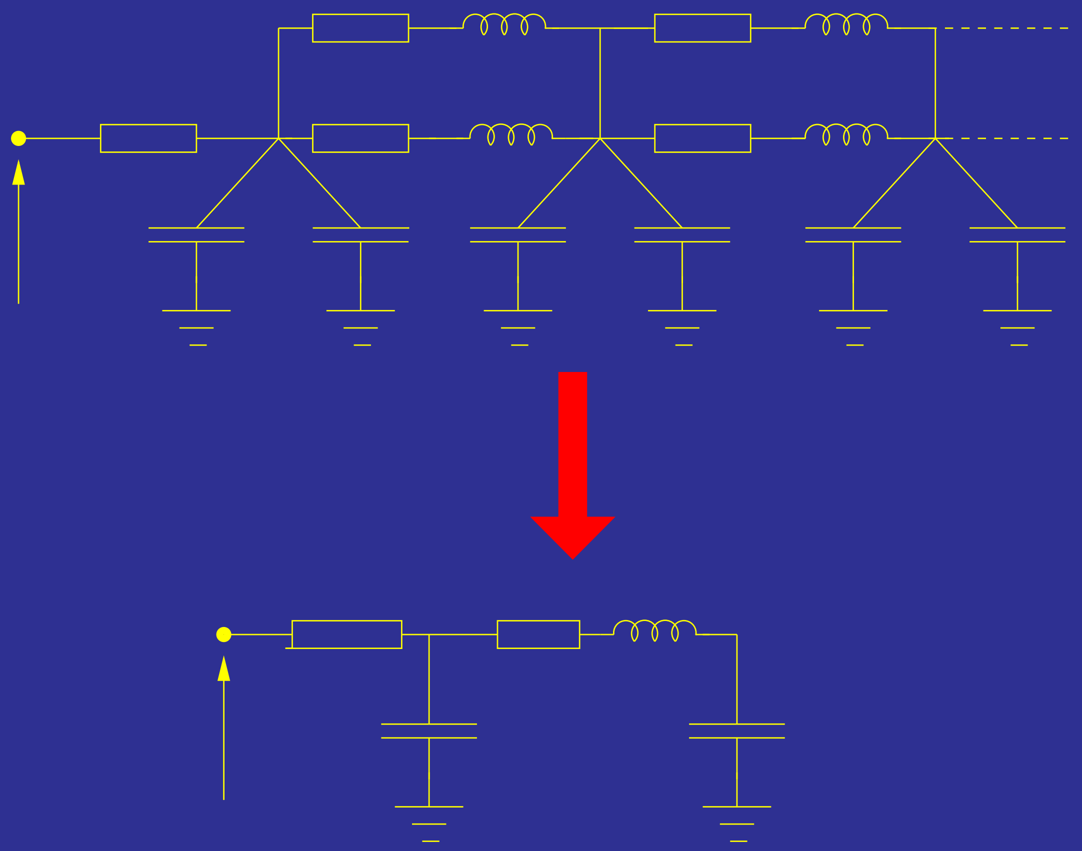
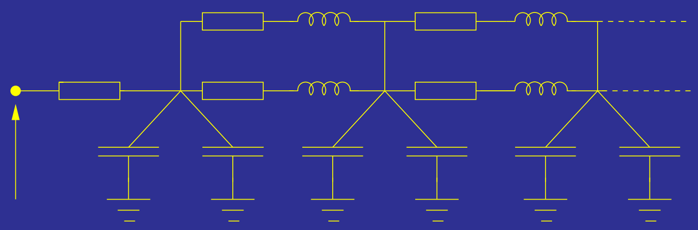
Problem: reduction of RCL networks



What we really want

- Original RCL network:



$$\Longleftrightarrow \quad \mathbf{C},\ \mathbf{G},\ \mathbf{B}$$

- SPRIM model order reduction:

$$\mathbf{C},\ \mathbf{G},\ \mathbf{B} \quad \Rightarrow \quad \mathbf{C}_n,\ \mathbf{G}_n,\ \mathbf{B}_n$$

- Reduced RCL network corresponding to $\mathbf{C}_n,\ \mathbf{G}_n,\ \mathbf{B}_n$?

‘Flaw’ of modified nodal analysis

- Matrices of original RCL network:

$$\mathbf{C} = \begin{bmatrix} \mathcal{C}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathcal{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \mathcal{G}_1 & \mathcal{A}_l & \mathcal{A}_v \\ -\mathcal{A}_l^\top & \mathbf{0} & \mathbf{0} \\ -\mathcal{A}_v^\top & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{0} & \mathcal{A}_i \\ \mathbf{0} & \mathbf{0} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}$$

  where $\mathcal{C}_1 = \mathcal{A}_c \mathcal{C} \mathcal{A}_c^\top$ and $\mathcal{G}_1 = \mathcal{A}_r \mathcal{R}^{-1} \mathcal{A}_r^\top$

- The voltage sources are input quantities and thus will not be reduced, yet they appear via $\mathcal{A}_v$ in $\mathbf{G}$

- $\mathbf{G}$ can be made symmetric if there are no voltage sources

How to handle $\mathcal{A}_v$

- $\mathcal{A}_v$ is an incidence matrix with full column rank
- 'Easy' transformation $\quad \mathcal{A}_v \quad \longrightarrow \quad \begin{bmatrix} I \\ 0 \end{bmatrix}$
- Matrices of RCL network:

$$\mathbf{C} = \begin{bmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & \mathbf{0} & \mathbf{0} \\ \mathcal{C}_{12}^{\mathsf{T}} & \mathcal{C}_{22} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathcal{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \ \mathbf{G} = \begin{bmatrix} \mathcal{G}_{11} & \mathcal{G}_{12} & \mathcal{G}_{13} & \mathbf{I} \\ \mathcal{G}_{12}^{\mathsf{T}} & \mathcal{G}_{22} & \mathcal{G}_{23} & \mathbf{0} \\ -\mathcal{G}_{13}^{\mathsf{T}} & -\mathcal{G}_{23}^{\mathsf{T}} & \mathbf{0} & \mathbf{0} \\ -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} \mathbf{0} & \mathcal{B}_{11} \\ \mathbf{0} & \mathcal{B}_{12} \\ \mathbf{0} & \mathbf{0} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}$$

- 'Eliminate' first and last block rows and columns

How to handle $\mathcal{A}_v$

- Result of elimination:

$$\mathbf{C} \quad \longrightarrow \quad \mathbf{E} := \begin{bmatrix} \mathcal{C}_{22} & \mathbf{0} \\ \mathbf{0} & -\mathcal{L} \end{bmatrix}$$

$$\mathbf{G} \quad \longrightarrow \quad \mathbf{F} := \begin{bmatrix} \mathcal{G}_{22} & \mathcal{G}_{23} \\ \mathcal{G}_{23}^{\top} & \mathbf{0} \end{bmatrix}$$

$$\mathbf{A} = (s_0\,\mathbf{C} + \mathbf{G})^{-1}\,\mathbf{C} \quad \longrightarrow \quad \mathbf{A}_1 := (s_0\,\mathbf{E} + \mathbf{F})^{-1}\,\mathbf{F}$$

- $\mathbf{E}$ and $\mathbf{F}$ are symmetric and $\mathbf{A}_1$ is $\mathbf{F}$-symmetric:

$$\mathbf{A}_1^{\top}\,\mathbf{F} = \mathbf{F}\,\mathbf{A}_1$$

How to handle $\mathcal{A}_v$

- Effect on transfer function:

$$\mathbf{H}(s) = \mathbf{B}^{\top}\,(s\,\mathbf{C} + \mathbf{G})^{-1}\,\mathbf{B}$$

$$= \mathbf{B}^{\top}\Big(\mathbf{I} + (s - s_0)\,\mathbf{A}\Big)^{-1}\mathbf{R}$$

$$= \mathbf{D}_0 + (s - s_0)\,\mathbf{D}_1 + (s - s_0)^2\Big(\mathbf{F}\,\mathbf{R}_1\Big)^{\top}\Big(\mathbf{I} + (s - s_0)\,\mathbf{A}_1\Big)^{-1}\mathbf{R}_1$$

  where $\mathbf{D}_0 = \mathbf{B}^{\top}\mathbf{R}$ and $\mathbf{D}_1 = -\mathbf{B}^{\top}\mathbf{A}\mathbf{R}$

- Since $\mathbf{A}_1$ is $\mathbf{F}$-symmetric, we can use the $\mathbf{F}$-symmetric band Lanczos process to compute Padé approximants of $\mathbf{H}(s)$ very efficiently

Consequences for SPRIM

- We need a matrix $\widehat{\mathbf{V}}_{\widehat{n}}$ such that

$$\text{Range}\,\widehat{\mathbf{V}}_{\widehat{n}} = \mathcal{K}_{\widehat{n}}(\mathbf{A},\mathbf{R}) = \text{colspan}_{\widehat{n}}\left[\,\mathbf{R}\quad \mathbf{AR}\quad \mathbf{A}^2\mathbf{R}\quad \cdots\,\right]$$

- Setting up the projection matrix:

$$\widehat{\mathbf{V}}_{\widehat{n}} = \begin{bmatrix} \widehat{\mathbf{V}}^{(1)} \\ \widehat{\mathbf{V}}^{(2)} \\ \widehat{\mathbf{V}}^{(3)} \end{bmatrix} \quad \Rightarrow \quad \mathbf{V}_n = \begin{bmatrix} \widehat{\mathbf{V}}^{(1)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \widehat{\mathbf{V}}^{(2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}$$

- We only need to construct $\widehat{\mathbf{V}}^{(1)}$ and $\widehat{\mathbf{V}}^{(2)}$, but not $\widehat{\mathbf{V}}^{(3)}$

Consequences for SPRIM

- Recall:

$$\mathbf{A} = (s_0\,\mathbf{C} + \mathbf{G})^{-1}\,\mathbf{C} \quad \text{and} \quad \mathbf{R} = (s_0\,\mathbf{C} + \mathbf{G})^{-1}\,\mathbf{B}$$

where

$$\mathbf{C} = \begin{bmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & \mathbf{0} & \mathbf{0} \\ \mathcal{C}_{12}^{\mathsf{T}} & \mathcal{C}_{22} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathcal{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \mathcal{G}_{11} & \mathcal{G}_{12} & \mathcal{G}_{13} & \mathbf{I} \\ \mathcal{G}_{12}^{\mathsf{T}} & \mathcal{G}_{22} & \mathcal{G}_{23} & \mathbf{0} \\ -\mathcal{G}_{13}^{\mathsf{T}} & -\mathcal{G}_{23}^{\mathsf{T}} & \mathbf{0} & \mathbf{0} \\ -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{0} & \mathcal{B}_{11} \\ \mathbf{0} & \mathcal{B}_{12} \\ \mathbf{0} & \mathbf{0} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}$$

- Corresponding smaller matrices after elimination:

$$\mathbf{A}_1 = (s_0\,\mathbf{E} + \mathbf{F})^{-1}\,\mathbf{F} \quad \text{and} \quad \mathbf{R}_1$$

Consequences for SPRIM

- Structure of block Krylov sequence:

$$\mathbf{R} = \begin{bmatrix} [\mathbf{I}\ \ \mathbf{0}] \\ \mathbf{R}_0 \\ \star \end{bmatrix}, \ \mathbf{A}\mathbf{R} = \begin{bmatrix} \mathbf{0} \\ \mathbf{R}_1 \\ \star \end{bmatrix}, \ \mathbf{A}^2\mathbf{R} = \begin{bmatrix} \mathbf{0} \\ \mathbf{A}_1\mathbf{R}_1 \\ \star \end{bmatrix}, \dots, \mathbf{A}^k\mathbf{R} = \begin{bmatrix} \mathbf{0} \\ \mathbf{A}_1^{k-1}\mathbf{R}_1 \\ \star \end{bmatrix}, \dots$$

- We can use the $\mathbf{F}$-symmetric band Lanczos process to effiicently compute the needed parts $\widehat{\mathbf{V}}^{(1)}$ and $\widehat{\mathbf{V}}^{(2)}$ of the projection matrix $\mathbf{V}_n$

- Note that

$$\widehat{\mathbf{V}}^{(1)} = \begin{bmatrix} [\ \mathbf{I} \quad \mathbf{0}\ ] \\ \star \end{bmatrix}$$

New and improved SPRIM

- Set

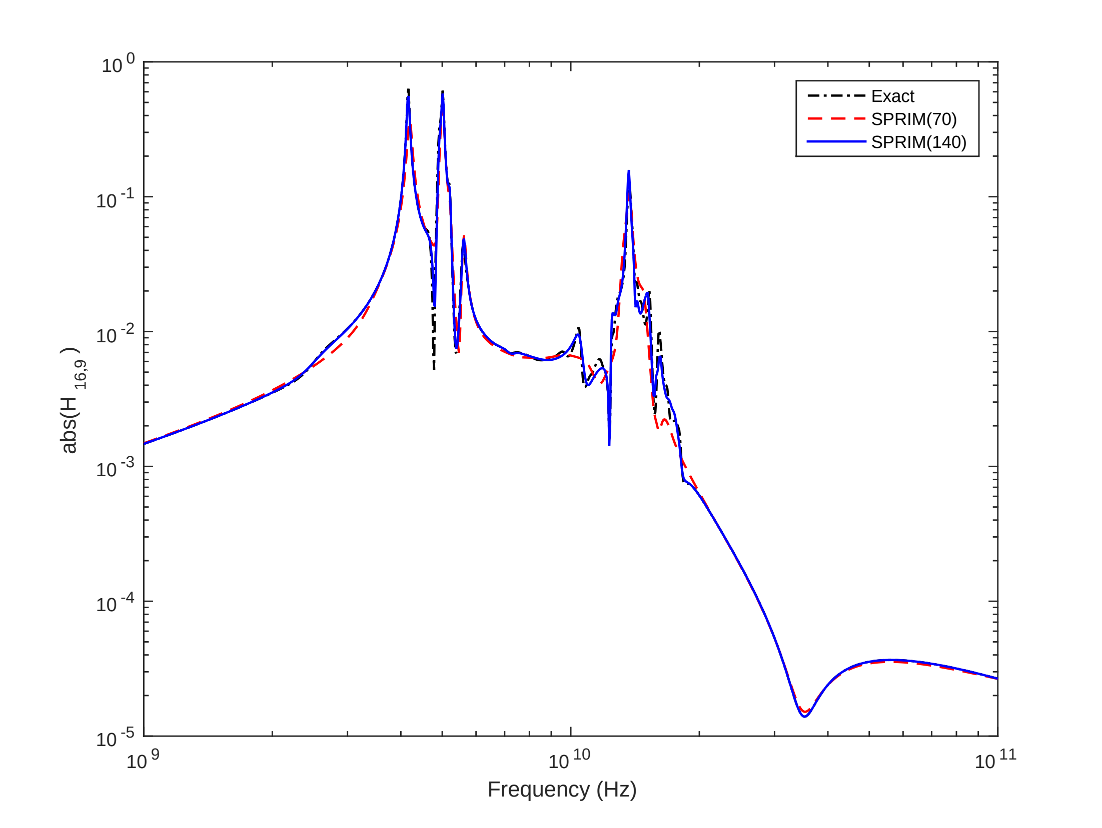
$$\mathbf{V}_n = \begin{bmatrix} \widehat{\mathbf{V}}^{(1)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \widehat{\mathbf{V}}^{(2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad \text{where} \quad \widehat{\mathbf{V}}^{(1)} = \begin{bmatrix} [\ \mathbf{I} & \mathbf{0}\ ] \\ \multicolumn{2}{c}{\star} \end{bmatrix}$$

- Reduced-order matrices:

$$\mathbf{C}_n = \begin{bmatrix} \tilde{\mathcal{C}}_1 & 0 & 0 \\ 0 & \tilde{\mathcal{C}}_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{G}_n = \begin{bmatrix} \tilde{\mathcal{G}}_1 & \tilde{\mathcal{G}}_2 & \mathcal{A}_v \\ -\tilde{\mathcal{G}}_2^\top & \mathbf{0} & \mathbf{0} \\ -\mathcal{A}_v^\top & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{B}_n = \begin{bmatrix} \mathbf{0} & \tilde{\mathcal{B}}_1 \\ \mathbf{0} & \mathbf{0} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}$$

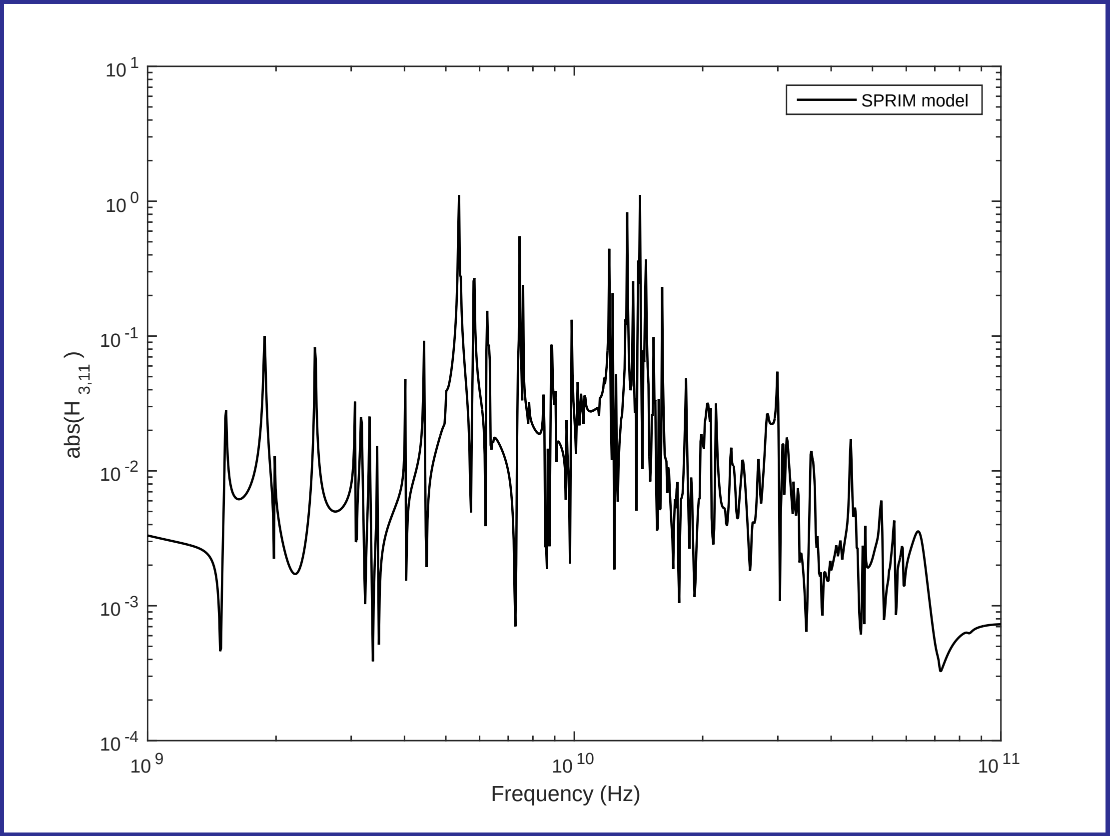
- The incidence matrix $\mathcal{A}_v$ of the voltage sources is preserved!

A package example



Exact and SPRIM models corresponding to $\hat{n} = 70$ and $\hat{n} = 140$

A much larger example



SPRIM model corresponding to $\hat{n} = 300$

Outline

- RCL network equations
- Projection onto Krylov subspaces
- PRIMA and SPRIM
- SPRIM revisited
- *Open problems*

Open problems

- ‘True’ RCL reduction via Krylov subspace-based methods?
- The new version of SPRIM resolves the issue with voltage sources; enough to guarantee RCL reduced-order models?
- Is there a downside to using $\mathbf{F}$-symmetric Lanczos to generate SPRIM models?
- I really should finish and release **BANDITS** (a Matlab package of band Krylov subspace iterations)

Thank you!