Continuum Thermodynamics, complete and reduced systems

Josef Málek

Charles University in Prague
Faculty of Mathematics and Physics, Mathematical institute
Sokolovská 83, 186 75 Praha 8

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Part #1

Do we indeed need such a workshop on
Model reduction in continuum thermodynamics: Modeling, analysis and computation
?

Progress in the theories during last twenty years

1. Continuum thermodynamics - new approaches (K.R. Rajagopal)
   - implicit constitutive theory (particular models since Barus 1893)
   - natural states associated with the body (Eckhart 1941)
   - maximization of the rate of the entropy production (Ziegler 1963)
   - theory of interacting continua (Darcy 1860, Fick 1855, Truesdell 1960)
2. Theoretical analysis of relevant boundary value problems (E. Feireisl, . . .)
   - large data analysis for compressible Newtonian fluids
   - large data analysis for certain classes of non-Newtonian fluids
   - qualitative theory (long-time behavior, limits for vanishing non-dimensional numbers, regularity, etc.)
3. Approximation of the problems (D. Silvester, Z. Strakoš, M. Vohralík)
   - aposteriori analysis towards guaranteed estimates, efficient stopping criteria and adaptativity
   - necessity to incorporate at all kind of errors together (model, discretization, algebraic, rounding)
4. Computational possibilities (D. Silvester, J. Hron)
   - Computational power (hardware and software)
   - Increasing requirements from applications

What do we mean by **Model reduction**?

$$\boxed{\mathcal{P} \qquad \text{replaced by} \qquad \mathcal{P}_{\textbf{simple}}}$$

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$\mathcal{P}$ replaced by $\mathcal{P}_{\textbf{simple}}$

$\mathcal{P}, \mathcal{P}_{\textbf{simple}}$ keeps same required information

$\mathcal{P}_{\textbf{simple}}$ is tractable

Aim

• To put together the experts who are looking at the problems from the larger scales and working in one of these particular discipline, to inform mutually about these developments and to suggest further "right" steps

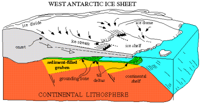
• What is "right" is hard to evaluate in traditional ways $\Longrightarrow$ looking at all these issues as the whole is neccessary

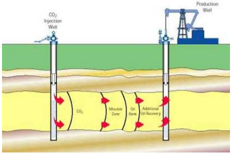
• Implicitly constituted material models: from theory through model reduction to efficient numerical methods (5 year ERC-CZ project)

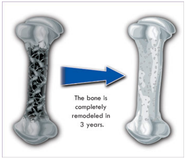
- Team members: Z. Strakoš, E. Feireisl, E. Süli (Oxford), M. Bulíček, J. Hron, V. Průša, O. Souček and M. Vohralík (INRIA)
- Advisory board members: M. Benzi (Atlanta), K.R. Rajagopal (Texas A&M University), R. Rannacher (Heidelberg), G. Seregin (Oxford, St. Petersburg)
- two open postdoc positions since February 2013

Modeling of real-world problems/1

1. Quality and wear/durability of highways and airport runways, containers for nuclear waste
2. Evolution of glacier and ice sheets, transport processes in permafrost soil, expansion/protection of teritories (Singapur)
3. Flows through porous media (enhanced oil recovery, alternative sources of energy, environment protection - permanent nuclear waste deposition)
4. Processes in human body (bone remodulation, aneurysm rupture, formation and resolution of blood clots, processes in joints and cartelages, growth and remodulation of living tissues)

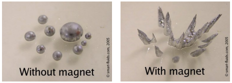






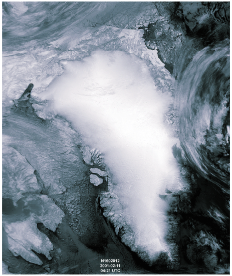
Modeling of real-world problems/2

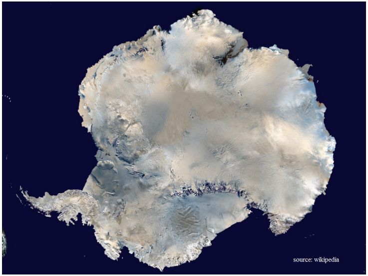
5. Shape memory materials (elastomers, smart materials reacting on heat, light, electric, magnetic stimuli)
6. Journal bearing in satelites and other devices (ER)
7. Materials with almost ideal properties (HPT)
8. Liquefaction - sediments in soils are transformed during earthbrakes (due to seismic waves) into the material that behaves as a liquid
9. Damage, corosion, cracks



Problem - Ice sheets and their dynamics

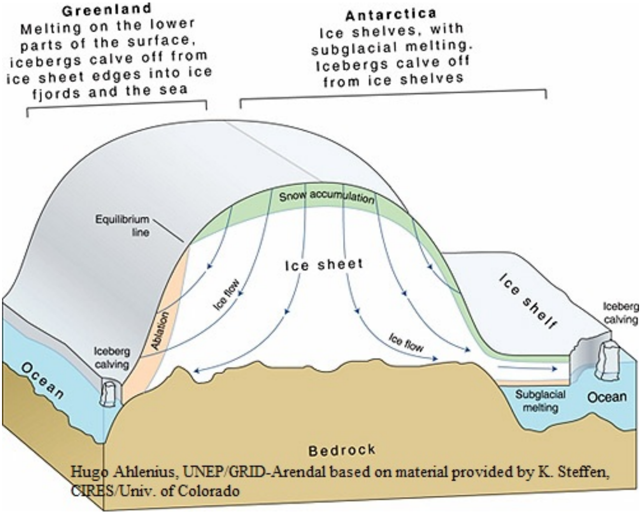
- Ice sheet = continental glacier - area greater than 50 000 km^2.
- Current biggest ice sheets:
    - Antarctic ice sheet
    - Greenland ice sheet



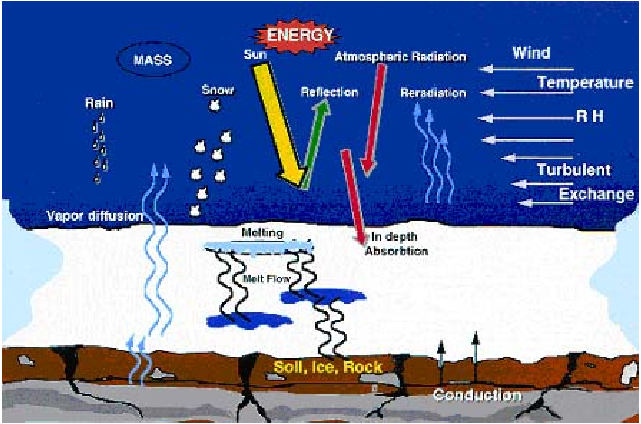


- project GRACE `http://www.csr.utexas.edu/grace/` measure changes in the gravitional forces over the Earth
- Combination of the climate (growth/melting of glaciers) and thermomechanical effects of litosphere (post-glacial rebound)

Scheme of an ice sheet

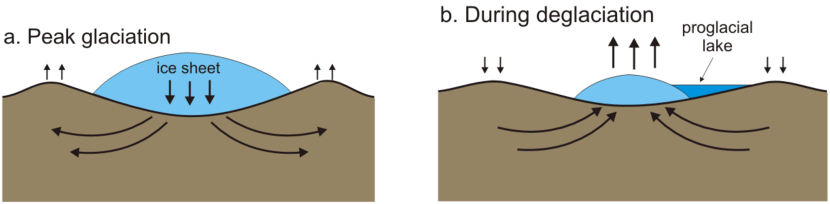


Transport processes in an ice sheet



Specification of the problem

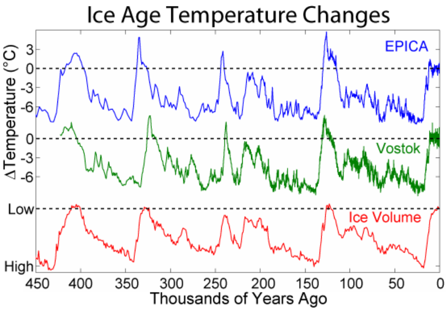
- Precise modeling of surface mass load for post-glacial rebound
- To separate climate effects from the effects in the lithosphere requires to model as best as possible the behavior of the lithospheric layer (Z. Martinec, O. Souček)



First important step

Understanding of long time dynamics of ice-sheets over the large areas and long time period up to 500 000 years

- "Periodic" changes in Earth glaciation (ice-ages) (100 ky, 40 ky)
- Last glacial maximum (20 ky b.p.)
- Deformation of glaciated regions - visco-elastic response of earth lithosphere (GIA)
- Coupled system: ice-sheets + GIA + gravity + sea-level (+ ocean circulation + rotational dynamics + climate)



source: wikipedia

Impossible to solve for a complete system - not enough data, not enough capacity

Part #2

Basic setting, (compressible) Navier-Stokes-Fourier and Korteweg-Fourier fluids

Continuum thermodynamics

- Balance equations for mass, linear momentum and energy

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \boldsymbol{v}) = 0$$

$$\frac{\partial(\varrho \boldsymbol{v})}{\partial t} + \operatorname{div}(\varrho \boldsymbol{v} \otimes \boldsymbol{v}) = \operatorname{div} \mathbf{T} + \varrho \boldsymbol{f}$$

$$\frac{\partial \varrho(e+|\boldsymbol{v}|^2/2)}{\partial t} + \operatorname{div}(\varrho(e + |\boldsymbol{v}|^2/2)\boldsymbol{v}) = \operatorname{div}(\mathbf{T}\boldsymbol{v} + \mathbf{q}) + \varrho \boldsymbol{f} \cdot \mathbf{v} + \varrho r$$

Five nonlinear partial differential equations for 14 unknowns
$(\varrho, e, \boldsymbol{v} = (v_1, v_2, v_3), \mathbf{T} = (T_{11}, T_{12}, T_{13}, T_{22}, T_{23}, T_{33}), \mathbf{q} = (q_1, q_2, q_3))$

- Constitutive equations

$$\mathbf{G}(\varrho, e, \mathbf{T}, \mathbf{D}(\boldsymbol{v})) = \mathbf{0} \text{ and } \mathbf{r}(\varrho, e, \mathbf{q}, \nabla e) = \mathbf{0} \qquad \mathbf{D} := \frac{1}{2}(\nabla \boldsymbol{v}) + (\nabla \boldsymbol{v})^T)$$

- Boundary conditions
- Knowledge of the initial state of the system

Aim: **to predict evolution of the density, forces, temperature, ...**

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Q. Can we describe the response of complex materials with minimal additional assumptions (micro-meso-macro, new balance equations, new internal parameters)?

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A. Recent systematic approaches

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- concept of natural configuration associated with the current configurtaion of the body (KR Rajagopal, AS Srinivasa since 1995)
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Q. Can we specify the structure of appropriate boundary conditions?

- interaction of two materials

Compressible Navier-Stokes-Fourier models

Balance equations

$$\dot{\varrho} = -\varrho \operatorname{div} \mathbf{v} \qquad \varrho\dot{\mathbf{v}} = \operatorname{div} \mathbf{T} + \varrho \boldsymbol{f} \qquad \varrho\dot{\mathrm{E}} = \operatorname{div}(\mathbf{T}\mathbf{v} - \mathbf{q}) + \varrho \boldsymbol{f} \cdot \mathbf{v} + \varrho r \qquad \mathrm{E} := e + |\mathbf{v}|^2/2$$

$$\boxed{\dot{z} = \frac{\partial z}{\partial t} + \nabla z \cdot \boldsymbol{v} \qquad \dot{\mathbf{z}} = \frac{\partial \mathbf{z}}{\partial t} + [\nabla \mathbf{z}]\mathbf{v}}$$

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Two explicit constitutive theories:

- standard: $\mathbf{T} = \tilde{\mathbf{T}}(\varrho, e, \mathbf{D}) \qquad \mathbf{q} = \tilde{\mathbf{q}}(\varrho, e, \nabla e)$
- new: $\mathbf{D} = \tilde{\mathbf{D}}(\varrho, e, \mathbf{T}) \qquad \nabla e = \mathbf{h}(\varrho, e, \mathbf{q})$

Navier-Stokes-Fourier

$$\mathbf{T} = -p(\varrho, e)\mathbf{I} + 2\mu(\varrho, e)\mathbf{D} + \lambda(\varrho, e)\,(\operatorname{div} \mathbf{v})\mathbf{I}$$
$$\mathbf{q} = -\kappa(\varrho, e)\nabla e$$

Framework for incompressible fluids

Balance equations

$$\begin{aligned} \operatorname{div} \mathbf{v} &= 0 \\ \varrho \dot{\mathbf{v}} &= \operatorname{div} \mathbf{T} + \varrho \boldsymbol{f} \\ \varrho \dot{\mathrm{E}} &= \operatorname{div}(\mathbf{T}\mathbf{v} - \mathbf{q}) + \varrho \boldsymbol{f} \cdot \mathbf{v} + \varrho r \end{aligned} \qquad \mathrm{E} := e + |\mathbf{v}|^2/2$$

Explicit constitutive theory (ϱ is a positive constant)

$$\boxed{\mathbf{T} = m\mathbf{I} + \mathbf{T}^d(e, \mathbf{D}) \qquad \mathbf{q} = \tilde{\mathbf{q}}(e, \nabla e)}$$

Navier-Stokes-Fourier

$$\mathbf{T} = m\mathbf{I} + 2\mu(e)\mathbf{D} \qquad \mathbf{q} = -\kappa(e)\nabla e$$

- $\boxed{m := \frac{1}{3} \operatorname{tr} \mathbf{T}}$ mean normal stress
- $\boxed{p(\varrho, e)}$ thermodynamic pressure

Main conquests of the maximization of the rate of entropy production principle

Maximization of the rate of the entropy production principle

- knowledge of the constitutive equation for two scalars, the entropy η (or any thermodynamical potential) and the rate of the entropy production ζ, suffices to determine the constitutive equations for the Cauchy stress and the energy flux as well as the structure of the boundary conditions
- second law of thermodynamics is automatically met
- choice of constitutive equations gives a priori estimates
- transparent and simple approach provided that you know how the body stores the energy and what are the entropy producing mechanisms
- a general approach based on a nontrivial physical tasks - specification of the costitutive eqs for the entropy and the entropy production.

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Nontrivial (simple) cases

- compressible Navier-Stokes-Fourier fluids
- compressible Korteweg-Fourier fluids

Korteweg fluids (1901)

$$\boxed{\mathbf{T} = (-p + \alpha_0|\nabla\varrho|^2 + \alpha_1\Delta\varrho)\mathbf{I} + \beta(\nabla\varrho\otimes\nabla\varrho) + 2\mu\mathbf{D} + \lambda(\operatorname{div}\mathbf{v})\mathbf{I}}$$

- capilarity effects
- phase transition phenomena
- flows of granular materials $\frac{1}{\varrho}$

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Restriction:

$$\boxed{\mathbf{v}\cdot\mathbf{n} = 0 \quad \text{on } (0,T)\times\Gamma \qquad\qquad \Gamma := \partial\Omega}$$

Q3. Can one develop a method in order to include thermal effects or non-Newtonian phenomena?

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References

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Entropy

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Continuum thermodynamics (Callen (1985), Rajagopal, Srinivasa (2004)): there is η (specific entropy density) being a function of state variables, one of them is internal energy $\boxed{\eta = \tilde{\eta}(e, y_1, y_2, \dots)}$, fulfilling:

- $\tilde{\eta}$ is increasing function of $e \implies \frac{1}{\theta} =: \frac{\partial \tilde{\eta}}{\partial e}$ or $e = \tilde{e}(\eta, \varrho) \implies \theta = \frac{\partial \tilde{e}}{\partial \eta}$
- $\eta \to 0+$ as $\theta \to 0+$
- $S(t) := \int_\Omega \varrho^* \eta(t, \cdot) dx$ goes to its maximum as $t \to \infty$ provided that the body is thermally and mechanically isolated

$$\begin{aligned}
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Constitutive equation for the entropy

$$\boxed{\eta = \tilde{\eta}(e, \varrho)} \quad \Longleftrightarrow \quad e = \tilde{e}(\eta, \varrho) \qquad \boxed{\theta := \frac{\partial \tilde{e}}{\partial \varrho} > 0}$$

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Applying the material derivative to the last Eq.

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Inserting the balance equations:

$$\varrho\theta\dot\eta = \mathbf{T}\cdot\mathbf{D} + \mathrm{div}\,\mathbf{q} + p\,\mathrm{div}\,\mathbf{v} \qquad \boxed{p := \varrho^2\frac{\partial\tilde{e}}{\partial\varrho}}$$

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Consequently

$$\boxed{\varrho \dot{\eta} - \operatorname{div}\left(\frac{\mathbf{q}}{\theta}\right) = \frac{1}{\theta}\left[\mathbf{T}^d \cdot \mathbf{D}^d + (m + p) \operatorname{div} \mathbf{v} + \mathbf{q} \cdot \frac{\nabla \theta}{\theta}\right]} \qquad \begin{cases} m : & = \frac{1}{3} \operatorname{tr} \mathbf{T} \\ \mathbf{C}^d : & = \mathbf{C} - \frac{1}{3}(\operatorname{tr} \mathbf{C})\mathbf{I} \end{cases}$$

$$\varrho\dot{\eta} - \operatorname{div}\left(\frac{\mathbf{q}}{\theta}\right) = \zeta^{\Omega} \qquad \zeta^{\Omega} = \frac{1}{\theta}\left[\mathbf{T}^{d}\cdot\mathbf{D}^{d} + (m+p)\operatorname{div}\mathbf{v} + \mathbf{q}\cdot\frac{\nabla\theta}{\theta}\right]$$

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$$= \frac{1}{\theta}\sum_{\alpha} J_{\alpha}A_{\alpha}$$

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J_{α} thermodynamical fluxes $\qquad$ A_{α} thermodynamical affinities

$$\varrho\dot{\eta} - \operatorname{div}\left(\frac{\mathbf{q}}{\theta}\right) = \zeta^{\Omega} \qquad \zeta^{\Omega} = \frac{1}{\theta}\left[\mathbf{T}^{d}\cdot\mathbf{D}^{d} + (m+p)\operatorname{div}\mathbf{v} + \mathbf{q}\cdot\frac{\nabla\theta}{\theta}\right] = \frac{1}{\theta}\sum_{\alpha} J_{\alpha}A_{\alpha} \geq 0$$

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Compressible NS fluid:

$$\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D} + \lambda(\operatorname{div}\mathbf{v})\mathbf{I} = -p\mathbf{I} + 2\mu\mathbf{D}^{d} + \frac{2\mu+3\lambda}{3}(\operatorname{div}\mathbf{v})\mathbf{I}$$

$$\mathbf{q} = \kappa\nabla\theta$$

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Two ways how to express ζ^{Ω}

$$\zeta^{\Omega} = \frac{1}{\theta}\left[2\mu|\mathbf{D}^d|^2 + \frac{2\mu+3\lambda}{3}(\operatorname{div}\mathbf{v})^2 + \kappa\frac{|\nabla\theta|^2}{\theta}\right]$$

$$\zeta^{\Omega} = \frac{1}{\theta}\left[\frac{1}{2\mu}|\mathbf{T}^d|^2 + \frac{3}{2\mu+3\lambda}(m+p)^2 + \frac{1}{\kappa}\frac{|\mathbf{q}|^2}{\theta}\right]$$

(I) Mechanically and **energetically** isolated body

$$\boxed{\mathbf{v} \cdot \mathbf{n} = 0 \quad \text{and} \quad (\mathbf{T}\mathbf{v} + \mathbf{q}) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma}$$

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(II) Mechanically and **thermally** isolated body

$$\boxed{\mathbf{v} \cdot \mathbf{n} = 0 \quad \text{and} \quad \mathbf{q} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma}$$

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(II) Mechanically and **thermally** isolated body

$$\boxed{\mathbf{v}\cdot\mathbf{n}=0 \quad \text{and} \quad \mathbf{q}\cdot\mathbf{n}=0 \quad \text{on } \Gamma}$$

Drawbacks of **(II)**:

- **(II)** is not compatible with Navier's slip bc: $\boxed{(1-\lambda)\alpha(\mathbf{T}\mathbf{n})_{\boldsymbol{\tau}}+\lambda\mathbf{v}_{\boldsymbol{\tau}}=\mathbf{0}}$
- **(II)** excludes Poiseuille flow as the admissible flow between two parallel plates if no slip bc should hold at plates   Bulíček, Málek, Rajagopal (2010)

It follows from **(I)**:

$$\begin{aligned}-\mathbf{q}\cdot\mathbf{n} &= \mathbf{T}\mathbf{v}\cdot\mathbf{n} = \mathbf{T}\cdot(\mathbf{v}\otimes\mathbf{n}) = \mathbf{T}\cdot(\mathbf{n}\otimes\mathbf{v}) \\ &= (\mathbf{T}\mathbf{n})\cdot\mathbf{v} = (\mathbf{T}\mathbf{n})_{\boldsymbol{\tau}}\cdot\mathbf{v}_{\boldsymbol{\tau}}\end{aligned}$$

Since $\boxed{-\mathbf{q}\cdot\mathbf{n} = (\mathbf{Tn})_{\boldsymbol{\tau}}\cdot\mathbf{v}_{\boldsymbol{\tau}}}$, the integration (over Ω) of

$$\varrho\dot{\eta} - \operatorname{div}\left(\frac{\mathbf{q}}{\theta}\right) = \zeta^{\Omega} =: \frac{\xi^{\Omega}}{\theta} \qquad\qquad \xi^{\Omega} = \mathbf{T}^d\cdot\mathbf{D}^d + (m+p)\operatorname{div}\mathbf{v} + \mathbf{q}\cdot\frac{\nabla\theta}{\theta}$$

leads to Eq. for $\boxed{S(t) := \int_{\Omega}\varrho\eta\,dx}$:

$$\begin{aligned}\frac{d}{dt}S(t) &= \int_{\Omega}\frac{\xi^{\Omega}}{\theta} + \int_{\Gamma}\frac{\xi^{\Gamma}}{\theta} = \int_{\Omega}\frac{\xi^{\Omega}}{\theta} + \int_{\Gamma}\frac{\mathbf{q}\cdot\mathbf{n}}{\theta} = \int_{\Omega}\frac{\xi^{\Omega}}{\theta} + \int_{\Gamma}\frac{-(\mathbf{Tn})_{\boldsymbol{\tau}}\cdot\mathbf{v}_{\boldsymbol{\tau}}}{\theta}\\ &= (\mathbf{T}^d,\mathbf{D}^d)_{\Omega} + ((m+p),\operatorname{div}\mathbf{v})_{\Omega} + \left(\mathbf{q},\frac{\nabla\theta}{\theta}\right)_{\Omega} + (-(\mathbf{Tn})_{\boldsymbol{\tau}},\mathbf{v}_{\boldsymbol{\tau}})_{\Gamma}\\ &:= X \geq 0\end{aligned}$$

$$(f,g)_G := \int_G \tfrac{1}{\theta} f\cdot g$$

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$$(f,g)_{G} := \int_{G}\tfrac{1}{\theta}f\cdot g$$

$$X = (\mathbf{T}^d, \mathbf{D}^d)_\Omega + ((m+p), \operatorname{div} \mathbf{v})_\Omega + \left(\mathbf{q}, \frac{\nabla\theta}{\theta}\right)_\Omega + (-(\mathbf{Tn})_{\boldsymbol{\tau}}, \mathbf{v}_{\boldsymbol{\tau}})_\Gamma$$

One set of sufficient relations that guarantee that $X \geq 0$

$$\mathbf{T}^d = 2\nu \mathbf{D}^d \qquad \mathbf{D}^d = \frac{1}{2\nu}\mathbf{T}^d$$

$$m + p = \frac{2\nu + 3\lambda}{3} \operatorname{div} \mathbf{v} \qquad \operatorname{div} \mathbf{v} = \frac{3}{2\nu + 3\lambda}(m+p)$$

$$\mathbf{q} = \kappa \frac{\nabla\theta}{\theta} \qquad \frac{\nabla\theta}{\theta} = \frac{1}{\kappa}\mathbf{q}$$

$$(\mathbf{Tn})_{\boldsymbol{\tau}} = -\frac{1}{\alpha}\mathbf{v}_\tau \qquad \mathbf{v}_{\boldsymbol{\tau}} = -\alpha(\mathbf{Tn})_{\boldsymbol{\tau}}$$

with ν, $2\nu + 3\lambda$, κ and α positive

$$\boxed{\begin{aligned} \xi^{\Omega} &= \mathbf{T}^d \cdot \mathbf{D}^d + (m+p)\,\mathrm{div}\,\mathbf{v} + \mathbf{q} \cdot \frac{\nabla\theta}{\theta} \\ \xi^{\Gamma} &= -(\mathbf{T}\mathbf{n})_{\tau} \cdot \mathbf{v}_{\tau} \end{aligned}} \tag{1}$$

Constitutive equation for X:

$$\boxed{\tilde{X} = \int_{\Omega} \frac{|\mathbf{T}^d|^2}{\nu} + \frac{3}{2\nu + 3\lambda}(m+p)^2 + \frac{1}{\kappa}|\mathbf{q}|^2 + \int_{\Gamma} \frac{1}{\alpha}|(\mathbf{T}\mathbf{n})_{\tau}|^2}$$

with ν, $2\nu + 3\lambda$, κ and α positive

Maximization of $\tilde{X}$ with respect to $\mathbf{T}^d$, $m+p$, $\mathbf{q}$ and $(\mathbf{T}\mathbf{n})_{\tau}$ keeping (1) as a **constraint** $\Longrightarrow$ the above constitutive equations relating $\mathbf{T}^d$ and $\mathbf{D}^d$, $\mathbf{q}$ and $\nabla\theta$ etc.

$$\xi^{\Omega} = \mathbf{T}^d \cdot \mathbf{D}^d + (m + p)\,\mathrm{div}\,\mathbf{v} + \mathbf{q} \cdot \frac{\nabla\theta}{\theta}$$
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Constitutive equation for X:

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$$\mathbf{q} = \kappa\nabla\theta$$

$$\xi^{\Omega} = \mathbf{T}^d \cdot \mathbf{D}^d + (m+p)\,\mathrm{div}\,\mathbf{v} + \mathbf{q} \cdot \frac{\nabla\theta}{\theta}$$

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with ν, $2\nu+3\lambda$, κ and α positive

Reduction due to constraints:

- (i) rigid body dynamics $\mathbf{D}^d(\mathbf{v}) = 0$ (ii) incompressibility $\mathrm{div}\,\mathbf{v} = 0$
- (iii) isothermal processes $\nabla\theta = 0$ (iv) no-slip $\mathbf{v}_{\boldsymbol{\tau}} = 0$

Reduction due to missing dissipative mechanism

- (i) $\mathbf{T}^d = \mathbf{0}$ (ii) $m = -p$
- (iii) $\mathbf{q} = \mathbf{0}$ (iv) $(\mathbf{Tn})_{\boldsymbol{\tau}} = \mathbf{0}$

Constitutive theory for mechanically and energetically isolated body

- Evolution eqs for certain quantities (balance equations, their consequences, etc. $\varrho \dot{y}_i = \dots$)
- Constitutive eq 1 for η: $\boxed{\eta = \tilde{\eta}(e, \varrho, y_2, y_3, \dots)} \iff e = \tilde{e}(\eta, \varrho, y_2, y_3, \dots)$

Write down eq. $\varrho \dot{\eta}$ and $S(t)$

$$\varrho \dot{\eta} - \operatorname{div}\left(\frac{\mathbf{q}_\eta}{\theta}\right) = \frac{\xi^\Omega}{\theta} \implies \frac{d}{dt} S(t) = \int_\Omega \frac{\xi^\Omega}{\theta} + \int_\Gamma \frac{\xi^\Gamma}{\theta} = X$$

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where

$$\boxed{\xi^\Omega = \sum_\alpha J_\alpha A_\alpha \quad \text{and} \quad \xi^\Gamma = \sum_\beta J_{\beta,\Gamma} A_{\beta,\Gamma}} \tag{2}$$

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- Constitutive eq 2 for X: $\boxed{X = \tilde{X}(J_\alpha, A_\alpha, J_{\beta,\Gamma}, A_{\beta,\Gamma})}$

Constitutive theory for mechanically and energetically isolated body

- Evolution eqs for certain quantities (balance equations, their consequences, etc. $\varrho \dot{y}_i = \dots$)
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$$\boxed{\xi^\Omega = \sum_\alpha J_\alpha A_\alpha \quad \text{and} \quad \xi^\Gamma = \sum_\beta J_{\beta,\Gamma} A_{\beta,\Gamma}} \tag{2}$$

- Constitutive eq 2 for X: $\boxed{X = \tilde{X}(J_\alpha, A_\alpha, J_{\beta,\Gamma}, A_{\beta,\Gamma})}$

Maximization of $\tilde{X}$ with respect to J_α and $J_{\beta,\Gamma}$ requiring that (2) as a **constraint** $\implies$ the constitutive equations for other involved quantities

$$
\begin{aligned}
\dot{\varrho} &= -\varrho \operatorname{div} \mathbf{v} \\
\varrho \dot{\mathbf{v}} &= \operatorname{div} \mathbf{T} + \varrho \boldsymbol{f} \qquad \mathbf{T} = \mathbf{T}^T \\
\varrho \dot{\mathrm{E}} &= \operatorname{div}(\mathbf{T}\mathbf{v} - \mathbf{q}) + \varrho \boldsymbol{f} \cdot \mathbf{v} \qquad \mathrm{E} := e + |\mathbf{v}|^2/2
\end{aligned}
$$

$$\boxed{\begin{aligned} \dot{\varrho} &= -\varrho \operatorname{div} \mathbf{v} \\ \varrho \dot{\mathbf{v}} &= \operatorname{div} \mathbf{T} + \varrho \boldsymbol{f} \qquad \mathbf{T} = \mathbf{T}^T \\ \varrho \dot{\mathrm{E}} &= \operatorname{div}(\mathbf{T}\mathbf{v} - \mathbf{q}) + \varrho \boldsymbol{f} \cdot \mathbf{v} \qquad \mathrm{E} := e + |\mathbf{v}|^2/2 \end{aligned}}$$

and

$$\boxed{\frac{d}{dt}(\nabla \varrho) =: \dot{\overline{\nabla \varrho}} = -(\nabla \mathbf{v})\nabla \varrho - \nabla(\varrho \operatorname{div} \mathbf{v})}$$

$$\dot{\varrho} = -\varrho \operatorname{div} \mathbf{v}$$

$$\varrho\dot{\mathbf{v}} = \operatorname{div} \mathbf{T} + \varrho \boldsymbol{f} \qquad \mathbf{T} = \mathbf{T}^T$$

$$\varrho\dot{\mathrm{E}} = \operatorname{div}(\mathbf{T}\mathbf{v} - \mathbf{q}) + \varrho \boldsymbol{f} \cdot \mathbf{v} \qquad \mathrm{E} := e + |\mathbf{v}|^2/2$$

and

$$\frac{d}{dt}(\nabla\varrho) =: \dot{\overline{\nabla\varrho}} = -(\nabla\mathbf{v})\nabla\varrho - \nabla(\varrho \operatorname{div} \mathbf{v})$$

Constitutive equation for the entropy

$$\boxed{\eta = \tilde{\eta}(e, \varrho, \nabla\varrho)} \quad \Longleftrightarrow \quad e = \tilde{e}(\eta, \varrho, |\nabla\varrho|)$$

$$\boxed{\begin{aligned} \dot{\varrho} &= -\varrho \operatorname{div} \mathbf{v} \\ \varrho \dot{\mathbf{v}} &= \operatorname{div} \mathbf{T} + \varrho \boldsymbol{f} \qquad \mathbf{T} = \mathbf{T}^T \\ \varrho \dot{\mathrm{E}} &= \operatorname{div}(\mathbf{T}\mathbf{v} - \mathbf{q}) + \varrho \boldsymbol{f} \cdot \mathbf{v} \qquad \mathrm{E} := e + |\mathbf{v}|^2/2 \end{aligned}}$$

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$$\boxed{\frac{d}{dt}(\nabla \varrho) =: \dot{\overline{\nabla \varrho}} = -(\nabla \mathbf{v})\nabla \varrho - \nabla(\varrho \operatorname{div} \mathbf{v})}$$

Constitutive equation for the entropy

$$\boxed{\eta = \tilde{\eta}(e, \varrho, \nabla \varrho)} \quad \Longleftrightarrow \quad e = \tilde{e}(\eta, \varrho, |\nabla \varrho|)$$

It leads to

$$\boxed{\varrho \dot{\eta} - \operatorname{div}\left(\frac{\mathbf{q} + \varrho^2 \operatorname{div} \mathbf{v} \partial_{\mathbf{z}} \tilde{e}}{\theta}\right) = \frac{1}{\theta}\left[\mathbf{T}^d_{diss} \cdot \mathbf{D}^d + t_{diss} \operatorname{div} \mathbf{v} + \mathbf{q}_{diss} \cdot \frac{\nabla \theta}{\theta}\right]}$$

$$\begin{aligned} \mathbf{T}_{diss} &:= \mathbf{T} + \varrho \partial_{\mathbf{z}} \tilde{e} \otimes \nabla \varrho \\ t_{diss} &:= m + p + \tilde{m} - \varrho \operatorname{div}(\varrho \partial_{\mathbf{z}} \tilde{e}) \\ \mathbf{q}_{diss} &:= \mathbf{q} + \varrho^2 \operatorname{div} \mathbf{v} \partial_{\mathbf{z}} \tilde{e}. \end{aligned}$$

Since $\mathbf{q} \cdot \mathbf{n} = -(\mathbf{T}\mathbf{n})_{\boldsymbol{\tau}} \cdot \mathbf{v}_{\boldsymbol{\tau}}$

$$\boxed{\begin{aligned} \frac{d}{dt} S(t) = & \int_\Omega \frac{1}{\theta} \left[\mathbf{T}^d_{diss} \cdot \mathbf{D}^d + t_{diss} \operatorname{div} \mathbf{v} + \mathbf{q}_{diss} \cdot \frac{\nabla \theta}{\theta} \right] \\ & + \int_\Gamma \frac{1}{\theta} \left[-(\mathbf{T}\mathbf{n})_{\boldsymbol{\tau}} \cdot \mathbf{v}_{\boldsymbol{\tau}} + \varrho^2 \operatorname{div} \mathbf{v} (\partial_{\mathbf{z}} \cdot \mathbf{n}) \right] \\ & =: X \end{aligned}}$$

Constitutive eq. for $X = \tilde{X}(\mathbf{T}_{diss}, t_{diss}, \mathbf{q}_{diss}, (\mathbf{T}\mathbf{n})_{\boldsymbol{\tau}}, \varrho \partial_{\mathbf{z}} \tilde{e} \cdot \mathbf{n})$ and Constrained maximization:

$$\mathbf{T}^d_{diss} = 2\nu \mathbf{D}^d \quad \text{in } \Omega \qquad\qquad (\mathbf{T}\mathbf{n})_{\boldsymbol{\tau}} = -\alpha \mathbf{v}_{\boldsymbol{\tau}} \quad \text{on } \Gamma$$

$$t_{diss} = \frac{2\nu + 3\lambda}{3} \operatorname{div} \mathbf{v} \qquad\qquad \varrho \operatorname{div} \mathbf{v} (\partial_{\mathbf{z}} \cdot \mathbf{n}) = \gamma \varrho \operatorname{div} \mathbf{v}$$

$$\mathbf{q}_{diss} = \kappa \frac{\nabla \theta}{\theta}$$

with ν, $2\nu + 3\lambda$, κ, α and γ positive

In particular: for $\boxed{e = e_0(\eta, \varrho) + \frac{\beta}{2\varrho}|\nabla\varrho|^2}$ we obtain:

In Ω

$$\boxed{\begin{aligned} \mathbf{T} &= (-p + \beta\varrho\Delta\varrho)\mathbf{I} - \beta(\nabla\varrho \otimes \nabla\varrho) + 2\nu\mathbf{D} + \lambda(\operatorname{div}\mathbf{v})\mathbf{I} \\ \mathbf{q} &= \kappa\frac{\nabla\theta}{\theta} - \beta\varrho\operatorname{div}\mathbf{v}\nabla\varrho \end{aligned}}$$

On Γ

$$\boxed{\begin{aligned} (\mathbf{Tn})_{\boldsymbol{\tau}} &= -\alpha\mathbf{v}_{\boldsymbol{\tau}} \\ \frac{\partial\varrho}{\partial\mathbf{n}} &= \frac{\gamma\varrho}{\beta}\operatorname{div}\mathbf{v} \end{aligned}}$$

Consequences towards theoretical analysis

- the approach (MREP) provides the starting apriori estimates that are in place $\Longrightarrow$ choice of the concept of solution, specification of the function spaces
- balance equations come out from basic considerations concerning heat transfer, balance of linear momentum in classical mechanics

$$\sup_{t\in[0,T]} \int_\Omega \varrho(e + |\boldsymbol{v}|^2/2) + \int_0^T \tilde{X} < \infty$$

where

$$\tilde{X} = \int_\Omega \frac{|\mathbf{T}^d|^2}{\nu} + \frac{3}{2\nu + 3\lambda}(m+p)^2 + \frac{1}{\kappa}|\mathbf{q}|^2 + \int_\Gamma \frac{1}{\alpha}|(\mathbf{Tn})_{\boldsymbol{\tau}}|^2$$

“Equivalent” formulations of the balance of energy

(I) Balance equations for mass, linear momentum and energy

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \boldsymbol{v}) = 0$$
$$\frac{\partial (\varrho \boldsymbol{v})}{\partial t} + \operatorname{div}(\varrho \boldsymbol{v} \otimes \boldsymbol{v}) = \operatorname{div} \mathbf{T} + \varrho \boldsymbol{f}$$
$$\frac{\partial \varrho (e + |\boldsymbol{v}|^2/2)}{\partial t} + \operatorname{div}(\varrho (e + |\boldsymbol{v}|^2/2) \boldsymbol{v}) = \operatorname{div} (\mathbf{T} \boldsymbol{v} + \mathbf{q}) + \varrho \boldsymbol{f} \cdot \mathbf{v} + \varrho r$$

(II) Balance equations for mass, linear momentum and energy

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \boldsymbol{v}) = 0$$
$$\frac{\partial (\varrho \boldsymbol{v})}{\partial t} + \operatorname{div}(\varrho \boldsymbol{v} \otimes \boldsymbol{v}) = \operatorname{div} \mathbf{T} + \varrho \boldsymbol{f}$$
$$\frac{\partial (\varrho e)}{\partial t} + \operatorname{div}(\varrho e \boldsymbol{v}) = \mathbf{T} \cdot \mathbf{D} + \operatorname{div} \mathbf{q} + \varrho r$$

“Equivalent” formulations of the balance of energy

(III) Balance equations for mass, linear momentum and energy

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \boldsymbol{v}) = 0$$

$$\frac{\partial (\varrho \boldsymbol{v})}{\partial t} + \operatorname{div}(\varrho \boldsymbol{v} \otimes \boldsymbol{v}) = \operatorname{div} \mathbf{T} + \varrho \boldsymbol{f}$$

$$\frac{d}{dt} \int_\Omega \varrho (e + |\boldsymbol{v}|^2/2) = 0$$

$$\frac{\partial (\varrho \eta)}{\partial t} + \operatorname{div}(\varrho \eta \boldsymbol{v}) - \operatorname{div}\left(\frac{\mathbf{q}}{\theta}\right) = \xi$$

where

$$\xi \geq \frac{1}{\theta}\left[2\mu |\mathbf{D}^d|^2 + \frac{2\mu + 3\lambda}{3} (\operatorname{div} \mathbf{v})^2 + \kappa \frac{|\nabla \theta|^2}{\theta}\right]$$

Part #7

Do we need more complex fluid models beyond compressible NSF?

Navier-Stokes fluid model cannot describe several phenomena that have observed and documented experimentally:

- shear thinning, shear thickenning
- pressure thickening
- the presence of activation or deactivation criteria
- the presence of the normal stress differences at simple shear flows
- stress relaxation
- non-linear creep
- responses of anisotropic materials
- responses of inhomogeneous materials

Navier-Stokes fluid model cannot describe several phenomena that have observed and documented experimentally:

- shear thinning, shear thickenning
- pressure thickening
- the presence of activation or deactivation criteria
- the presence of the normal stress differences at simple shear flows
- stress relaxation
- non-linear creep
- responses of anisotropic materials
- responses of inhomogeneous materials

Example of non-Newtonian fluids:

- the lithosphere
- glacier, ice sheets

Part #8

What can one understand by *Model reduction*?

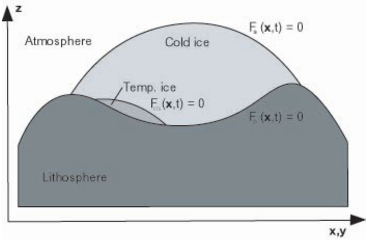
Types of Model Reduction

1. constraints
   - rigid-body dynamics
   - incompressibility
   - isothermal
   - no-slip
2. steady flows
3. neglecting inertia
4. finite-dimensional
   - ansatz - special flows/deformation (ODEs)
   - discretization
5. geometry
   - thin film - Reynolds approximation
   - boundary layer - Prandtl
   - shallow water, shallow ice (K. Hutter), shallow shell approximations
   - Oberbeck-Boussinesq approximation
   - linearized elasticity

Problem - long time dynamics of ice sheets

- Simplified single-component ice-sheet model
- Scaling and dimensionless form
- Shallow-Ice limit
- Solvability of the SIA

Simple Ice-sheet model



$$F_s, F_b : \mathbb{R}^2 \times (0,\tau) \to \mathbb{R}, \qquad F_s, F_b \in C^1(\mathbb{R}^2 \times (0,\tau))$$

$$\Omega(t) := \left\{ \boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3; \quad F_b(x_1, x_2, t) < x_3 < F_s(x_1, x_2, t) \right\}$$

$$\boldsymbol{n}_s := \frac{\left(-\frac{\partial F_s}{\partial x_1}, -\frac{\partial F_s}{\partial x_2}, 1\right)}{\sqrt{1+\left(\frac{\partial F_s}{\partial x_1}\right)^2+\left(\frac{\partial F_s}{\partial x_2}\right)^2}} \qquad \boldsymbol{n}_b := \frac{\left(\frac{\partial F_b}{\partial x_1}, \frac{\partial F_b}{\partial x_2}, -1\right)}{\sqrt{1+\left(\frac{\partial F_b}{\partial x_1}\right)^2+\left(\frac{\partial F_b}{\partial x_2}\right)^2}}$$

Balance equations

- Balance equations for mass, linear and angular momentum, energy

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \boldsymbol{v}) = 0$$

$$\frac{\partial(\varrho \boldsymbol{v})}{\partial t} + \operatorname{div}(\varrho \boldsymbol{v} \otimes \boldsymbol{v}) = \operatorname{div} \mathbf{T} + \varrho \boldsymbol{f} \qquad \mathbf{T} = \mathbf{T}^T$$

$$\frac{\partial \varrho(e + |\boldsymbol{v}|^2/2)}{\partial t} + \operatorname{div}(\varrho(e + |\boldsymbol{v}|^2/2)\boldsymbol{v}) = \operatorname{div}(\mathbf{T}\boldsymbol{v} + \mathbf{q}) + \varrho \boldsymbol{f} \cdot \mathbf{v} + \varrho r$$

Balance equations, incompressibility and uniform density

- Balance equations

$$\begin{aligned} \operatorname{div} \boldsymbol{v} &= 0 \\ \varrho\left(\boldsymbol{v}_{,t} + [\nabla \boldsymbol{v}]\boldsymbol{v}\right) &= -\nabla p + \operatorname{div} \mathbf{S} + \varrho \mathbf{g} \\ \varrho c_v \left(\theta_{,t} + \nabla\theta \cdot \boldsymbol{v}\right) &= \mathbf{S} \cdot \boldsymbol{D}(\boldsymbol{v}) + k \triangle \theta \end{aligned}$$

- Constitutive equations + constraints

$$\begin{aligned} \varrho &= \mathrm{const} \\ \mathbf{S} &= 2\eta(\theta, |\mathbf{D}|^2)\mathbf{D} \\ \eta &= \frac{1}{2}\mathcal{A}(\theta)^{-\frac{1}{n}} |\mathbf{D}|^{\frac{1-n}{n}} \\ \mathbf{D} &= \mathcal{A}(\theta)|\mathbf{S}|^{2^{\frac{n-1}{2}}}\mathbf{S} \\ r &= 0 \\ \boldsymbol{q} &= -k\nabla\theta \\ e &= c_v\theta \end{aligned}$$

$$\begin{aligned} \mathbf{D} &= \frac{1}{2}(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T) \\ |\mathbf{D}|^2 &= \mathbf{D} \cdot \mathbf{D} \\ |\mathbf{S}|^2 &= \mathbf{S} \cdot \mathbf{S} \\ \mathcal{A}(\theta) &= A \exp\left(-\frac{Q}{k_B \theta}\right) \end{aligned}$$

Boundary conditions

- Kinematic Γ_s

$$-\frac{\partial F_s(x_1,x_2,t)}{\partial t} + \boldsymbol{v}\cdot\left(-\frac{\partial F_s}{\partial x_1}, -\frac{\partial F_s}{\partial x_2}, 1\right) = -a^s(x_1,x_2,t)\sqrt{1+\left(\frac{\partial F_s}{\partial x_1}\right)^2+\left(\frac{\partial F_s}{\partial x_2}\right)^2}$$

- Dynamic Γ_s

$$\begin{aligned} (-p\boldsymbol{I}+\mathbf{S})\,\boldsymbol{n}_s &= \mathbf{0} \\ \theta &= \theta^s \end{aligned}$$

- Dynamic Γ_b

$$\begin{aligned} \beta^2(p, \boldsymbol{n}_b\cdot\mathbf{S}\boldsymbol{n}_b, \theta)\boldsymbol{v}_{\boldsymbol{\tau}_b} &= -(\mathbf{S}\boldsymbol{n}_b)_{\boldsymbol{\tau}_b} \\ k\nabla\theta\cdot\boldsymbol{n}_b &= -\boldsymbol{q}^{geo}\cdot\boldsymbol{n}_b \end{aligned}$$

Scaling

- Scales

$$
\begin{aligned}
(x_1, x_2) &= [L](\tilde{x}_1, \tilde{x}_2) \\
x_3 &= [H]\tilde{x}_3 \\
(v_1, v_2) &= [v_h](\tilde{v}_1, \tilde{v}_2) \\
v_3 &= [v_v]\tilde{v}_3 \\
t &= \frac{[L]}{[v_h]}\tilde{t} \\
T &= [T]\,\tilde{\theta} \\
\mathcal{A}(\theta) &= [\mathcal{A}]\,\tilde{\mathcal{A}}(\tilde{\theta})
\end{aligned}
$$

Scaling

- Scales

$$
\begin{aligned}
a^s &= \varrho[v_v]\,\tilde{a}^s \\
(q_1, q_1^{geo}, q_2, q_2^{geo}) &= \frac{k[\theta]}{[L]}\,(\tilde{q}_1, \tilde{q}_1^{geo}, \tilde{q}_2, \tilde{q}_2^{geo}) \\
(q_3, q_3^{geo}) &= \frac{k[\theta]}{[H]}\,(\tilde{q}_3, \tilde{q}_3^{geo}) \\
\beta(p, \boldsymbol{n}_b \cdot \mathbf{S}\boldsymbol{n}_b, \theta, \ldots) &= [\beta]\,\tilde{\beta}(\tilde{p}, \tilde{\boldsymbol{n}}_b \cdot \tilde{\mathbf{S}}\tilde{\boldsymbol{n}}_b, , \tilde{\theta}, \ldots) \\
p &= [H]\varrho g\tilde{p} \\
(S_{13}, S_{23}) &= \frac{[H]^2\varrho g}{[L]}(\tilde{S}_{13}, \tilde{S}_{23}) \\
(S_{11}, S_{22}, S_{12}) &= \frac{[H]^3\varrho g}{[L]^2}(\tilde{S}_{11}, \tilde{S}_{22}, \tilde{S}_{12})
\end{aligned}
$$

- Constants: $k, c_v, \varrho, g, \frac{Q}{k_b}$
- Taking $\frac{[v_v]}{[v_h]} := \frac{[H]}{[L]}$ $\Rightarrow$ **6 independent scales** + **5 constants** - **4** (rank of dimension matrix) = **7** (Buckingham's Pi Theorem) independent dimensionless numbers

Dimensionless numbers

$$\epsilon = \frac{[H]}{[L]} \qquad \mathcal{C} = \frac{g[H]}{c_v[\theta]}$$

$$\mathcal{K} = \frac{[\mathcal{A}]\varrho^n g^n [H]^{2n+1}}{[L]^n [v_h]} \qquad \mathcal{B} = \frac{\varrho g [H]^2}{[L][v_h][\beta]^2}$$

$$\mathcal{P} = \frac{k[L]}{\varrho c_v [v_h][H]^2} \qquad \mathcal{Q} = \frac{Q}{k_b[T]}$$

$$\mathcal{F} = \frac{[v_h]^2}{[L]g}$$

$[T] = 273.15\ \mathrm{K}$ $\quad g = 9.81\ \mathrm{m\ s^{-2}}$ $\quad \varrho = 910\ \mathrm{kg\ m^{-3}}$ $\quad c_v = 2000\ \mathrm{J\ kg^{-1}\ K^{-1}}$

$[v_h] = 1\ \mathrm{m\ a^{-1}}$ $\quad k = 2\ \mathrm{W\ m^{-1}\ K^{-1}}$ $\quad \frac{Q}{k_B} = 7216\ \mathrm{K}$ $\quad [\beta]^2 = 1000\ \mathrm{kg\ m^{-2}\ s^{-1}}$

Dimensionless numbers

$$\epsilon = \frac{[H]}{[L]} \quad \sim 10^{-2} - 10^{-3} \qquad \mathcal{C} = \frac{g[H]}{c_v[\theta]} \quad \sim 0.018$$

$$\mathcal{K} = \frac{[\mathcal{A}]\varrho^n g^n [H]^{2n+1}}{[L]^n [v_h]} \quad n = 3 \ \sim 9 \times 10^{12} \qquad \mathcal{B} = \frac{\varrho g [H]^2}{[L][v_h][\beta]^2} \quad \sim 2.8 \times 10^9$$

$$\mathcal{P} = \frac{k[L]}{\varrho c_v [v_h][H]^2} \quad \sim 3.44 \qquad \mathcal{Q} = \frac{Q}{k_b[T]} \quad \sim 26.42$$

$$\mathcal{F} = \frac{[v_h]^2}{[L]g} \quad \sim 1 \times 10^{-23}$$

$[T] = 273.15 \text{ K}$ $\quad g = 9.81 \text{ m s}^{-2}$ $\quad \varrho = 910 \text{ kg m}^{-3}$ $\quad c_v = 2000 \text{ J kg}^{-1} \text{ K}^{-1}$

$[v_h] = 1 \text{ m a}^{-1}$ $\quad k = 2 \text{ W m}^{-1} \text{ K}^{-1}$ $\quad \frac{Q}{k_B} = 7216 \text{ K}$ $\quad [\beta]^2 = 1000 \text{ kg m}^{-2} \text{ s}^{-1}$

Dimensionless form and Shallow-Ice limit $\epsilon, \frac{\mathcal{F}}{\epsilon} \to 0+$

- Incompressibility

$$\operatorname{div} \boldsymbol{v} = 0$$

$$\frac{\partial \tilde{v}_1}{\partial \tilde{x}_1} + \frac{\partial \tilde{v}_2}{\partial \tilde{x}_2} + \frac{\partial \tilde{v}_3}{\partial \tilde{x}_3} = 0$$

Dimensionless form and Shallow-Ice limit $\epsilon, \frac{\mathcal{F}}{\epsilon} \to 0+$

- Balance of linear momentum

$$\varrho\left(\boldsymbol{v}_{,t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v}\right) = -\nabla p + \operatorname{div}\mathbf{S} + \varrho\mathbf{g}$$

$$\begin{aligned}
\frac{\mathcal{F}}{\epsilon}\left(\frac{\partial\tilde{v}_1}{\partial\tilde{t}} + \tilde{v}_k\frac{\partial\tilde{v}_1}{\partial\tilde{x}_k}\right) &= -\frac{\partial\tilde{p}}{\partial\tilde{x}_1} + \frac{\partial\tilde{S}_{13}}{\partial\tilde{x}_3} + \epsilon^2\left(\frac{\partial\tilde{S}_{11}}{\partial\tilde{x}_1} + \frac{\partial\tilde{S}_{12}}{\partial\tilde{x}_2}\right)\\
\frac{\mathcal{F}}{\epsilon}\left(\frac{\partial\tilde{v}_2}{\partial\tilde{t}} + \tilde{v}_k\frac{\partial\tilde{v}_2}{\partial\tilde{x}_k}\right) &= -\frac{\partial\tilde{p}}{\partial\tilde{x}_2} + \frac{\partial\tilde{S}_{23}}{\partial\tilde{x}_3} + \epsilon^2\left(\frac{\partial\tilde{S}_{12}}{\partial\tilde{x}_1} + \frac{\partial\tilde{S}_{22}}{\partial\tilde{x}_2}\right)\\
\epsilon\mathcal{F}\left(\frac{\partial\tilde{v}_3}{\partial\tilde{t}} + \tilde{v}_k\frac{\partial\tilde{v}_3}{\partial\tilde{x}_k}\right) &= -\frac{\partial\tilde{p}}{\partial\tilde{x}_3} - 1 + \epsilon^2\left(\frac{\partial\tilde{S}_{13}}{\partial\tilde{x}_1} + \frac{\partial\tilde{S}_{23}}{\partial\tilde{x}_2} + \frac{\partial\tilde{S}_{33}}{\partial\tilde{x}_3}\right)
\end{aligned}$$

Dimensionless form and Shallow-Ice limit $\epsilon, \frac{\mathcal{F}}{\epsilon} \to 0+$

- Balance of linear momentum

$$\varrho\,(\boldsymbol{v}_{,t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v}) \quad = \quad -\nabla p + \operatorname{div}\mathbf{S} + \varrho\mathbf{g}$$

$$\begin{aligned}
0 &= -\frac{\partial \tilde{p}^0}{\partial \tilde{x}_1} + \frac{\partial \tilde{S}^0_{13}}{\partial \tilde{x}_3} \\
0 &= -\frac{\partial \tilde{p}^0}{\partial \tilde{x}_2} + \frac{\partial \tilde{S}^0_{23}}{\partial \tilde{x}_3} \\
0 &= -\frac{\partial \tilde{p}^0}{\partial \tilde{x}_3} - 1
\end{aligned}$$

Shallow-Ice limit $\epsilon, \frac{\mathcal{F}}{\epsilon} \longrightarrow 0+$ - balance eq.

- Constitutive equations

$$\mathbf{S} = 2\eta(T, |\mathbf{D}|^2)\mathbf{D} = \mathcal{A}(\theta)^{-\frac{1}{n}}|\mathbf{D}|^{\frac{1-n}{n}}\mathbf{D}$$

$$\begin{aligned}
\tilde{S}_{13} &= \tilde{\eta}\left(\frac{\partial \tilde{v}_1}{\partial \tilde{x}_3} + \epsilon^2 \frac{\partial \tilde{v}_3}{\partial \tilde{x}_1}\right) & \tilde{D}_{13} &= \frac{1}{2}\left(\frac{\partial \tilde{v}_1}{\partial \tilde{x}_3} + \epsilon^2 \frac{\partial \tilde{v}_3}{\partial \tilde{x}_1}\right) \\
\tilde{S}_{23} &= \tilde{\eta}\left(\frac{\partial \tilde{v}_2}{\partial \tilde{x}_3} + \epsilon^2 \frac{\partial \tilde{v}_3}{\partial \tilde{x}_2}\right) & \tilde{D}_{23} &= \frac{1}{2}\left(\frac{\partial \tilde{v}_2}{\partial \tilde{x}_3} + \epsilon^2 \frac{\partial \tilde{v}_3}{\partial \tilde{x}_2}\right) \\
\tilde{S}_{12} &= \tilde{\eta}\left(\frac{\partial \tilde{v}_1}{\partial \tilde{x}_2} + \frac{\partial \tilde{v}_2}{\partial \tilde{x}_1}\right) & \tilde{D}_{12} &= \frac{1}{2}\left(\frac{\partial \tilde{v}_1}{\partial \tilde{x}_2} + \frac{\partial \tilde{v}_2}{\partial \tilde{x}_1}\right) \\
\tilde{\mathbf{S}}_{11} &= 2\tilde{\eta}\frac{\partial \tilde{v}_1}{\partial \tilde{x}_1} & \tilde{D}_{11} &= \frac{\partial \tilde{v}_1}{\partial \tilde{x}_1} \\
\tilde{S}_{22} &= 2\tilde{\eta}\frac{\partial \tilde{v}_2}{\partial \tilde{x}_2} & \tilde{D}_{22} &= \frac{\partial \tilde{v}_2}{\partial \tilde{x}_2}
\end{aligned}$$

$$\begin{aligned}
\tilde{\eta} &= \frac{1}{2}(\tilde{\mathcal{A}}(\tilde{\theta})\mathcal{K})^{-\frac{1}{n}}\left(\tilde{D}_{13}^2 + \tilde{D}_{23}^2 + \epsilon^2\left(\tilde{D}_{11}^2 + \tilde{D}_{22}^2 + \tilde{D}_{11}\tilde{D}_{22} + \tilde{D}_{12}^2\right)\right)^{\frac{1-n}{2n}} \\
\tilde{\mathcal{A}} &= \frac{A}{[A]}\exp\left(-\frac{\mathcal{Q}}{\tilde{\theta}}\right)
\end{aligned}$$

Shallow-Ice limit $\epsilon, \frac{\mathcal{F}}{\epsilon} \longrightarrow 0+$ - balance eq.

- Constitutive equation

$$\mathbf{S} = 2\eta(T, |\mathbf{D}|^2)\mathbf{D} = \mathcal{A}(\theta)^{-\frac{1}{n}}|\mathbf{D}|^{\frac{1-n}{n}}$$

$$\tilde{S}^0_{13} = \tilde{\eta}^0 \frac{\partial \tilde{v}^0_1}{\partial \tilde{x}_3} \qquad \tilde{D}^0_{13} = \frac{1}{2}\frac{\partial \tilde{v}^0_1}{\partial \tilde{x}_3}$$

$$\tilde{S}^0_{23} = \tilde{\eta}^0 \frac{\partial \tilde{v}^0_2}{\partial \tilde{x}_3} \qquad \tilde{D}^0_{23} = \frac{1}{2}\frac{\partial \tilde{v}^0_2}{\partial \tilde{x}_3}$$

$$\tilde{S}^0_{12} = \tilde{\eta}^0 \left(\frac{\partial \tilde{v}^0_1}{\partial \tilde{x}_2} + \frac{\partial \tilde{v}^0_2}{\partial \tilde{x}_1}\right) \qquad \tilde{D}^0_{12} = \frac{1}{2}\left(\frac{\partial \tilde{v}^0_1}{\partial \tilde{x}_2} + \frac{\partial \tilde{v}^0_2}{\partial \tilde{x}_1}\right)$$

$$\tilde{S}^0_{11} = 2\tilde{\eta}^0 \frac{\partial \tilde{v}^0_1}{\partial \tilde{x}_1} \qquad \tilde{D}^0_{11} = \frac{\partial \tilde{v}^0_1}{\partial \tilde{x}_1}$$

$$\tilde{S}^0_{22} = 2\tilde{\eta}^0 \frac{\partial \tilde{v}^0_2}{\partial \tilde{x}_2} \qquad \tilde{D}^0_{22} = \frac{\partial \tilde{v}^0_2}{\partial \tilde{x}_2}$$

$$\tilde{\eta}^0 = (2\tilde{\mathcal{A}}(\tilde{\theta})\mathcal{K})^{-\frac{1}{n}} \left(\left(\frac{\partial \tilde{v}^0_1}{\partial \tilde{x}_3}\right)^2 + \left(\frac{\partial \tilde{v}^0_2}{\partial \tilde{x}_3}\right)^2\right)^{\frac{1-n}{2n}}$$

Solvability of SIA

- The solution is found semi-analytically and the problem obtained is semi-local, we only need to integrate over vertical lines and differentiate.
- Natural numerical implementation by finite differences: integration = weighted summation, derivatives = differences

Another meaning of Model Reduction

Theory and Numerics of Model Reduction

Most numerical simulations are based on complex mathematical models, often described by partial differential equations (PDEs). A typical use of such simulations is the measurement and control of output quantities such as heat, noise, and stress at critical points of the domain with respect to a selected set of input parameters. The fundamental idea of model reduction is that this input-output behaviour can often be well approximated by a much simpler model than needed for describing the entire state of the simulation. In this lecture, we consider automatic model reduction techniques that are primarily based on numerical control theory. In contrast to classical approaches, such techniques require little or no understanding of the underlying model.

Once model reduction has been performed, the original model can be replaced by the resulting simpler model, leading to reduced simulation times and greatly facilitating the further analysis and design of a control system. For instance, often only a low-order model allows for the use of more sophisticated robust and optimal control techniques. With the advances of modern control theory, model reduction has become an important and rapidely changing field with a large diversity of application areas, including structural and fluid dynamics, biosystems, circuit simulation, and micro-electro-mechanical systems.

Lecturers (Martin Gutknecht, Daniel Kressner)

Part #11

Mixtures

PDE in theory of Mixtures

$$
\begin{aligned}
\partial_t \varrho^s + \operatorname{div}(\varrho^s \mathbf{v}^s) &= m^s \\
\partial_t \varrho^f + \operatorname{div}(\varrho^f \mathbf{v}^f) &= m^f \\
\partial_t(\varrho^s \mathbf{v}^s) + \operatorname{div}(\varrho^s \mathbf{v}^s \otimes \mathbf{v}^s) &= \operatorname{div} \mathbf{T}^s + \varrho^s \boldsymbol{f}^s + \mathbf{m}^s \\
\partial_t(\varrho^f \mathbf{v}^f) + \operatorname{div}(\varrho^f \mathbf{v}^f \otimes \mathbf{v}^f) &= \operatorname{div} \mathbf{T}^f + \varrho^f \boldsymbol{f}^f + \mathbf{m}^f
\end{aligned}
$$

PDE in theory of Mixtures

$$
\begin{aligned}
\partial_t \varrho^s + \operatorname{div}(\varrho^s \mathbf{v}^s) &= m^s \\
\partial_t \varrho^f + \operatorname{div}(\varrho^f \mathbf{v}^f) &= m^f \\
\partial_t(\varrho^s \mathbf{v}^s) + \operatorname{div}(\varrho^s \mathbf{v}^s \otimes \mathbf{v}^s) &= \operatorname{div} \mathbf{T}^s + \varrho^s \boldsymbol{f}^s + \mathbf{m}^s \\
\partial_t(\varrho^f \mathbf{v}^f) + \operatorname{div}(\varrho^f \mathbf{v}^f \otimes \mathbf{v}^f) &= \operatorname{div} \mathbf{T}^f + \varrho^f \boldsymbol{f}^f + \mathbf{m}^f
\end{aligned}
$$

Comments

- total mass balance: $m^f = -m^s$
- Newton's third law: $\mathbf{m}^s + \mathbf{m}^f + m^f \mathbf{v}^f + m^s \mathbf{v}^s = 0$
- In particular: if $m^s = m^f = 0$ then $\mathbf{m}^s = -\mathbf{m}^f$.
- m^s, $\mathbf{m}^s$, $\mathbf{T}^s$, $\mathbf{T}^f$ are constitutive quantities
- one of many mechanisms (drag): $\mathbf{m}^s = \alpha(\cdot)(\mathbf{v}^f - \mathbf{v}^s)$
- α can depend on $|\mathbf{v}^f - \mathbf{v}^s|$, p, etc.

PDE in theory of Mixtures

Theory of mixtures

- a source of new classes of PDE
- significant drawbacks:
  - many constitutive quantities
  - unclear structure of the constitutive equations
  - specification of boundary conditions
  - thermal effects
- Recent approaches give hope to fix these issues

Questions as conclusion

- Scale and discretize or discretize and scale?

$\boxed{\mathcal{P}}$ $\boxed{\mathcal{P}_{\text{simple}}}$

$\boxed{\mathcal{P}^h}$ $\boxed{\mathcal{P}^h_{\text{simple}}}$

- Is extension a Model Reduction?

$$\mathbf{G}(\mathbf{T}, \mathbf{D}(\boldsymbol{v})) = \mathbf{0} \qquad \text{vs.} \qquad \mathbf{G}(\mathbf{T}, \mathbf{B}) = \mathbf{0} \text{ and } \mathbf{D}(\boldsymbol{v}) = \mathbf{B}$$

- Model reduction for the mixtures?