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RATE OF CONVERGENCE TO EQUILIBRIUM AND ŁOJASIEWICZ-TYPE ESTIMATES

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Abstract. A well known result states that the Łojasiewicz gradient inequality implies some estimates of the rate of convergence to equilibrium for solutions of gradient systems. We generalize this result to gradient-like systems satisfying certain angle condition and Kurdyka-Łojasiewicz inequality and to even more general situation. We apply the results to a broad class of second order equations with damping.

1. Introduction

In this paper, we study rate of convergence to equilibrium of solutions to gradient-like ordinary differential equations based on some generalizations of the Łojasiewicz gradient inequality.

General assumptions. *Throughout this paper, we assume that* (M, g) *is a smooth Riemannian manifold,* $\|\cdot\|$ *is the norm on the tangent bundle* TM *induced by the Riemannian metric* g *and* $d(\cdot,\cdot)$ *is the distance on* M *induced by* g*. We assume that* $F : M \to TM$ *is a continuous vector field. We consider an ordinary differential equation*

$$\dot{u} + F(u) = 0, \tag{1}$$

its bounded solution $u \in W^{1,1}_{loc}([0,+\infty), M)$*, and a point* φ *in the omega-limit set of* u*,*

$$\omega(u) = \{\varphi \in M : \exists\, t_n \nearrow +\infty,\ u(t_n) \to \varphi\}.$$

Moreover, we assume that a continuously differentiable function $\mathcal{E} : M \to \mathbb{R}$ *is a a strict Lyapunov function to* (1)*, i.e.,*

$$\langle \nabla\mathcal{E}(u), F(u)\rangle > 0 \quad \textit{whenever } u \in M,\ F(u) \neq 0. \tag{2}$$

There are many results saying that under additional conditions on $\mathcal{E}$ we have $u(t) \to \varphi$ as $t \to +\infty$. The main goal of this paper is to find the rate of

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convergence, i.e. a function R as small as possible such that

$$d(u(t),\varphi) \le R(t) \quad \text{for all } t \ge 0.$$

Let us start with a special case when $F = \nabla\mathcal{E}$, then (1) becomes a *gradient system*

$$\dot{u} + \nabla\mathcal{E}(u) = 0.$$

The classical result of Łojasiewicz (see [9]) states that $u(t)$ converges to φ if the *Łojasiewicz gradient inequality*

$$\text{(LI)} \qquad |\mathcal{E}(u) - \mathcal{E}(\varphi)|^{1-\theta} \le C\|\nabla\mathcal{E}(u)\| \quad \text{for all } u \in N(\varphi)$$

holds with some $C > 0$ and $\theta \in (0, \frac{1}{2}]$ (by $N(\varphi)$ we denote some neigborhood of φ). Moreover, there exist $a, K > 0$ such that for all $t \ge 0$ it holds that

$$d(u(t),\varphi) \le Ke^{-at} \quad \text{if } \theta = \frac{1}{2}$$

and

$$d(u(t),\varphi) \le K(1+t)^{\frac{-\theta}{1-2\theta}} \quad \text{if } \theta < \frac{1}{2}.$$

The result by Łojasiewicz was later generalized in several ways. First, the inequality (LI) was generalized to the so called *Kurdyka-Łojasiewicz inequality* (see [7])

$$\text{(KLI)} \qquad \Theta(|\mathcal{E}(u) - \mathcal{E}(\varphi)|) \le \|\nabla\mathcal{E}(u)\| \quad \text{for all } u \in N(\varphi)$$

with a function Θ positive on $(0,+\infty)$ and satisfying $\Theta(0) = 0$ and $\frac{1}{\Theta} \in L^1_{loc}([0,1))$ (if we take $\Theta(s) = s^{1-\theta}$, then (KLI) becomes (LI)). In this case, we have again convergence $u(t) \to \varphi$ as $t \to +\infty$ with convergence rate given by

$$\text{(3)} \qquad d(u(t),\varphi) \le K\Phi(\psi^{-1}(t-t_0)) \quad \text{for some } K, t_0 > 0 \text{ and all } t > t_0,$$

with

$$\Phi(t) = \int_0^t \frac{1}{\Theta(s)}ds \quad \text{and} \quad \psi(t) = -\int \frac{1}{\Theta^2(t)}dt.$$

The rate of convergence was proved by Chill and Fiorenza in [5].

Second, it was generalized to gradient-like system, i.e. ordinary differential equations wiht a strict Lyapunov function $\mathcal{E}$ (not neccessarily satisfying $F = \nabla\mathcal{E}$). In this case, it is not sufficient to assume that $\mathcal{E}$ satisfies the Łojasiewicz or Kurdyka-Łojasiewicz inequality to obtain convergence $u(t) \to \varphi$. We need to add so called *angle condition* (see [1, Theorem 2.2], [6, Proposition 5(a), Theorem 4], [8, Definition 1.1])

$$\text{(AC)} \quad \langle\nabla\mathcal{E}(u), F(u)\rangle \ge \alpha\|\nabla\mathcal{E}(u)\|\,\|F(u)\| \quad \text{for some } \alpha > 0 \text{ and all } u \in N(\varphi).$$

If we moreover assume the *comparability condition*

$$\text{(C)} \quad c_1\|F(u)\| \le \|\nabla\mathcal{E}(u)\| \le c_2\|F(u)\| \quad \text{for some } c_1, c_2 > 0 \text{ and all } u \in N(\varphi),$$

then we can obtain the same rate of convergence as for gradient systems (see Corollary 2 below). It is not surprising, since the orbits depend on the direction of F only, the size of F determines how quickly the solution moves along the orbit. Let us mention that (AC) and (C) together are equivalent to the *angle and comparability condition*

$$\text{(AC+C)} \qquad \langle \nabla \mathcal{E}(u), F(u) \rangle \geq c(\|\nabla \mathcal{E}(u)\|^2 + \|F(u)\|^2) \quad \text{for all } u \in N(\varphi).$$

Further, in [2] we introduced a generalized Łojasiewicz condition

$$\text{(GLI)} \qquad \Theta(|\mathcal{E}(u) - \mathcal{E}(\varphi)|) \leq \frac{1}{\|F(u)\|} \langle \nabla \mathcal{E}(u), F(u) \rangle \quad \text{for all } u \in N(\varphi),$$

that generalizes (AC+C) and (KLI) (i.e., (AC+C) and (KLI) imply (GLI)) and is sufficient to obtain convergence $u(t) \to \varphi$. We give an estimate of the convergence rate for this case in Theorem 1, which is the main result of this paper. We apply this result to a second order equation with a general (weak) damping function (Theorems 5 and 6) and generalize the result by Chergui [4, Theorem 1.3].

Finally, we present better estimates of the convergence rate in some cases. In fact, all the estimates mentioned above use the inequlality

$$d(u(t), \varphi) \leq \int_t^{+\infty} \|\dot{u}(s)\| ds,$$

i.e., estimate the distance to the equilibrium by the length of the remaining trajectory. This estimate is far from being optimal if the solution u has a shape of a spiral, which is exactly the case if we consider a second order equation with a weak damping (smaller than linear). We show that it can be better to estimate $d(u(t), \varphi)$ directly by a function of $\mathcal{E}(u(t))$.

Section 2 is devoted to the abstract results, while in Section 3 we apply these results to a damped second order equation and Section 4 contains some technical Lemmas.

2. Main results

We formulate the main result of this paper (keeping in mind the general assumptions introduced in the previous section).

Theorem 1. *Let $\mathcal{E}$ and F satisfy* (GLI) *with a function $\Theta : [0, 1) \to \mathbb{R}_+$ such that $\frac{1}{\Theta} \in L^1_{loc}([0, 1))$ and $\Theta(s) > 0$ for $s > 0$. Then u has finite length in (M, g) and, in particular, $\lim_{t \to +\infty} u(t) = \varphi$ in (M, g). Moreover, if $\alpha : (0, 1) \to (0, +\infty)$ is nondecreasing and satisfies*

$$(4) \qquad \alpha(\mathcal{E}(u(t))) \leq \|F(u(t))\| \quad \textit{for all } t \textit{ large enough},$$

then there exist $t_0 > 0$ *such that*

$$\|u(t) - \varphi\| \le \Phi(\psi^{-1}(t - t_0)) \quad \textit{for all } t > t_0,$$

where

$$\Phi(t) := \int_0^t \frac{1}{\Theta(s)} ds \quad \textit{and} \quad \psi(t) := \int_t^{1/2} \frac{1}{\Theta(s)\alpha(s)} ds.$$

Let us remark that an example of such function α (and, in fact, the best one) is

$$\alpha(s) := \min\{\|F(u(t))\| : \ \mathcal{E}(u(t)) - \mathcal{E}(\varphi) \ge s\}.$$

This function is well defined. Since $\{u(t) : \ t \in \mathbb{R}_+\} \cup \{\varphi\}$ is compact and so the level set $\{u(t) : \ \mathcal{E}(u(t)) \ge s + \mathcal{E}(\varphi)\}$ is also compact. Therefore $\|F(u(t))\|$ attains its minimum on this set. Positivity and monotonicity of α and (4) follow immediately.

Proof. We have proved convergence in [2], Theorem 5. It remains to show the moreover part. Without loss of generality we may assume $\mathcal{E}(\varphi) = 0$. Since α is nondecreasing, the Lebesgue integral in the definition of ψ exists and ψ is decreasing, therefore invertible. We show below that $\lim_{s\to 0+} \psi(s) = +\infty$, which implies that ψ^{-1} is defined on a neighborhood of $+\infty$.

Let $\varepsilon > 0$ be small enough. For all t large enough we have $\mathcal{E}(u(t)) \in (0, \varepsilon)$ and for almost all such t's it holds that (by definition of ψ, (GLI) and (4))

$$\begin{aligned}
\frac{d}{dt}\psi(\mathcal{E}(u(t))) &= -\frac{1}{\Theta(\mathcal{E}(u(t)))\alpha(\mathcal{E}(u(t)))}\langle \nabla\mathcal{E}(u(t)), \dot{u}(t)\rangle \\
&= \frac{1}{\Theta(\mathcal{E}(u(t)))\alpha(\mathcal{E}(u(t)))}\langle \nabla\mathcal{E}(u(t)), F(u(t))\rangle \\
&\ge \frac{1}{\Theta(\mathcal{E}(u(t)))\alpha(\mathcal{E}(u(t)))}\Theta(\mathcal{E}(u(t)))\|F(u(t))\| \\
&\ge \frac{1}{\Theta(\mathcal{E}(u(t)))\alpha(\mathcal{E}(u(t)))}\Theta(\mathcal{E}(u(t)))\alpha(\mathcal{E}(u(t))) \\
&= 1.
\end{aligned}$$

Fix t_0 large enough (such that $\psi(\mathcal{E}(u(t_0))) > 0$) and integrate this inequality from t_0 to $t > t_0$

$$\psi(\mathcal{E}(u(t))) \ge (t - t_0) + \psi(\mathcal{E}(u(t_0))) \ge t - t_0.$$

From this inequality it follows that $\lim_{s\to 0+} \psi(s) = +\infty$. Since ψ is decreasing, we have $\mathcal{E}(u(t)) \le \psi^{-1}(t - t_0)$. Further, by (GLI), $\dot{u} = -F(u)$ and by definition of Φ, we have

$$
\begin{aligned}
d(u(t),\varphi) &\leq \int_t^{+\infty} \|\dot{u}(s)\| \\
&\leq \int_t^{+\infty} -\frac{1}{\Theta(\mathcal{E}(u(s)))}\langle \nabla\mathcal{E}(u(s)), \dot{u}(s)\rangle \\
&= -\lim_{s\to+\infty} \Phi(\mathcal{E}(u(s))) + \Phi(\mathcal{E}(u(t))) \\
&= \Phi(\mathcal{E}(u(t))) \\
&\leq \Phi(\psi^{-1}(t-t_0)).
\end{aligned}
$$

□

Corollary 2. *Let $\mathcal{E}$ and F satisfy* (AC), *eqrefC and* (KLI) *with a function $\Theta : [0,1) \to \mathbb{R}_+$ such that $\frac{1}{\Theta} \in L^1_{loc}([0,1))$ and $\Theta(s) > 0$ for $s > 0$. Then u has finite length in (M, g) and, in particular, $\lim_{t\to+\infty} u(t) = \varphi$ in (M, g). Moreover, there exists $t_0 > 0$ such that*

$$d(u(t),\varphi) \leq \Phi_1(\psi_1^{-1}(t-t_0)) \quad \textit{for all } t > t_0,$$

where

$$\Phi_1(t) := c_1 \int_0^t \frac{1}{\Theta(s)} ds \quad \textit{and} \quad \psi_1(t) := c_2 \int_t^{1/2} \frac{1}{\Theta^2(s)} ds$$

for appropriate positive constants c_1, c_2.

Proof. Conditions (AC) and (KLI) imply

$$\frac{1}{\|F(u)\|}\langle \nabla\mathcal{E}(u), F(u)\rangle \geq \alpha\|\nabla\mathcal{E}(u)\| \geq \alpha\Theta(|\mathcal{E}(v) - \mathcal{E}(\varphi)|),$$

so (GLI) holds with Θ replaced by $\tilde{\Theta} := \alpha\Theta$. Since $\Theta(|\mathcal{E}(u) - \mathcal{E}(\varphi)|) \leq \|\nabla\mathcal{E}(u)\| \leq c_2\|F(u)\|$ by (KLI) and (C), we can take $\alpha(s) = \frac{1}{c_2}\Theta(s)$ and apply Theorem 1. □

The above results estimate the distance from the equilibrium by the length of the remaining trajectory, i.e.

$$\|u(t) - \varphi\| = \int_t^{\infty} \dot{u}(s) ds \leq \int_t^{\infty} \|\dot{u}(s)\| ds. \tag{5}$$

This estimate seems to be quite bad if the trajectory looks like a spiral; then the remaining trajectory can be much longer than the distance to the equilibrium.

Let us assume that (M, g) be an open subset of $\mathbb{R}^n$ with the Euclidean metric. We denote the Euclidean norm by $|\cdot|$. It is easy to show that if

$$\langle F(u), u\rangle \geq \alpha|F(u)|\,|u| \tag{6}$$

for some fixed $\alpha > 0$, then the estimate (5) is optimal. In fact,

$$-\frac{d}{dt}|u(t)| = -\langle \frac{u(t)}{|u(t)|}, \dot{u}(t)\rangle = \langle \frac{u(t)}{|u(t)|}, F(u(t))\rangle \geq \alpha|F(u(t))| = \alpha|\dot{u}(t)|$$

and after integration from T to $+\infty$ we obtain

$$|u(T)| \geq \alpha \int_T^{+\infty} |\dot{u}(t)| dt.$$

The estimate (6) means that $\mathcal{E}(u) = |u|^2$ is a Lyapunov function and $\nabla\mathcal{E}$ and F satisfy (AC). So, the estimate (5) is optimal even for some spirals (logarithmic spiral). However, in many cases the estimate (5) is not optimal and the following corollary yields a better result.

Corollary 3. *Let the assumptions of Theorem 1 hold and let $\gamma : (0,1) \to (0,+\infty)$ be a nondecreasing function satisfying $\gamma(\mathcal{E}(u) - \mathcal{E}(\varphi)) \geq |u - \varphi|$ for all u in a neigborhood of φ. Then there exist $t_0 > 0$ such that*

$$|u(t) - \varphi| \leq \gamma(\psi^{-1}(t - t_0)) \quad \textit{for all } t > t_0.$$

Proof. As in the proof of Theorem 1 (assuming $\mathcal{E}(\varphi) = 0$) we obtain $\mathcal{E}(u(t)) \leq \psi^{-1}(t - t_0)$. Further,

$$|u(t) - \varphi| \leq \gamma(\mathcal{E}(u(t))) \leq \gamma(\psi^{-1}(t - t_0))$$

since γ is nondecreasing. □

Let us mention that an example (and, in fact, the best one) of a function γ from Corollary 3 is

$$\gamma(s) := \sup\{|u(t) - \varphi| : \mathcal{E}(u(t)) - \mathcal{E}(\varphi) \leq s\}.$$

Let us recall the Example 7 from [2]:

Example 4. Let $M \subseteq \mathbb{R}^2$ be the open unit disk, equipped with the Euclidean metric. Let $\alpha \geq 0$, and let $F(u) = F(u_1, u_2) = (|u|^\alpha u_1 - u_2, u_1 + |u|^\alpha u_2)$ and $\mathcal{E}(u) = \frac{1}{2}(u_1^2 + u_2^2)$. Then

$$\langle \nabla\mathcal{E}(u), F(u)\rangle = |u|^{2+\alpha}, \ |F(u)| = |u| \cdot \sqrt{1 + |u|^{2\alpha}} \quad \text{and} \quad |\nabla\mathcal{E}(u)| = |u|.$$

The function $\mathcal{E}$ satisfies the Łojasiewicz inequality (LI) near the origin for $\theta = \frac{1}{2}$. But the angle condition (AC) does not hold on any neighbourhood of the critical point $(0,0)$, so Corollary 2 does not apply (unless $\alpha = 0$). On the other side, we have

$$\frac{1}{|F(u)|}\langle \mathcal{E}'(u), F(u)\rangle = \frac{|u|^{1+\alpha}}{\sqrt{1 + |u|^{2\alpha}}} \geq \frac{1}{\sqrt{2}}|u|^{2(1-\theta)} \geq \frac{1}{\sqrt{2}}\mathcal{E}(u)^{1-\theta}$$

provided $0 < \theta \leq \frac{1-\alpha}{2}$. Hence, if $0 \leq \alpha < 1$, then $\mathcal{E}$ satisfies (GLI) with $\Theta(s) = \frac{1}{\sqrt{2}}s^{1-\theta}$, $\theta = \frac{1-\alpha}{2}$.

We can apply Theorem 1 with $\alpha(s) = 2\sqrt{s}$ since for small s we have

$$\inf\{|u(t)|\sqrt{1+|u(t)|^{2\alpha}} : \frac{1}{2}|u(t)|^2 \geq s\} = \sqrt{2s}\sqrt{1+(2s)^\alpha} \leq 2\sqrt{s}.$$

Hence, $\psi'(s) = cs^{\theta-3/2}$ and $\psi(s) = cs^{\theta-1/2}$ and $\psi^{-1}(s) = cs^{\frac{1}{\theta-1/2}}$ and $\Phi = cs^\theta$ (with various constants c). Then,

$$|u(t)| \leq \Phi(\psi^{-1}(t-t_0)) \leq C(t-t_0)^{\frac{\theta}{\theta-1/2}} = C(t-t_0)^{\frac{1}{\alpha}-1}.$$

If (AC+C) condition were satisfied, the decay of u would be exponential due to the Łojasiewicz exponent equal to $\frac{1}{2}$. Since the (AC+C) condition is not satisfied, the decay is only polynomial.

However, the above estimate is not optimal and we can get a better one from Corollary 2. In fact, taking

$$\gamma(s) = \sup\left\{\sqrt{x^2+y^2} : \frac{1}{2}(x^2+y^2) \leq s\right\} = \sqrt{2s}$$

we obtain

$$|u(t)| \leq \sqrt{2C(t-t_0)^{\frac{1}{\theta-1/2}}} = \tilde{C}(t-t_0)^{-\frac{1}{\alpha}}.$$

This is a better result since $-\frac{1}{\alpha} < \frac{1}{\alpha} - 1$. Moreover, transformation to polar coordinates show that this result is optimal. In fact, we obtain $r' = -r^{\alpha+1}$, which yields $r(t) = c(t-t_0)^{-1/\alpha}$.

3. Second order equation with damping

In this section we apply the previous results to a damped second order equation

$$\ddot{u} + G(u,\dot{u}) + \nabla E(u) = 0. \tag{7}$$

We studied such equations in [2] (see [2, Theorem 4]), where we proved convergence to equilibrium under appropriate assumptions on E and G. We have further generalized the result in [3, Theorem 6.1]. A special case $G(u,v) = |v|^\alpha v$ and E satisfying the Łojasiewicz inequality with an exponent θ was considered by Chergui, he proved in [4, Theorem 1.3] the convergence rate estimates

$$|u(t)-\varphi| + |\dot{u}(t)| \leq C(1+t)^{-\frac{\theta-\alpha(1-\theta)}{1-2\theta+\alpha(1-\theta)}}. \tag{8}$$

We generalize this estimate to more general dampings and more general E and we obtain a better estimate for some special functions E.

Of course, equation (7) can be reduced to a first order system

$$\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix} + F(u,v) = 0 \quad \text{with} \quad F(u,v) = \begin{pmatrix}-v\\ G(u,v)+\nabla E(u)\end{pmatrix}.$$

Now, we formulate our assumptions on E and G. We start with a first set of assumptions and apply the approach from [2]. Then we introduce more general assumptions and use the result from [3].

(E) Let $E \in C^2(\mathbb{R}^n, \mathbb{R})$ satisfy (KLI) with a function $\Theta : [0, 1) \to [0, +\infty)$ which is nondecreasing, sublinear ($\Theta(s+t) \le \Theta(s)+\Theta(t)$), and it holds that $\frac{1}{\Theta} \in L^1_{loc}([0, 1))$ and $0 < \Theta(s) \le c\sqrt{s}$ for all $s \in (0, 1)$ and some $c > 0$.

(G) The function $G : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and there exists a function $h : [0, +\infty) \to [0, +\infty)$, which is concave and nondecreasing and it holds that

- **(g1)** there exists $C_2 > 0$ such that $|G(w, z)| \le C_2|z|h(|z|)$ for all $z, w \in \mathbb{R}^n$,
- **(g2)** there exists $C_3 > 0$ such that $|G(w, z)| \ge C_3|z|h(|z|)$ for all $z, w \in \mathbb{R}^n$,
- **(g3)** there exists $C_4 > 0$ such that $\langle G(w, z), z\rangle \ge C_5|G(w, z)||z|$ for all $w, z \in \mathbb{R}^n$.
- **(g4)** there exists $C_5 > 0$ such that $|\nabla G(w, z)| \le C_5 h(|z|)$ for all $w, z \in \mathbb{R}^n$.
- **(g5)** the function $s \mapsto \frac{1}{\Theta(s)h(\Theta(s))}$ belongs to $L^1((0, \tau))$,

Let us comment on these assumptions. First, function $\Theta(s) = s^{1-\theta}$, $\theta \in (0, \frac{1}{2}]$ satisfies the assumptions in (E), in this case (KLI) reduces to (LI). Concerning assumptions on G, let us first consider $G(w, z) = g(|z|)z$. Then conditions (g1),(g2) say that the damping function g is between two multiples of a concave function h (g can even oscilate between them but not much, due to (g4)). Condition (g5) is a connection between E and G. If $\Theta(s) = s^{1-\theta}$ and $g(z) = |z|^\alpha$, then (g5) reduces to $\alpha < \frac{\theta}{1-\theta}$, which is the condition from [4]. In this case, the following theorem gives the same rate of convergence as [4, Theorem 1.3].

If $G(w, z) = g(|z|)z$, then the damping force acts in the direction opposite to velocity. For general G, condition (g3) is an angle condition which says that the angle between the damping force and minus velocity is less than $\frac{\pi}{2}$ uniformly.

Theorem 5. *Let E and G satisfy (E) and (G). Let $u \in W^{1,\infty}((0, +\infty), \mathbb{R}^n) \cap W^{2,1}_{loc}([0, +\infty), \mathbb{R}^n)$ be a solution to* (7) *and $\varphi \in \omega(u)$. Then there exists $t_0 > 0$ such that*

$$|\dot u(t)| + |u(t) - \varphi| + \int_t^{+\infty} |\dot u(s)|ds \le \Phi(\psi^{-1}(t - t_0)), \tag{9}$$

holds for all $t > t_0$, some C_1, $C_2 > 0$ and

$$\Phi(t) = C_1 \int_0^t \frac{1}{\Theta(s)h(\Theta(s))}ds \quad \text{and} \quad \psi(t) = C_2 \int_t^{\frac{1}{2}} \frac{1}{\Theta^2(s)h(\Theta(s))}ds. \tag{10}$$

Proof. By [2, Theorem 4], the left-hand side of (9) tends to zero as $t \to +\infty$. Let us assume without loss of generality that $\varphi = 0$ and $E(\varphi) = 0$ and denote

$v(t) := \dot{u}(t)$. In the proof of [2, Theorem 4] we have shown that (for $\varepsilon > 0$ small enough)

$$\mathcal{E}(u,v) = \frac{1}{2}|v|^2 + E(u) + \varepsilon\langle G(u,\nabla E(u)), v\rangle$$

is a strict Lyapunov function and satisfies ([2, last inequality of the proof])

$$\tilde{\Theta}(\mathcal{E}(u,v)) \leq \frac{1}{|F(u,v)|}\langle \nabla\mathcal{E}(u,v), F(u,v)\rangle$$

with $\tilde{\Theta}(s) := \Theta(s)h(\Theta(s))$. Further, we have shown ([2, p.71, first inequality])

$$\Theta(\mathcal{E}(u,v)) \leq C(\|v\| + \|\nabla E(u)\|). \tag{11}$$

From the definition of F we have immediately

$$|F(u,v)| = (|v|^2 + |G(u,v) + \nabla E(u)|^2)^{1/2} \geq \frac{1}{2}(|v| + |G(u,v) + \nabla E(u)|).$$

Now, we show that

$$|F(u,v)| \geq c(|v| + |\nabla E(u)|). \tag{12}$$

By the assumptions on G we have $|G(u,v)| \leq C|v|h(|v|)$. Now, we distingiush two cases:

1. If (u,v) is such that $C|v|h(|v|) \leq (1-\alpha)|\nabla E(u)|$ for some $\alpha \in (0,1)$, then

$$|F(u,v)| \geq \frac{1}{2}(|v| + |\nabla E(u)| - (1-\alpha)|\nabla E(u)|) \geq \frac{\alpha}{2}(|v| + |\nabla E(u)|).$$

2. If (u,v) is such that $C|v|h(|v|) \geq (1-\alpha)|\nabla E(u)|$, then for $|v|$ small enough (we are interested for small $|v|$ only) we have $h(|v|) \leq c$. Then

$$|v| + |\nabla E(u)| \leq |v| + \frac{cC}{1-\alpha}|v| = \tilde{C}\cdot\frac{1}{2}|v| \leq \tilde{C}|F(u,v)|$$

We now have (12) and using (11) we obtain

$$|F(u,v)| \geq \frac{c}{C}\Theta(\mathcal{E}(u,v)).$$

It remains to apply Theorem 1 with $\alpha(s) = \frac{c}{C}\Theta(s)$ and the proof is complete (with $C_1 = 1$ and $C_2 = \frac{C}{c}$). □

Now, let us further relax the assumptions on the damping function G. In particular, if $G(w,z) = g(|z|)z$, then the following assumption say that g is still bigger than a concave function h on $(0,\tau)$, but not necessarily less than a multiple of h. Since we do not have any condition on ∇G, function g can oscilate arbitrarily between h and a constant function on $(0,\tau)$.

(GG) The function $G : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and there exists $\tau > 0$ such that
- **(gg1)** there exists $C_2 > 0$ such that $|G(w,z)| \leq C_2|z|$ for all $|z| < \tau$, $w \in \mathbb{R}^n$,
- **(gg2)** there exists $C_3 > 0$ such that $C_3|z| \leq |G(w,z)|$ for all $z \geq \tau$, $w \in \mathbb{R}^n$,

(gg3) there exists $C_5 > 0$ such that $\langle G(w,z), z\rangle \geq C_5|G(w,z)||z|$ for all $w, z \in \mathbb{R}^n$.

(HH) For τ from condition (G) there exists a function $h : [0,+\infty) \to [0,+\infty)$, which is concave and nondecreasing on $[0,\tau]$ and satisfies

(hh1) $|G(w,z)| \geq h(|z|)|z|$ for all $|z| < \tau$, $w \in \mathbb{R}^n$,

(hh2) the function $s \mapsto \frac{1}{\Theta(s)h(\Theta(s))}$ belongs to $L^1((0,\tau))$,

(hh3) the function $\psi : s \mapsto sh(\sqrt{s})$ is convex on $[0,\tau^2]$.

Theorem 6. *Theorem 5 remains valid if E and G satisfy weaker assumptions (E), (GG), (HH).*

Proof. By [3, Theorem 6.1], the left-hand side of (9) tends to zero as $t \to +\infty$. Let us assume without loss of generality that $\varphi = 0$ and $E(\varphi) = 0$. Denote $v(t) := \dot{u}(t)$. In the proof of [3, Theorem 6.1] we have defined for $\varepsilon > 0$ small enough

$$H(u,v) := \frac{1}{2}|v|^2 + E(u) + \varepsilon h(|v|)\langle \nabla E(u), v\rangle$$

In fact, $\tilde{h}$ used in [3] is equal to h on a neighborhood of zero and E in [3] has the opposite sign. In the following we write u, v instead of $u(t)$, $v(t)$. In [3] we have shown that (see [3, inequality (10)])

$$-\frac{d}{dt}H(u,v) \geq ch(|v|)(|v| + |\nabla E(u)|)^2$$

and (see [3, inequality (12)])

$$\Theta(H(u,v))h(\Theta(H(u,v))) \leq C(|v| + |\nabla E(u)|)h(|v| + |\nabla E(u)|),$$

and (see [3, the inequality below (13)])

$$-\frac{1}{\Theta(H(u,v))h(\Theta(H(u,v)))} \cdot \frac{d}{dt}H(u,v) \geq c|v|.$$

Since

$$-\frac{d}{dt}H(u,v) = -\langle \nabla H(u,v), (\dot{u},\dot{v})\rangle = \langle \nabla H(u,v), F(u,v)\rangle,$$

we have

$$\frac{\langle \nabla H(u,v), F(u,v)\rangle}{c \cdot \Theta(H(u,v)) \cdot h(\Theta(H(u,v)))} \geq |v|, \tag{13}$$

which is almost (GLI) with $\mathcal{E} = H$, $\tilde{\Theta}(s) = c\Theta(s)h(\Theta(s))$; the only difference is that there is $|v|$ instead of $|F(u,v)|$ on the right-hand side of (13). It holds that

$$\Theta(H(u,v)) \leq d(|v| + |\nabla E(u)|) \leq \tilde{d}|F(u,v)|, \tag{14}$$

the first inequality is proven in [3, the inequality before (12)], the second inequality follows by the same arguments as in the proof of Theorem 5.

Therefore, we can continue similarly to the proof of Theorem 1

$$\begin{aligned}\frac{d}{dt}\psi(H(u,v)) &= -\psi'(H(u,v))\langle\nabla H(u,v), F(u,v)\rangle \\ &\geq \frac{cC_2}{\Theta(H(u,v))}|v| \\ &\geq \frac{cC_2}{d}\cdot\frac{|v|}{|v|+|\nabla E(u)|},\end{aligned}$$

where the first inequality follows from (13) and the definition of ψ and the second inequality follows from (14). In Lemma 8 we show that for an appropriate L, $t_0 > 0$ we have

$$\int_{t_0}^{t}\frac{|v|}{|v|+|\nabla E(u)|} \geq L(t-t_0) \quad \text{for all } t > t_0.$$

Then we can complete the proof similarly to Theorem 1. We have (taking t_0 large enough and $C_2 := d(cL)^{-1}$)

$$\psi(H(u(t),v(t))) \geq cC_2Ld^{-1}(t-t_0) + \psi(H(u(t_0),v(t_0))) \geq t - t_0,$$

so $H(u(t),v(t)) \leq \psi^{-1}(t-t_0)$. Finally, we complete the proof by the estimates

$$\begin{aligned}|u(t)|+|v(t)| &\leq \int_t^{+\infty}|\dot u(s)|+|\dot v(s)| \\ &\leq \sqrt{2}\int_t^{+\infty}|F(u(s),v(s))| \\ &\leq \sqrt{2}K\int_t^{+\infty}|v(s)| \\ &\leq \sqrt{2}K\int_t^{+\infty}\frac{\langle\nabla H(u(s),v(s)),F(u(s),v(s))\rangle}{c\Theta(H(u(s),v(s)))h(\Theta(H(u(s),v(s)))} \\ &= \Phi(H(u(t),v(t))) - \lim_{s\to+\infty}\Phi(H(u(s),v(s))) \\ &\leq \Phi(\psi^{-1}(t-t_0)).\end{aligned}$$

Here the first and second inequality are obvious, we applied Lemma 9 in the third inequality, in the forth inequality we used (13) and in the next equality we set $C_1 := \frac{K}{c}\sqrt{2}$ in the definition of Φ. □

Now, we improve the above estimates by applying Corollary 3. In the simplest case

$$\ddot u + |u|^{\alpha}u + u = 0,$$

where u is a scalar function, the solutions are spirals. So, it is reasonable to assume that the length of the remaining trajectory is much bigger than the distance from the equilibrium. So, we improve the estimates from Theorem

6 in case that the energy function E is nice. In particular, in addition to the Kurdyka–Łojasiewicz inequality (KLI) we assume that also the opposite is true. Moreover, we assume that $|u|$ can be estimated by an appropriate function of E.

(E1) Let E and Θ be the functions from (E) and φ be the point from Theorem 5. Then there exists $c > 0$ such that

$$\Theta(E(u) - E(\varphi)) \geq c|\nabla E(u)| \quad \text{for all } u \in N(\varphi).$$

Theorem 7. *Let the assumptions of Theorem 5 hold with (E), (G) replaced by (E), (E1), (GG), (HH). Moreover, let γ be a nondecreasing function satisfying*

$$\gamma(E(u) - E(\varphi)) \geq |u - \varphi| \quad \textit{for all } u \in N(\varphi). \tag{15}$$

Then

$$|\dot{u}(t)| \leq C\sqrt{\psi^{-1}(t - t_0)} \quad \textit{and} \quad |u(t) - \varphi| \leq C\gamma(\psi^{-1}(t - t_0)), \tag{16}$$

holds for all $t > t_0$ and some $C > 0$, ψ defined as in (10).

Before we prove this theorem, let us mention that if $p \geq 2$, then $E(u) = \sum_{i=1}^{n} |u_i|^p$ is a prototype of an energy satisfying the Łojasiewicz estimate with $\theta = \frac{1}{p}$ (i.e., $\Theta(s) = s^{\frac{p-1}{p}}$). Moreover, this function E satisfies (E1) and function $\gamma(t) = \frac{t}{\Theta(t)} = t^{\frac{1}{p}}$ satisfies (15). If we take a damping function $G(u, v) = |v|^{\alpha} v$ for $\alpha < \frac{1}{p-1}$ as in [4] or any larger admissible function, we obtain

$$|\dot{u}(t)| \leq C(1 + t)^{-\frac{\frac{1}{2}}{1-2\theta+\alpha(1-\theta)}} \quad \text{and} \quad |u(t)| \leq C(1 + t)^{-\frac{\theta}{1-2\theta+\alpha(1-\theta)}},$$

which improves the convergence rate from [4, Theorem 1.3]. In particular, if $\alpha = 0$ (linear damping), then this result is equal to the one in [4, Theorem 1.3] and the weaker is the damping, the bigger is the difference between the two results.

Proof. As in the proof of Theorem 6 we show $H(u(t), v(t)) \leq \psi^{-1}(t - t_0)$. Now, we show that (assuming WLOG $\varphi = 0$, $E(\varphi) = 0$)

$$H(u, v) \geq c(|v|^2 + E(u)). \tag{17}$$

By definition of H (see proof of Theorem 6) we have

$$H(u, v) \geq \frac{1}{2}|v|^2 + E(u) - \varepsilon h(|v|)|\nabla E(u)||v|.$$

Further,

$$|\nabla E(u)||v| \leq C\Theta(E(u))|v| \leq C\sqrt{E(u)}|v| \leq \frac{C}{2}(|v|^2 + E(u)).$$

Since h is bounded on a neighborhood of 0, by taking $\varepsilon > 0$ small enough we obtain (17).

Now, we have

$$|v(t)| \leq \frac{1}{\sqrt{c}}\sqrt{H(u(t),v(t))} \leq \frac{1}{\sqrt{c}}\sqrt{\psi^{-1}(t-t_0)},$$

which is the first estimate in (16). Further, by monotonicity of γ we have

$$|u| \leq \gamma(E(u)) \leq \gamma(c^{-1}H(u,v)) \leq \gamma(c^{-1}\psi^{-1}(t-t_0)),$$

which is the second estimate in (16), when we change the constant C_2 in the definition of ψ. □

4. Appendix

In this section we prove two technical Lemmas.

Lemma 8. *There exist* $L, t_0 > 0$ *such that*

$$\int_{t_0}^{t} \frac{|v(s)|}{|v(s)| + |\nabla E(u(s))|} ds \geq L(t-t_0) \quad \textit{for all } t > t_0.$$

Proof. We will show that if $p(t) := \frac{|v(t)|}{|\nabla E(u(t))|}$ is small on an interval I, then we can find an equally long interval immediately before I, where $p(t)$ is large. In particular, assume that for some $t_p > t_0$ it holds that $|v(t_p)| < \kappa|\nabla E(u(t_p))|$, where κ is a small constant, which will be specified later. Define

$$t_1 := \sup\{t < t_p : |v(t)| \geq \kappa|\nabla E(u(t))|\},$$
$$t_8 := \sup\{t < t_p : |v(t)| \geq 8\kappa|\nabla E(u(t))|\}.$$

We show that $t_p - t_1 < t_1 - t_8$. If we start in t_0 where $|v(t_0)| > 8\kappa|\nabla E(u(t_0))|$, then $t_8 > t_0$. Therefore, for any $t > t_0$ we have $|v(s)| \geq \kappa|\nabla E(u(s))|$ for $s \in M_t \subset (t_0, t)$ and measure of M_t is at least $\frac{1}{2}(t-t_0)$. Therefore

$$\begin{aligned}\int_{t_0}^{t} \frac{|v(s)|}{|v(s)| + |\nabla E(u(s))|} ds &\geq \int_{M_t} \frac{|v(s)|}{|v(s)| + |\nabla E(u(s))|} ds \\ &\geq \int_{M_t} \frac{\kappa|\nabla E(u(s))|}{\kappa|\nabla E(u(s))| + |\nabla E(u(s))|} ds \\ &\geq \frac{1}{2}(t-t_0)\frac{\kappa}{\kappa+1},\end{aligned}$$

what we wanted to prove (in the second inequality we used the fact that $x \mapsto \frac{x}{x+a}$ is increasing for $x \geq 0$ if $a > 0$).

So, it remains to show $t_p - t_1 \leq t_1 - t_8$. The idea is that in the points where $v(t)$ is almost zero and $\nabla E(u(t))$ is large in comparison to $v(t)$ it holds that

$$\dot{v}(t) = -\nabla E(u(t)) + O(|v|) \quad \text{and} \quad \frac{d}{dt}\nabla E(u(t)) = \nabla^2 E(u(t))v(t) = O(|v|)$$

(the first equality follows from the differential equation (7) and $|G(u,v)| \le c|v|$). It means that $\nabla E(u)$ changes very slowly and the change of v is relatively fast and (almost) constant concerning size and also direction.

Let t_p be such that $|v(t_p)| < \kappa|\nabla E(u(t_p))|$. Then

$$\langle \dot v(t), -\nabla E(u(t_p))\rangle = \langle G(u(t),v(t)), -\nabla E(u(t_p))\rangle + \langle -\nabla E(u(t)), -\nabla E(u(t_p))\rangle \tag{18}$$

First of all, let us denote $K = |\nabla E(u(t_p))|$ and estimate

$$\left|\frac{d}{dt}|\nabla E(u(t))|\right| \le |\nabla^2 E(u(t))||v(t)| \le C\kappa|\nabla E(u(t))|$$

and therefore for $t \in (t_1, t_p)$

$$e^{-C\kappa(t_p-t)}K \le |\nabla E(u(t))| \le e^{C\kappa(t_p-t)}K.$$

To estimate the right-hand side of (18) we employ

$$\begin{aligned}|\langle G(u(t),v(t)), -\nabla E(u(t_p))\rangle| &\le |G(u(t),v(t))|\cdot|\nabla E(u(t_p))|\\ &\le c|v(t)||\nabla E(u(t))|\\ &\le c\kappa K^2 e^{2C\kappa(t_p-t)}\end{aligned}$$

and

$$\begin{aligned}\langle \nabla E(u(t)), \nabla E(u(t_p))\rangle &= |\nabla E(u(t_p))|^2 - \langle \nabla E(u(t_p)) - \nabla E(u(t)), \nabla E(u(t_p))\rangle\\ &\ge K^2 - K|\nabla^2 E(u)(\xi)||v(\xi)||t_p - t|\\ &\ge K^2 - KC\kappa(t_p-t)Ke^{C\kappa(t_p-t)}\\ &= K^2(1-\kappa C(t_p-t)e^{C\kappa(t_p-t)}).\end{aligned}$$

If $t \in (t_1, t_p)$, $t > t_p - 1$ and $\kappa < (4eC)^{-1}$, $\kappa < (4ec)^{-1}$, then right-hand side of (18) is larger than

$$K^2(1-\kappa(C(t_p-t)e^{C\kappa(t_p-t)} + ce^{2C\kappa(t_p-t)})) \ge \frac{1}{2}K^2.$$

Integrating (18) from t to t_p, where we obtain

$$\begin{aligned}\frac{1}{2}K^2(t_p-t) &\le \langle v(t_p) - v(t), -\nabla E(u(t_p))\rangle\\ &\le K|v(t) - v(t_p)|\\ &\le K(|v(t)| + |v(t_p)|)\\ &\le K(\kappa K e^{C\kappa(t_p-t)} + \kappa K)\\ &\le \kappa K^2(1 + e^{C\kappa(t_p-t)})\\ &\le \frac{5}{2}\kappa K^2\end{aligned}$$

if κ is small enough. So,

$$t_p - t \leq 5\kappa.$$

It means that taking κ small enough we have $t_p - t << 1$ and the restriction $t > t_p - 1$ plays no role and we have the estimate

$$t_p - t_1 \leq 5\kappa.$$

Now, we need to do similar estimates on (t_8, t_1). Let us denote $K_1 := |\nabla E(u(t_1))|$. Similarly as above we obtain

$$e^{-8C\kappa(t_1-t)}K_1 \leq |\nabla E(u(t))| \leq e^{8C\kappa(t_1-t)}K_1$$

Further, we have

$$\begin{aligned}
|\dot{v}(t)| &\leq |G(u(t), v(t))| + |\nabla E(u(t))| \\
&\leq c|v(t)| + |\nabla E(u(t))| \\
&\leq c8\kappa|\nabla E(u(t))| + |\nabla E(u(t))| \\
&\leq (1 + 8c\kappa)e^{8C\kappa(t_1-t)}K_1 \\
&\leq \frac{6}{5}K_1,
\end{aligned}$$

provided $t > t_1 - 1$ and κ is small enough. Integrating this inequality we obtain

$$|v(t) - v(t_1)| \leq \int_t^{t_1} |\dot{v}(s)| \leq \frac{6}{5}K_1(t_1 - t)$$

If $t_8 > t_1 - 1$, then

$$\begin{aligned}
|v(t_8) - v(t_1)| &\geq |v(t_8)| - |v(t_1)| \\
&= 8\kappa|\nabla E(u(t_8))| - \kappa|\nabla E(u(t_1))| \\
&\geq 8\kappa K_1 e^{-8\kappa c(t_1-t_8)} - \kappa K_1 \\
&\geq 6\kappa K_1
\end{aligned}$$

and together with the previous inequality we have

$$t_1 - t_8 \geq 5\kappa.$$

If $t_8 \leq t_1 - 1$, then $t_1 - t_8 \geq 1 \geq 5\kappa$ if κ is small enough. Together with the upper estimate of $t_p - t_1$ we have

$$t_1 - t_8 \geq 5\kappa \geq t_p - t_1$$

and the proof is complete. □

Lemma 9. *There exists K, $t_0 > 0$ such that for all $t > t_0$ it holds that*

$$\int_t^{+\infty} |F(u(s), v(s))| \leq K \int_t^{+\infty} |v(s)|.$$

Proof. Since $|F(u,v)| \le C(|v| + |\nabla E(u)|)$, it remains to estimate $\int_t^{+\infty} |\nabla E(u(s))|$. Let $\kappa > 0$ be small enough. Let us decompose $(t,+\infty)$ into two parts

$$M_1 := \{s > t : |v(s)| \le \kappa |\nabla E(u(s))|\},$$
$$M_2 := \{s > t : |v(s)| > \kappa |\nabla E(u(s))|\}.$$

Then

$$\int_{M_2} |\nabla E(u(s))| ds \le \frac{1}{\kappa} \int_{M_2} |v(s)| ds.$$

If $t_p \in M_1$, then we can find

$$t_1 := \inf\{t > t_p : |v(s)| \ge \kappa |\nabla E(u(s))|\}$$
$$t_8 := \inf\{t > t_p : |v(s)| \ge 8\kappa |\nabla E(u(s))|\}$$

and we can proof that $t_8 - t_1 \ge 5\kappa \ge t_1 - t_p$ similarly to the proof of Lemma 8 (here we have $t_8 > t_1 > t_p$ unlike in Lemma 8, where we had $t_8 < t_1 < t_p$, but the situation is symmetric). Denote $K_1 := |\nabla E(u(t_1))|$. Similarly to the proof of Lemma 8 we can prove on (t_p, t_1) the inequality

$$|\nabla E(u(t))| \le K_1 e^{C\kappa(t_1 - t)}$$

and since the length of the interval (t_p, t_1) is less that 5κ we obtain

$$|\nabla E(u(t))| \le 2K_1$$

if $5C\kappa^2 < \ln 2$. On the other hand, on $(t_1, t_1 + 5\kappa) \subset (t_1, t_8)$ it holds that

$$|\nabla E(u(t))| \ge K_1 e^{-8C\kappa(t - t_1)} \ge K_1 e^{-8 \cdot 5C\kappa^2} \ge 2^{-8} K_1$$

if $5C\kappa^2 < \ln 2$. It follows that

$$\begin{aligned}
\int_{t_p}^{t_1} |\nabla E(u(s))| &\le (t_1 - t_p) 2K_1 \\
&\le 5\kappa 2^9 \cdot 2^{-8} K_1 \\
&\le 2^9 \int_{t_1}^{t_1 + 5\kappa} 2^{-8} K_1 \\
&\le 2^9 \int_{t_1}^{t_1 + 5\kappa} |\nabla E(u(s))|
\end{aligned}$$

The set M_1 is (at most countable) union of intervals (t_{pn}, t_{1n}) which are followed by corresponding intervals $(t_{1n}, t_{1n} + 5\kappa) \subset M_2$. Therefore, we have

$$\int_{M_1} |\nabla E(u(s))| ds \le 2^9 \int_{M_2} |\nabla E(u(s))| ds \le \frac{2^9}{\kappa} \int_{M_2} |v(s)| ds$$

and the proof is complete. ◻

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