Weak solutions to the barotropic Navier-Stokes system with slip boundary conditions in time dependent domains

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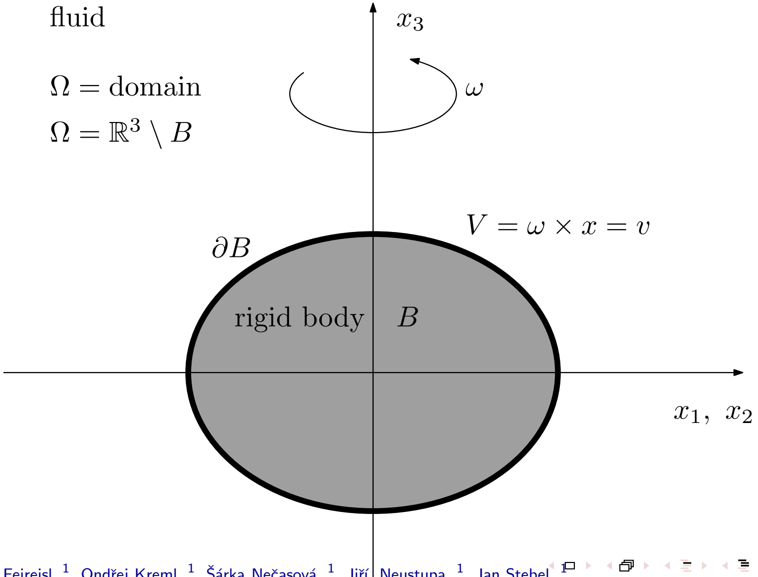
- The motion of viscous fluids around a rotating body (M.Krbec, R.Farwig, R.B. Guenther, ,S.N., E.A. Thomann)
- Weak solutions to the barotropic Navier-Stokes system with slip boundary conditions in time dependent domains (Eduard Feireisl, Ondřej Kreml, Šárka Nečasová, Jiří Neustupa, Jan Stebel,)

fluid

Ω = domain

$\Omega = \mathbb{R}^3 \setminus B$



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Brenner, H.,

(1959)

-**Thomson, W. (Lord Kelvin),**(1882)

-**Serre, D.,**(1987)

-**Weinberger, H. F.,**(1973),(1972)

-**Borchers, W.,**(1992),

Farwig, R., Hishida, T., Geissert, M., Heck M., Hieber M., Galdi, G. P.,, Silvestre A., Shibata, Neustupa, Krbec, M., Kračmar, S., Schumacher K., M.Kyed, Difficulty- **Problem is described in domain which is changing with respect to time**

How to solve problem?

- transformation of coordinate to frame of body (local or global transformation)
- working in domain, which is changing with respect to time

$D(t) \subset \mathbb{R}^n$ $(n = 2, 3)$

$$\begin{array}{rcll} \partial_t v - \nu \Delta v + (v \cdot \nabla) v + \nabla q & = & \tilde{f} & \text{in } \mathcal{D}(t), \\ \operatorname{div} v & = & 0 & \text{in } \mathcal{D}(t), \\ v(y,t) & = & \omega \times y & \text{on } \partial\mathcal{D}(t), \\ v(y,t) & \to & v_\infty & \text{as } |y| \to \infty \end{array} \qquad (1)$$

Global Transformation

in a time-dependent exterior domain $\mathcal{D}(t) \subset \mathbb{R}^3$,$t \in (0, \infty)$.

$$D(t) = O_\omega(t)D,$$

where $D \subset \mathbb{R}^n$ is a fixed exterior domain and $O_\omega(t)$ denotes the orthogonal matrix

$$O_\omega(t) = \begin{pmatrix} \cos|\omega|t & -\sin|\omega|t & 0 \\ \sin|\omega|t & \cos|\omega|t & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2}$$

or

$$O_\omega(t) = \begin{pmatrix} \cos|\omega|t & -\sin|\omega|t \\ \sin|\omega|t & \cos|\omega|t \end{pmatrix} \quad \text{if} \quad n = 2. \qquad (3)$$

$$x := O_\omega(t)^T y$$

$$u(x,t) := O_\omega^T \tilde{v}(y,t) - v_\infty$$

$$p(x,t) := \tilde{q}(y,t)$$

$$f(x,t) := O_\omega(t)^T \tilde{f}(y,t)$$

$$
\begin{aligned}
& u_t - \nu\Delta u + (u\cdot\nabla)u - ((\omega\wedge x)\cdot\nabla)u \\
& +((O_\omega(t)^T v_\infty\cdot\nabla)u + \omega\wedge u + \nabla p = f \qquad \text{in } D,\ t>0 \\
& \text{div } u = 0 \qquad \text{in } D,\ t>0 \\
& u(x,t) + O_\omega(t)^T v_\infty = \omega\wedge x \quad \text{on } \partial D,\ t>0 \\
& u(x,t)\to 0 \qquad \text{as } |x|\to\infty\,.
\end{aligned}
\tag{4}
$$

$((\omega\wedge x)\cdot\nabla)u$ drift term with unbounded coefficient

$((O_\omega(t)^T v_\infty\cdot\nabla)u$ it depends if ω is parallel to v_∞or not

$\omega\wedge u$ Coriolis force

Local transformation

$$
\begin{aligned}
\frac{\partial u}{\partial t} + \nu(\mathbf{L}u) + (\mathbf{M}u) + (\mathbf{N}u) &= 0 \text{ in } \Omega \times (0, T) \\
\text{div} &= 0 \text{ in } \Omega \times (0, T) \\
u(y, t) &= \omega \times y \text{ on } \partial D \times (0, T) \\
u(y, 0) &= v_0(y),\ y \in \Omega
\end{aligned} \tag{5}
$$

(**L**u) ... transformation of Δu and closed to Δu for small t
(**M**u) .... transformation of linear term in u and ∇u
(**N**u) is a non-linear term corresponding to $(v \cdot \nabla v)$
(**G**p) is transformation of p and closed to t for small t

Global transformation- existence of global solution- large time
Local transformation - existence of local solution- large time

Global transformation linearized steady versions

- either in the whole space $\mathbb{R}^n$ **the modified Stokes systems**,

$$
\begin{aligned}
-\nu\Delta u - ((\omega \wedge x)\cdot\nabla)u + \omega\wedge u + \nabla p = f \qquad & \text{in } \mathbb{R}^n,\\
\operatorname{div} u = 0 \text{ or } g \qquad & \text{in } \mathbb{R}^n,\\
u \rightarrow 0 \qquad & \text{as } |x| \rightarrow \infty,
\end{aligned}
\tag{6}
$$

where $n = 2$ or $n = 3$;

- or in an open set Ω **the modified Oseen systems**,

$$\begin{aligned} -\nu\Delta u + k\partial_3 u - ((\omega\wedge x)\cdot\nabla)u + \omega\wedge u + \nabla p = f \quad & \text{in } \Omega, \\ \text{div } u = 0 \text{ or } g \quad & \text{in } \Omega, \\ u(.,t) + u_\infty = \omega\wedge x \quad & \text{on } \partial\Omega, \\ u \to 0 \quad & \text{as } |x|\to\infty, \end{aligned} \tag{7}$$

with an appropriate choice of the constant translational velocity at infinity $u_\infty = ke_3 \neq 0$, therefore parallel to ω.

Goal: find L^p estimates and characterize behaviour at infinity
Problem:
Solving our equation we arrive to singular integral- they are not Calderon- Zygmund type - they have to be solved by Paley Littlewood decompositions

M.Krbec,R.Farwig,Š. N.

Strong solution

Stokes system

$$\|\nu\nabla^2 u\|_{w,q} + \|(\omega\wedge x)\cdot u - \omega\wedge u\|_{w,q} + \|\nabla p\|_{w,q} \leqslant c\|f\|_{w,q}. \quad (8)$$

$|x|^\alpha$, $(1+|x|)^\alpha$, $\alpha\in\mathbf{R}$, $(1+|x|)^\alpha(1+r)^\beta$, $\alpha,\beta\in\mathbf{R}$,
$r=\sqrt{x_1^2+x_2^2}$ *is the radial distance of* $x=(x_1,x_2,x_3)$ *from the axis of revolution.*

$$\begin{aligned} 2 \leqslant q < \infty \quad &: \quad -n < \alpha < \tfrac{nq}{2} \\ 1 < q < 2 \quad &: \quad -\tfrac{nq}{2} < \alpha < n(q-1). \end{aligned}$$

$$\begin{aligned} 2 \leqslant q < \infty \quad &: \quad -2 < \beta < q \quad &&\text{and} \quad -3 < \alpha + \beta < \tfrac{3q}{2} \\ 1 < q < 2 \quad &: \quad -q < \beta < 2(q-1) \quad &&\text{and} \quad -\tfrac{3q}{2} < \alpha + \beta < 3(q-1). \end{aligned}$$

Oseen system

$$\|\nu\nabla^2 u\|_{q,w} + \|\nabla p\|_{q,w} \leqslant c\|f\|_{q,w}, \tag{9}$$

$$\|k\partial_3 u\|_{q,w} + \|(\omega \wedge x)\cdot u - \omega \wedge u\|_{q,w} \leqslant c(k,\nu,\omega)\|f\|_{q,w}. \tag{10}$$

As an example of anisotropic weight functions we consider

$$w(x) = \eta^{\alpha}_{\beta}(x) = (1+|x|)^{\alpha}(1+s(x))^{\beta}, \quad s(x) = |(x_1,x_2,x_3)| - x_3, \tag{11}$$

$$
\begin{aligned}
2 \leqslant q < \infty \quad &: -\tfrac{q}{2} < \alpha < \tfrac{q}{2}, \qquad 0 \leqslant \beta < \tfrac{q}{2} \qquad \text{and} \quad \alpha + \beta > -1 \\
1 < q < 2 \quad &: -\tfrac{q}{2} < \alpha < q - 1, \quad 0 \leqslant \beta < q - 1 \quad \text{and} \quad \alpha + \beta > -\tfrac{q}{2}.
\end{aligned}
$$

the rigid body moving with rotation and translation

P.Deuring, S.Kračmar, S.N.

$$-\Delta u(z) - (U + \omega \times z) \cdot \nabla u(z) + \omega \times u(z) + \nabla \pi(z) = f(z),$$

$$\operatorname{div} u(z) = 0 \tag{12}$$

$$\text{for } \; z \in \mathbb{R}^3 \backslash \overline{\mathfrak{D}}.$$

$$s_\tau(x) := 1 + \tau \cdot (|x| - x_1) \quad \text{for} \;\; x \in \mathbb{R}^3.$$

Theorem

Let $p \in (1, \infty)$, $(u, \pi) \in \mathfrak{M}_p$. *Put* $F := L(u) + \nabla\pi$. *Suppose there are numbers* S_1, S, $\gamma \in (0, \infty)$, $A \in [2, \infty)$, $B \in \mathbb{R}$ *such that* $S_1 < S$, $\overline{\mathfrak{D}} \subset B_{S_1}$,

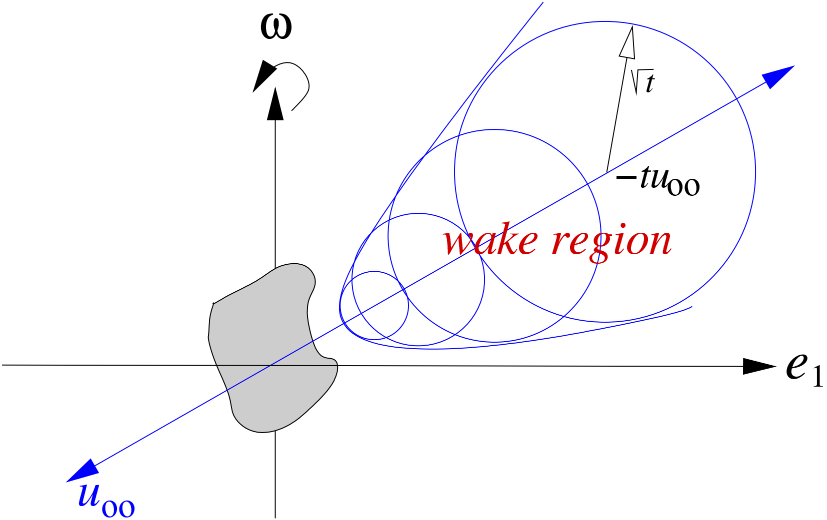
$$u|B_S^c \in L^6(B_S^c)^3, \;\; \nabla u|B_S^c \in L^2(B_S^c)^9, \;\; \pi|B_S^c \in L^2(B_S^c),$$

$$supp(\operatorname{div} u) \subset B_{S_1},$$

$$A + \min\{1, B\} \geqslant 3,$$
$$|F(z)| \leqslant \gamma\, |z|^{-A}\, s_\tau(z)^{-B} \text{ for } \; z \in B_{S_1}^c.$$

Put $\delta := dist(\overline{\mathfrak{D}}, \partial B_S)$. Let $i, j \in \{1, 2, 3\}$, $y \in B_S^c$. Then

$$|u_j(y)| \simeq \left(|y| \, s_\tau(y) \right)^{-1} \, l_{A,B}(y), \tag{13}$$


ω
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R.Farwig, E.Thomann, E.Guenther, Š.N.

$$
\begin{aligned}
&\partial_t u - \nu \Delta u + \nabla p - \\
&\quad -\left[\left(\omega \wedge x + O(t)^T u_\infty\right) \cdot \nabla\right] u + \omega \wedge u &=&\ f &\text{in}&\ \mathcal{D} \times (0,\infty) \\
&\text{div } u &=&\ 0 &\text{in}&\ \mathcal{D} \times (0,\infty) \\
&u &=&\ u_{\partial \mathcal{D}} &\text{on}&\ \partial\mathcal{D} \times (0,\infty) \\
&u(x,t) &\to&\ 0 &\text{as}&\ |x| \to \infty
\end{aligned} \tag{14}
$$

$$U(t) = O(t)^T u_\infty = e^{-t\Omega} u_\infty,$$

$\widetilde{U}(t)$ the solution of the differential equation

$$\frac{\partial \widetilde{U}}{\partial t} + \omega \wedge \widetilde{U} = U(t), \quad \widetilde{U}(0) = 0, \tag{15}$$

$\widetilde{U}(t) = te^{-t\Omega} u_\infty$,
fixed $z \in \mathbb{R}^3$

$$z(t) = e^{-t\Omega} z - \widetilde{U}(t). \tag{16}$$

Theorem

The fundamental tensor of the linearized problem (1.20) is given by

$$\Gamma(y,z,t) =$$
$$K(y-z(t),t)\Big\{ \Big[I - \frac{(y-z(t))\otimes(y-z(t))}{|y-z(t)|^2} \Big] -$$
$$- \Big(1, \frac{5}{2}, \frac{|y-z(t)|^2}{4t} \Big) \times$$
$$\times \Big[\frac{1}{3} I - \frac{(y-z(t))\otimes(y-z(t))}{|y-z(t)|^2} \Big] \Big\} e^{-t\Omega}$$

$$Q(y,z,t) = -\frac{1}{4\pi}\nabla_y \frac{1}{|y - z(t)|}\,\delta_0(t) \equiv Q^*(y,z(t))\,\delta_0(t)\,. \qquad (17)$$

Theorem

Let $T > 0$ and assume that for some $1 < p < \infty$, $0 < \alpha < 1$, $u_0 \in L^p(\mathbb{R}^3)^3$, $F \in C([0,T]; L^p(\mathbb{R}^3)^3) \cap C^{1+\alpha}(\mathbb{R}^3 \times [0,T])^3$ and $\nabla \cdot u_0 = 0$. Then the unique solution $(v, p) \in C([0,T], (L^p(\mathbb{R}^3))^4$ of

$$\begin{array}{rcll} \frac{\partial v}{\partial t} - (U + \omega \wedge y) \cdot \nabla v + \omega \wedge v - \Delta v + \nabla \pi & = & F & \text{in } \mathbb{R}^3 \times (0,\infty) \\ \nabla \cdot v & = & 0 & \text{in } \mathbb{R}^3 \times (0,\infty) \end{array}$$

with initial data $v(0,y) = u_0(y)$ is given by

$$v(y,t) = \int_0^t \int_{\mathbb{R}^3} \Gamma(y,z,t-s) F(z,s)\, dz\, ds +$$
$$+ \int_{\mathbb{R}^3} \Gamma(y,z,t) u_0(z)\, dz,$$
$$p(y,t) = \int_{\mathbb{R}_3} Q^*(y,z(t)) \cdot F(z,t)\, dz.$$

Theorem

The fundamental solution $\Gamma(y,z,t-s)$ from Theorem previous is "unique". As a function of (y,t) it satisfies for $(y,t) \neq (z,s)$, $\mathcal{L}(\Gamma a) = 0$, $\nabla \cdot (\Gamma a) = 0$ for any $a \in \mathbb{R}^3$, and as a function of (z,s) its transpose, Γ' satisfies the adjoint problem $\mathcal{L}^*(\Gamma' a) = 0$, $\nabla_z \cdot (\Gamma' a) = 0$. Furthermore, for $F \in \mathcal{S}(\mathbb{R}^3)^3$

$$\lim_{(y,t)\to(y^0,0^+)} \int_{\mathbb{R}^3} \Gamma(y,z,t) F(z)\, dz = H(y^0)$$

where H is the projection of F onto divergence free vector fields and

$$\int_{\mathbb{R}^3} \Gamma(y, z', t-\tau)\, \Gamma(z', z, \tau - s)\, dz' = \Gamma(y, z, t-s).$$

Problems involving the motion of solid objects in fluids occur frequently in various applications of continuum fluid dynamics, where the boundary conditions on the interfaces play a crucial role.

no-slip condition, where the velocity of the fluid coincides with that of the adjacent solid body

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{18}$$

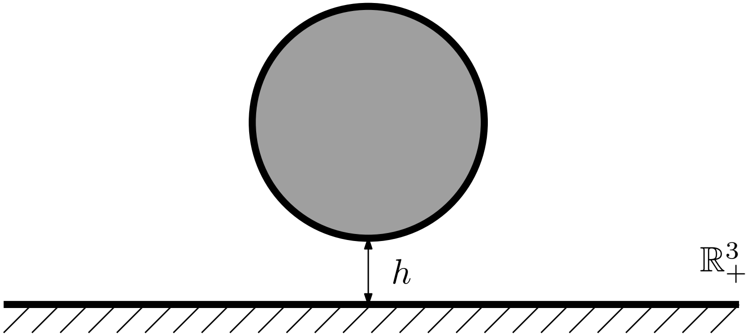
u – the velocity of the fluid

n – the outer normal vector on the boundary of a spatial domain $\Omega \subset R^N$, $N = 2, 3$, occupied by the fluid.

The behavior of the tangential component $[\mathbf{u}]_{tan}$ *is a more delicate issue.*

the *no-slip boundary condition*

$$\mathbf{u} = [\mathbf{u}]_{tan}|_{\partial\Omega} = 0 \tag{19}$$



Hesla, 2005

Hillairet, 2006

2D - absence of collisions in viscous fluids

Feireisl, Hillairet, N. 2008 non-Newtonian fluids
Neustupa, 2007, 2008 the existence of collisions depends on the boundary conditions

2D bounded domains with $C^{1,\alpha}$-Hölder regularity, $0 < \alpha \leqslant 1$, -boundary by Gérard - Varet and Hillairet(09)

- $\alpha \geqslant 1/2$ no collision can occur and strong solutions exist for all time;
- for $\alpha < 1/2$, one can find solutions for which collisions occur

Hillairet, Takahashi 2009 extend result of no collision(falling body in the half space)

Navier proposed the boundary conditions in the form

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0,$$
$$[\mathbb{S}\mathbf{n}]_{\tan} + \kappa\,[\mathbf{u}-\mathbf{V}]_{\tan}\,|_{\Gamma_\tau} = 0,\ \ \kappa \geqslant 0, \tag{20}$$

$\mathbb{S}$ is the viscous stress tensor, κ represents a "friction" coefficient, $\mathbf{u}$ and $\mathbf{V}$ denote the fluid and solid body velocities, Γ_τ is the position of the interface at a time τ,
outer normal vector $\mathbf{n}$.

If $\kappa = 0$,
we obtain the *complete slip* while the asymptotic limit $\kappa \to \infty$ gives rise to the standard no-slip boundary conditions.

The standard *Navier-Stokes system*:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{21}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \varrho \mathbf{f}, \tag{22}$$

where ϱ is the density, $p = p(\varrho)$ the (barotropic) pressure, $\mathbf{f}$ a given external force, and $\mathbb{S}$ is determined by the standard *Newton rheological law*

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \mu > 0, \ \eta \geqslant 0. \tag{23}$$

The boundary of the domain Ω_t - by means of a *given* velocity field $\mathbf{V}(t, \mathbf{x})$, $t \geqslant 0$ and $\mathbf{x} \in R^3$.
V ..... regular

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}(t,\mathbf{x}) = \mathbf{V}\Big(t, \mathbf{X}(t,\mathbf{x})\Big),\ t > 0,\ \mathbf{X}(0,\mathbf{x}) = \mathbf{x}, \tag{24}$$

$\Omega_\tau = \mathbf{X}(\tau, \Omega_0)$, where $\Omega_0 \subset R^3$ is a given domain, $\Gamma_\tau = \partial\Omega_\tau$, and $Q_\tau = \{(t,x) \mid t \in (0,\tau),\ x \in \Omega_\tau\}$.

The boundary Γ_τ is impermeable,

$$(\mathbf{u} - \mathbf{V}) \cdot \mathbf{n}|_{\Gamma_\tau} = 0 \text{ for any } \tau \geqslant 0. \tag{25}$$

The problem (20 - 25) is supplemented by the initial conditions

$$\varrho(0,\cdot) = \varrho_0,\ (\varrho\mathbf{u})(0,\cdot) = (\varrho\mathbf{u})_0 \text{ in } \Omega_0. \tag{26}$$

Our main goal: existence of global-in-time weak solutions to problem (20 - 26) for any finite energy initial data.

- The existence theory for the barotropic Navier-Stokes system on *fixed* spatial domains in the framework of weak solutions was developed - **P.L. Lions** extended - **E. Feireisl** + col. a class of physically relevant pressure-density state equations.
- The investigation of *incompressible* fluids in time dependent domains started with a seminal paper of Ladyzhenskaya, Fujita, Sauer, ..
- Neustupa, Penel
- *Compressible* fluid flows in time dependent domains, supplemented with the *no-slip* boundary conditions, were examined in Feireisl, Neustupa, Stebel -by means of Brinkman's penalization method.

A penalization method to the slip boundary conditions is more delicate.

no-slip........the fluid velocity coincides with the field $\mathbf{V}$ outside Ω_τ,
slip ......only its normal component $\mathbf{u} \cdot \mathbf{n}$

Penalty approach to nonslip condition - see eg. Angot et al. we add to the momentum equations the term

$$-\frac{1}{\epsilon}\chi(\mathbf{u}-\mathbf{V})$$

where

$$\chi(t,x)=\begin{cases} =0 \text{ if } t\in(0,T), x\in\Omega_\tau \\ =1 \text{ otherwise}\end{cases}$$

Penalty approach to slip conditions for *stationary incompressible* fluids was proposed by Stokes and Carey.
The variational (weak) formulation of the momentum equation is supplemented by a singular forcing term

$$\frac{1}{\varepsilon}\int_0^T\int_{\Gamma_t}(\mathbf{u}-\mathbf{V})\cdot\mathbf{n}\,\boldsymbol{\varphi}\cdot\mathbf{n}\,\mathrm{dS}_x\,\mathrm{d}t,\ \varepsilon>0 \text{ small}, \tag{27}$$

penalizing the normal component of the velocity on the boundary of the fluid domain.

The time-dependent geometries

Three level penalization scheme:

- In addition to (27), we introduce a *variable* shear viscosity coefficient $\mu = \mu_\omega$, where μ_ω remains strictly positive in the fluid domain Q_T but vanishes in the solid domain Q_T^c as $\omega \to 0$.
- Similarly to the existence theory developed in Feireisl, Novotný, Petzeltová we introduce the *artificial pressure*

$$p_\delta(\varrho) = p(\varrho) + \delta \varrho^\beta, \ \beta \geqslant 2, \ \delta > 0,$$

in the momentum equation (22).

- Keeping $\varepsilon, \delta, \ \omega > 0$ fixed, we solve the modified problem in a (bounded) reference domain $B \subset R^3$ chosen in such a way that

$$\overline{\Omega}_\tau \subset B \text{ for any } \tau \geqslant 0.$$

(we adapt the existence theory for the compressible Navier-Stokes system with variable viscosity coefficients developed in Feireisl)

- We take the initial density ϱ_0 vanishing outside Ω_0, and letting $\varepsilon \to 0$ for fixed $\delta, \ \omega > 0$ we obtain a "two-fluid" system, where the density vanishes in the solid part $((0,T) \times B) \setminus Q_T$ of the reference domain.
- Letting the viscosity vanish in the solid part, we perform the limit $\omega \to 0$, where the extra stresses disappear in the limit system. The desired conclusion results from the final limit process $\delta \to 0$.

$$\int_{\Omega_\tau} \varrho\varphi(\tau,\cdot)\,\mathrm{d}x - \int_{\Omega_0} \varrho_0\varphi(0,\cdot)\,\mathrm{d}x = \int_0^\tau \int_{\Omega_t} \left(\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi\right)\,\mathrm{d}x\,\mathrm{d}t \tag{28}$$

for any $\tau \in [0, T]$ and any test function $\varphi \in C_c^\infty([0, T] \times R^3)$.

Equation (21) -in the sense of renormalized solutions (DiPerna and Lions)

$$\int_{\Omega_\tau} b(\varrho)\varphi(\tau,\cdot)\,\mathrm{d}x - \int_{\Omega_0} b(\varrho_0)\varphi(0,\cdot)\,\mathrm{d}x =$$

$$\int_0^\tau \int_{\Omega_t} \left(b(\varrho)\partial_t\varphi + b(\varrho)\mathbf{u}\cdot\nabla_x\varphi + (b(\varrho) - b'(\varrho)\varrho)\,\mathrm{div}_x\mathbf{u}\varphi\right)\,\mathrm{d}x\,\mathrm{d}t \tag{29}$$

for any $\tau \in [0, T]$, any $\varphi \in C_c^\infty([0, T] \times R^3)$, and any $b \in C^1[0, \infty)$, $b(0) = 0$, $b'(r) = 0$ for large r. Of course, we suppose that $\varrho \geqslant 0$ a.a. in $(0, T) \times R^3$.

The momentum equation (22) is replaced by a family of integral identities

$$\int_{\Omega_\tau} \varrho \mathbf{u} \cdot \varphi(\tau, \cdot) \, \mathrm{d}x - \int_{\Omega_0} (\varrho \mathbf{u})_0 \cdot \varphi(0, \cdot) \, \mathrm{d}x \tag{30}$$

$$= \int_0^\tau \int_{\Omega_t} \left(\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho [\mathbf{u} \otimes \mathbf{u}] : \nabla_x \varphi + p(\varrho) \mathrm{div}_x \varphi \right) \mathrm{d}x \, \mathrm{d}t$$

$$+ \int_0^\tau \int_{\Omega_t} \left(-\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \varphi + \varrho \mathbf{f} \cdot \varphi \right) \, \mathrm{d}x \, \mathrm{d}t$$

for any $\tau \in [0, T]$ and any test function $\varphi \in C_c^\infty([0, T] \times R^3; R^3)$ satisfying

$$\varphi \cdot \mathbf{n}|_{\Gamma_\tau} = 0 \text{ for any } \tau \in [0, T]. \tag{31}$$

The impermeability condition (25) is satisfied in the sense of traces,

$$\mathbf{u} \in L^2(0,T; W^{1,2}(R^3;R^3)) \text{ and } (\mathbf{u}-\mathbf{V})\cdot\mathbf{n}(\tau,\cdot)|_{\Gamma_\tau} = 0 \text{ for a.a. } \tau \in [0,T]. \tag{32}$$

Theorem

Let $\Omega_0 \subset R^3$ be a bounded domain of class $C^{2+\nu}$, and let $\mathbf{V} \in C^1([0,T]; C^3_c(R^3;R^3))$ be given. Assume that the pressure $p \in C[0,\infty) \cap C^1(0,\infty)$ satisfies

$$p(0) = 0,\ \ p'(\varrho) > 0 \text{ for any } \varrho > 0,$$
$$\lim_{\varrho\to\infty} \frac{p'(\varrho)}{\varrho^{\gamma-1}} = p_\infty > 0 \text{ for a certain } \gamma > 3/2.$$

Let the initial data satisfy

$$\varrho_0 \in L^\gamma(R^3),\ \ \varrho_0 \geqslant 0,\ \ \varrho_0 \not\equiv 0,\ \ \varrho_0|_{R^3\setminus\Omega_0} = 0,$$
$$(\varrho\mathbf{u})_0 = 0 \text{ a.a. on the set } \{\varrho_0 = 0\},$$

$$\int_{\Omega_0} \frac{1}{\varrho_0} |(\varrho \mathbf{u})_0|^2 \, \mathrm{d}x < \infty.$$

Then the problem (20 - 27) admits a weak solution on any time interval $(0, T)$ *in the sense specified through (28 - 32).*

$\kappa = 0$, $\eta = 0$, and $\mathbf{f} = 0$.
Choose $R > 0$:

$$\mathbf{V}|_{[0,T]\times\{|\mathbf{x}|>R\}} = 0, \quad \overline{\Omega}_0 \subset \{|\mathbf{x}| < R\} \tag{33}$$

The reference domain $B = \{|\mathbf{x}| < 2R\}$.

$$\mu_\omega \in C_c^\infty([0,T]\times R^3),\ 0 < \underline{\mu}_\omega \leqslant \mu_\omega(t,x) \leqslant \mu \text{ in } [0,T]\times B,$$
$$\mu_\omega(\tau,\cdot)|_{\Omega_\tau} = \mu \text{ for any } \ \tau \in [0,T]. \tag{34}$$

initial data :

$$\varrho_0 = \varrho_{0,\delta},\ \varrho_{0,\delta} \geqslant 0,\ \varrho_{0,\delta} \not\equiv 0,\ \varrho_{0,\delta}|_{R^3\setminus\Omega_0} = 0,\ \int_B \left(\varrho_{0,\delta}^{\gamma} + \delta\varrho_{0,\delta}^{\beta}\right) \mathrm{d}x \leqslant c, \tag{35}$$

$$(\varrho\mathbf{u})_0 = (\varrho\mathbf{u})_{0,\delta},\ (\varrho\mathbf{u})_{0,\delta} = 0 \text{ a.a. on the set } \{\varrho_{0,\delta} = 0\}, \tag{36}$$

$$\int_{\Omega_0} \frac{1}{\varrho_{0,\delta}} |(\varrho\mathbf{u})_{0,\delta}|^2 \,\mathrm{d}x \leqslant c.$$

The weak formulation of the *penalized problem*:

$$\int_B \varrho\varphi(\tau,\cdot)\,\mathrm{d}x - \int_B \varrho_0\varphi(0,\cdot)\,\mathrm{d}x = \int_0^\tau \int_B (\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi)\,\mathrm{d}x\,\mathrm{d}t \tag{37}$$

for any $\tau \in [0, T]$ and any test function $\varphi \in C_c^\infty([0, T] \times R^3)$;

$$\int_B \varrho \mathbf{u} \cdot \boldsymbol{\varphi}(\tau, \cdot) \, \mathrm{d}x - \int_B (\varrho \mathbf{u})_0 \cdot \boldsymbol{\varphi}(0, \cdot) \, \mathrm{d}x \qquad (38)$$

$$= \int_0^\tau \int_B \left(\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho [\mathbf{u} \otimes \mathbf{u}] : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} + \delta \varrho^\beta \mathrm{div}_x \boldsymbol{\varphi} \right) \mathrm{d}x \, \mathrm{d}t -$$

$$\int_0^\tau \int_B \left(\mu_\omega \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) : \nabla_x \boldsymbol{\varphi} \right) \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \frac{1}{\varepsilon} \int_0^\tau \int_{\Gamma_t} \left((\mathbf{V} - \mathbf{u}) \cdot \mathbf{n} \, \boldsymbol{\varphi} \cdot \mathbf{n} \right) \, \mathrm{dS}_x \, \mathrm{d}t$$

for any $\tau \in [0, T]$ and any test function $\boldsymbol{\varphi} \in C_c^\infty([0, T] \times B; R^3)$,

$\mathbf{u} \in L^2(0,T;W_0^{1,2}(B;R^3))$,
$\mathbf{u}$ satisfies the no-slip boundary condition

$$\mathbf{u}|_{\partial B} = 0 \text{ in the sense of traces.} \tag{39}$$

ε, δ, and ω are positive parameters.

The *existence* of global-in-time solutions to the penalized problem - Feireisl

The *energy inequality*

$$\int_B \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) + \frac{\delta}{\beta-1}\varrho^\beta \right)(\tau,\cdot)\ \mathrm{d}x + \tag{40}$$

$$\frac{1}{2}\int_0^\tau \int_B \mu_\omega \left| \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I} \right|^2\ \mathrm{d}x\ \mathrm{d}t$$

$$+\frac{1}{\varepsilon}\int_0^\tau \int_{\Gamma_t} [(\mathbf{u}-\mathbf{V})\cdot\mathbf{n}]\,\mathbf{u}\cdot\mathbf{n}\ \mathrm{d}S_x\ \mathrm{d}t \leqslant$$

$$\int_B \left(\frac{1}{2\varrho_{0,\delta}}|(\varrho\mathbf{u})_{0,\delta}|^2 + P(\varrho_{0,\delta}) + \frac{\delta}{\beta-1}\varrho_{0,\delta}^\beta \right)\ \mathrm{d}x,$$

where

$$P(\varrho) = \varrho\int_1^\varrho \frac{p(z)}{z^2}\ \mathrm{d}z.$$

The quantity on the right-hand side of (40) representing the total energy of the system is finite because of (35), (36).

Since $\beta \geqslant 2$, the density is square integrable and we may use the regularization technique of DiPerna and Lions to deduce the renormalized version of (37), namely

$$\int_B b(\varrho)\varphi(\tau,\cdot)\ \mathrm{d}x - \int_B b(\varrho_0)\varphi(0,\cdot)\ \mathrm{d}x =$$

$$\int_0^\tau \int_B \left(b(\varrho)\partial_t\varphi + b(\varrho)\mathbf{u}\cdot\nabla_x\varphi + (b(\varrho) - b'(\varrho)\varrho)\,\mathrm{div}_x\mathbf{u}\varphi\right)\ \mathrm{d}x\ \mathrm{d}t \tag{41}$$

for any φ and b as in (29).

Since the vector field $\mathbf{V}$ vanishes on the boundary ∂B, it may be used as a test function in (38). Combining the resulting expression with the energy inequality (40), we obtain
the following bounds *independent* of the parameters ε, δ, and ω:

$$\mathrm{ess} \sup_{t \in (0,T)} \| \sqrt{\varrho} \mathbf{u}(t,\cdot) \|_{L^2(B;R^3)} \leqslant c, \tag{42}$$

$$\mathrm{ess} \sup_{t \in (0,T)} \int_B P(\varrho)(t,\cdot) \, \mathrm{d}x \leqslant c \text{ yielding } \mathrm{ess} \sup_{t \in (0,T)} \| \varrho(t,\cdot) \|_{L^\gamma(B)} \leqslant c, \tag{43}$$

$$\mathrm{ess} \sup_{t \in (0,T)} \delta \| \varrho(t,\cdot) \|^\beta_{L^\beta(B)} \leqslant c, \tag{44}$$

$$\int_0^T \int_B \mu_\omega \left| \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right|^2 \, \mathrm{d}x \, \mathrm{d}t \leqslant c, \tag{45}$$

and

$$\int_0^T \int_{\Gamma_t} |(\mathbf{u} - \mathbf{V}) \cdot \mathbf{n}|^2 \, \mathrm{dS}_x \, \mathrm{d}t \leqslant \varepsilon c. \tag{46}$$

The total mass is conserved:

$$\int_B \varrho(\tau, \cdot) \, \mathrm{d}x = \int_B \varrho_{0,\delta} \, \mathrm{d}x = \int_{\Omega_0} \varrho_{0,\delta} \, \mathrm{d}x \leqslant c \text{ for any } \tau \in [0, T]. \tag{47}$$

Relations (42), (45), (47), combined with the generalized version of Korn's inequality imply that

$$\int_0^T \|\mathbf{u}(t, \cdot)\|^2_{W_0^{1,2}(B;R^3)} \leqslant c(\omega). \tag{48}$$

Γ_τ are determined *a priori*, The so-called Bogovskii operator to deduce the uniform bounds

$$\int\int_{\mathcal{K}} \left(p(\varrho)\varrho^{\nu} + \delta\varrho^{\beta+\nu} \right) \, \mathrm{d}x \, \mathrm{d}t \leqslant c(\mathcal{K}) \text{ for a certain } \nu > 0 \quad (49)$$

for any compact $\mathcal{K} \subset [0,T] \times \overline{B}$ such that

$$\mathcal{K} \cap \left(\cup_{\tau\in[0,T]} \left(\{\tau\} \times \Gamma_\tau \right) \right) = \emptyset,$$

The singular limits $\varepsilon \to 0$, $\omega \to 0$, and $\delta \to 0$.
The parameters δ, ω fixed, our goal is to let $\varepsilon \to 0$ in (37), (38). Let $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon\}$ be the corresponding weak solution of the perturbed problem constructed in the previous section. To begin, the estimates (43), (48), combined with the equation of continuity (37), imply that

$$\varrho_\varepsilon \to \varrho \text{ in } C_{\rm weak}([0,T]; L^\gamma(B)),$$

and

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly in } L^2(0,T; W_0^{1,2}(B, R^3))$$

at least for suitable subsequences, where, as a direct consequence of (46),

$$(\mathbf{u} - \mathbf{V}) \cdot \mathbf{n}(\tau, \cdot)|_{\Gamma_\tau} = 0 \text{ for a.a. } \tau \in [0,T]. \tag{50}$$

(42), (43) and the compact embedding $L^\gamma(B) \hookrightarrow\hookrightarrow W^{-1,2}(B)$, we obtain

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to \varrho \mathbf{u} \text{ weakly-(*) in } L^\infty(0,T; L^{2\gamma/(\gamma+1)}(B;R^3)), \tag{51}$$

and, thanks to the embedding $W_0^{1,2}(B) \hookrightarrow L^6(B)$,

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \to \overline{\varrho \mathbf{u} \otimes \mathbf{u}} \text{ weakly in } L^2(0,T; L^{6\gamma/(4\gamma+3)}(B;R^3)),$$

From the momentum equation (38) that

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to \varrho \mathbf{u} \text{ in } C_{\rm weak}([T_1, T_2]; L^{2\gamma/(\gamma+1)}(O; R^3))$$

for any space-time cylinder

$$(T_1, T_2) \times O \subset [0, T] \times B, \ [T_1, T_2] \times \overline{O} \cap \cup_{\tau \in [0,T]} (\{\tau\} \times \Gamma_\tau) = \emptyset.$$

$L^{2\gamma/(\gamma+1)}(B) \hookrightarrow\hookrightarrow W^{-1,2}(B)$ we conclude, exactly as in (51), that

$$\overline{\varrho \mathbf{u} \otimes \mathbf{u}} = \varrho \mathbf{u} \otimes \mathbf{u} \text{ a.a. in } (0, T) \times B.$$

Passing to the limit in (37) we obtain

$$\int_B \varrho\varphi(\tau,\cdot)\,\mathrm{d}x - \int_B \varrho_{0,\delta}\varphi(0,\cdot)\,\mathrm{d}x = \int_0^\tau \int_B \left(\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi\right)\,\mathrm{d}x\,\mathrm{d}t \tag{52}$$

for any $\tau \in [0, T]$ and any test function $\varphi \in C_c^\infty([0, T] \times R^3)$.

The limit in the momentum equation (38) is more delicate- Feireisl's method

The momentum equation reads

$$\int_B \varrho\mathbf{u}\cdot\boldsymbol{\varphi}(\tau,\cdot)\ \mathrm{d}x - \int_B (\varrho\mathbf{u})_{0,\delta}\cdot\boldsymbol{\varphi}(0,\cdot)\ \mathrm{d}x \tag{53}$$

$$= \int_0^\tau \int_B \left(\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \varrho[\mathbf{u}\otimes\mathbf{u}] : \nabla_x\boldsymbol{\varphi} + p(\varrho)\mathrm{div}_x\boldsymbol{\varphi}\right)\ \mathrm{d}x dt +$$

$$\int_0^\tau \int_B \left(\delta\varrho^\beta\mathrm{div}_x\boldsymbol{\varphi} - \mu_\omega\left(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\right) : \nabla_x\boldsymbol{\varphi}\right)\ \mathrm{d}x\ \mathrm{d}t$$

for any test function $\boldsymbol{\varphi}$

$$\boldsymbol{\varphi} \in C^1([0,T]; W_0^{1,\infty}(B;R^3)),\ \mathrm{supp}[\mathrm{div}_x\boldsymbol{\varphi}(\tau,\cdot)]\cap\Gamma_\tau = \emptyset,$$

$$\boldsymbol{\varphi}\cdot\mathbf{n}|_{\Gamma_t} = 0 \text{ for all } \tau\in[0,T]. \tag{54}$$

In addition, as already observed, the limit solution $\{\varrho,\mathbf{u}\}$ satisfies also the renormalized equation (41).

Our next goal is to use the specific choice of the initial data $\varrho_{0,\delta}$ to get rid of the density-dependent terms in (53) supported by the "solid" part $((0, T) \times B) \setminus Q_T$. To this end, we show the following result, rather obvious for regular solutions but a bit more delicate in the weak framework, that may be of independent interest.

Lemma

Let $\varrho \in L^\infty(0,T;L^2(B))$, $\varrho \geqslant 0$, $\mathbf{u} \in L^2(0,T;W_0^{1,2}(B;R^3))$ *be a weak solution of the equation of continuity, specifically,*

$$\int_B \Big(\varrho(\tau,\cdot)\varphi(\tau,\cdot) - \varrho_0\varphi(0,\cdot)\Big)\mathrm{d}x = \int_0^\tau \int_B \Big(\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t \tag{55}$$

for any $\tau \in [0,T]$ *and any test function* $\varphi \in C_c^1([0,T]\times R^3)$. *In addition, assume that*

$$(\mathbf{u}-\mathbf{V})(\tau,\cdot)\cdot\mathbf{n}|_{\Gamma_\tau} = 0 \text{ for a.a. } \tau \in (0,T), \tag{56}$$

and that

$$\varrho_0 \in L^2(R^3),\ \ \varrho_0 \geqslant 0,\ \ \varrho_0|_{B\setminus\Omega_0} = 0.$$

Then

$$\varrho(\tau, \cdot)|_{B \setminus \Omega_\tau} = 0 \text{ for any } \tau \in [0, T].$$

Thus, by virtue of Lemma 6, the momentum equation (53) reduces to

$$\int_{\Omega_\tau} \varrho \mathbf{u} \cdot \boldsymbol{\varphi}(\tau, \cdot) \, \mathrm{d}x - \int_{\Omega_0} (\varrho \mathbf{u})_{0,\delta} \cdot \boldsymbol{\varphi}(0, \cdot) \, \mathrm{d}x \tag{57}$$

$$= \int_0^\tau \int_{\Omega_t} \left(\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho [\mathbf{u} \otimes \mathbf{u}] : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} + \right.$$

$$\left. \delta \varrho^\beta \mathrm{div}_x \boldsymbol{\varphi} - \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \tfrac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t \right.$$

$$- \int_0^\tau \int_{B \setminus \Omega_t} \mu_\omega \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t$$

for any test function $\boldsymbol{\varphi}$ as in (54). We remark that it was exactly this step when we needed the extra pressure term $\delta \varrho^\beta$ ensuring the density ϱ to be square integrable.

In order to get rid of the last integral in (57), we take the viscosity coefficient

$$\mu_\omega = \begin{cases} \mu = \mathrm{const} > 0 \text{ in } Q_T, \\ \\ \mu_\omega \to 0 \text{ a.a. in } ((0,T)\times B)\setminus Q_T. \end{cases}$$

Denoting $\{\varrho_\omega, \mathbf{u}_\omega\}$ the corresponding solution constructed in the previous section, we may use (45) to deduce that

$$\int_0^T \int_{\Omega_t} \left| \nabla_x \mathbf{u}_\omega + \nabla_x^t \mathbf{u}_\omega - \frac{2}{3} \mathrm{div}_x \mathbf{u}_\omega \mathbb{I} \right|^2 \mathrm{d}x \, \mathrm{d}t < c, \tag{58}$$

while

$$\int_0^T \int_{B\setminus\Omega_t} \mu_\omega \left| \nabla_x \mathbf{u}_\omega + \nabla_x^t \mathbf{u}_\omega - \frac{2}{3} \mathrm{div}_x \mathbf{u}_\omega \mathbb{I} \right|^2 \mathrm{d}x \, \mathrm{d}t \leqslant c,$$

where the latter estimates yields

$$\int_0^\tau \int_{B\setminus\Omega_t} \mu_\omega \left(\nabla_x \mathbf{u}_\omega + \nabla_x^t \mathbf{u}_\omega - \frac{2}{3} \mathrm{div}_x \mathbf{u}_\omega \mathbb{I} \right) : \nabla_x \varphi \ \mathrm{d}x \ \mathrm{d}t =$$

$$\int_0^\tau \int_{B\setminus\Omega_t} \sqrt{\mu_\omega}\sqrt{\mu_\omega} \left(\nabla_x \mathbf{u}_\omega + \nabla_x^t \mathbf{u}_\omega - \tfrac{2}{3} \mathrm{div}_x \mathbf{u}_\omega \mathbb{I} \right) : \nabla_x \varphi \ \mathrm{d}x \ \mathrm{d}t \to 0$$

as $\omega \to 0$

for any fixed φ.

We know from Lemma 6 that the density ϱ_ω is supported by the "fluid" region Q_T, we can still use (42), (58), together with Korn's inequality to obtain

$$\int_0^T \int_{\Omega_t} |\nabla_x \mathbf{u}_\omega|^2 \, \mathrm{d}x \, \mathrm{d}t \leqslant c.$$

Repeating step by step the arguments of the preceding section, we let $\omega \to 0$ to obtain the momentum equation in the form

$$\int_{\Omega_\tau} \varrho \mathbf{u} \cdot \boldsymbol{\varphi}(\tau, \cdot)\ \mathrm{d}x - \int_{\Omega_0} (\varrho \mathbf{u})_{0,\delta} \boldsymbol{\varphi}(0, \cdot)\ \mathrm{d}x \tag{59}$$

$$= \int_0^\tau \int_{\Omega_t} \Big(\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho [\mathbf{u} \otimes \mathbf{u}] : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} \Big)\ \mathrm{d}x dt +$$

$$\int_0^\tau \int_{\Omega_t} \delta \varrho^\beta \mathrm{div}_x \boldsymbol{\varphi} - \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \boldsymbol{\varphi} \Big)\ \mathrm{d}x\ \mathrm{d}t$$

for any test function $\boldsymbol{\varphi}$ as in (54). Note that compactness of the density is now necessary only in the "fluid" part Q_T so a possible loss of regularity of $\mathbf{u}_\omega$ outside Q_T is irrelevant.

The final step is standard, we let $\delta \to 0$ in (59) to get rid of the artificial pressure term $\delta \varrho^{\beta}$ and to adjust the initial conditions, see Feireisl. However, the momentum equation identity (59) holds only for the class of functions specified in (54). The last step of the proof of Theorem 5 is therefore to show that the class of admissible test functions can be extended by density arguments.

Consider a test function $\varphi \in C_c^\infty([0,T] \times R^3; R^3)$ such that

$$\varphi(\tau,\cdot)\cdot \mathbf{n}|_{\Gamma_\tau} = 0 \text{ for any } \tau. \tag{60}$$

Our goal is to show the existence of an approximating sequence of functions φ_n belonging to the class specified in (54) and such that

$$\|\varphi_n\|_{W^{1,\infty}((0,T)\times B;R^3)} \leqslant c,\ \varphi_n \to \varphi,\ \partial_t\varphi_n \to \partial_t\varphi, \quad \text{and } \nabla_x\varphi_n \to \nabla_x\varphi \text{ a.a. in } Q_T. \tag{61}$$

Combining (61) with Lebesgue dominated convergence theorem we may infer that φ belongs to the class of admissible test functions for (30).

We have to find a suitable *solenoidal extension* of the tangent vector field $\varphi|_{\Gamma_\tau}$ inside Ω_τ. Since Γ_τ is regular, there is an open neighborhood $\mathcal{U}_\tau$ of Γ_τ such that each point $\mathbf{x} \in \mathcal{U}_\tau$ admits a *single* closest point $\mathbf{b}_\tau(\mathbf{x}) \in \Gamma_\tau$.

Set

$$\mathbf{h}(\tau, \mathbf{x}) = \varphi(\tau, \mathbf{b}_\tau(\mathbf{x})) \text{ for all } \mathbf{x} \in \mathcal{U}_\tau.$$

We define

$$\mathbf{w}(\tau, \mathbf{x}) = \mathbf{h}(\tau, \mathbf{x}) + \mathbf{g}(\tau, \mathbf{x}),$$

where

$$\mathbf{g}(\tau, \mathbf{x}) = 0 \text{ whenever } \mathbf{x} \in \Gamma_\tau,$$

The local coordinate system at $\mathbf{x}$ so that e_3 coincides with $\mathbf{x} - \mathbf{b}_\tau(\mathbf{x})$, we set

$$\mathbf{g}(\tau, \mathbf{x}) = [0, 0, g^3(\tau, \mathbf{x})], \partial_{x_3} g^3(\tau, \mathbf{x}) = -\partial_{x_1} h^1(\tau, \mathbf{x}) - \partial_{x_2} h^2(\tau, \mathbf{x}).$$

We check that

$$\mathrm{div}_x \mathbf{w}(\tau,\cdot) = 0 \text{ in } \mathcal{U}_\tau,\ \mathbf{w}(\tau,\cdot)|_{\Gamma_\tau} = \varphi(\tau,\cdot)|_{\Gamma_\tau}.$$

Extending $\mathbf{w}(\tau,\cdot)$ inside Ω_τ, we may use smoothness of φ and Γ_τ to conclude that

$$\mathbf{w} \in W^{1,\infty}(Q_T).$$

Writing

$$\varphi = (\varphi - \mathbf{w}) + \mathbf{w},$$

the field $\mathbf{w}$ belongs to the class (54), while

$$(\varphi - \mathbf{w})(\tau, \cdot)|_{\partial\Omega_\tau} = 0 \text{ for any } \tau \geqslant 0.$$

Construct a sequence $\mathbf{a}_n$ such that

$$\mathbf{a}_n \in C_c^\infty([0,T] \times B; R^3),\ \mathrm{supp}[\mathbf{a}_n(\tau,\cdot)] \subset \Omega_\tau \text{ for any } \tau \in [0,T],$$

in particular $\mathbf{a}_n$ belongs to the class (54), and

$$\|\mathbf{a}_n\|_{W^{1,\infty}((0,T)\times B;R^3)} \leqslant c,\ \mathbf{a}_n \to (\varphi - \mathbf{w}),\ \partial_t \mathbf{a}_n \to \partial_t(\varphi - \mathbf{w}),$$

and $\nabla_x \mathbf{a}_n \to \nabla_x(\varphi - \mathbf{w})$ a.a. in Q_T.

The sequence

$$\varphi_n = \mathbf{a}_n + \mathbf{w}$$

complies with (61).