Mathematical Analysis of a Steady Navier–Stokes–Boussinesq BVP in an Inclined Rectangular Cavity

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Model reduction in continuum thermodynamics
Modelling, analysis and computation

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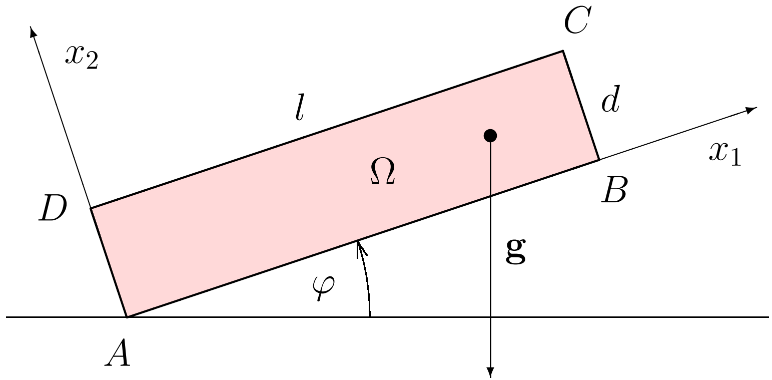
1. Motivation, equations and boundary conditions



$$\rho \;=\; \rho_0 \left[1 - \beta(T - T_0)\right], \tag{1}$$

ρ_0	...	reference density,	ρ	...	density
T_0	...	reference temperature,	T	...	temperature,
β	...	coefficient of thermal expansion			

The acting body force:

$$\rho \mathbf{g} \;=\; \rho_0 \left[1 - \beta(T - T_0)\right] \mathbf{g}. \tag{2}$$

The Navier–Stokes equation:

$$-\nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla \frac{p}{\rho_0} \;=\; \left[1 - \beta\,(T - T_0)\right] \mathbf{g}, \tag{3}$$

The condition of incompressibility:

$$\operatorname{div} \mathbf{u} \;=\; 0. \tag{4}$$

The equation of balance of internal energy:

$$\mathbf{u} \cdot \nabla T \;=\; \kappa \Delta T. \tag{5}$$

Boundary conditions for temperature:

$$T = T_{AB} \text{ on } AB \quad \text{and} \quad T = T_{CD} \text{ on } CD, \tag{6}$$

$$\partial_1 T = 0 \qquad \text{on } AD \cup CD. \tag{7}$$

Conditions of compatibility:

$$T'_{AB}(0) = T'_{AB}(l) = T'_{CD}(0) = T'_{CD}(l) = 0, \tag{8}$$

Boundary condition for velocity:

$$\mathbf{u} = \mathbf{0} \qquad \text{on } \partial\Omega. \tag{9}$$

We denote by (P_1) the boundary–value problem (3), (4), (5), (6), (7), (9).

In order to obtain a problem with homogeneous boundary conditions, we put

$$T = T_{\text{ext}} + \theta, \tag{10}$$

where

$$T_{\text{ext}}(x_1, x_2) = T_{AB}(x_1) + \frac{x_2}{d}\left[T_{CD}(x_1) - T_{AB}(x_1)\right],$$

and θ is a new unknown function. Substituting for T from (10) to (3) and (5), we obtain the equations

$$-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\frac{p}{\rho_0} = \left[1 - \beta\left(T_{\text{ext}} + \theta - T_0\right)\right]\mathbf{g}, \tag{11}$$

$$-\kappa\Delta\theta + \mathbf{u}\cdot\nabla\theta = \left[\kappa\Delta T_{\text{ext}} - \mathbf{u}\cdot\nabla T_{\text{ext}}\right]. \tag{12}$$

Function θ should now satisfy the homogeneous boundary conditions

$$\theta = 0 \qquad \text{on } AB\cup CD, \tag{13}$$

$$\partial_1\theta = 0 \qquad \text{on } AD\cup BC. \tag{14}$$

We denote by (P_2) the boundary–value problem (4), (9), (11), (12), (13), (14).

Problems (P_1) and (P_2) are related through formula (10).

Mathematical analysis of problem (P_2):

- Existence of a solution (with a general specific body force $\mathbf{f}$).

- Structure of the set of solutions for in dependence on the general specific body force $\mathbf{f}$.

- Structure of the set of solutions in the special case when the driving specific body force essentially equals the gravity. Dependence of solutions on the angle of inclination.

2. Existential theory for problem (P_2)

Some previous related results:

- **P. Rabinowitz (1968)**
  - Existence of a steady solution of the three dimensional Bénard problem between two parallel horizontal planes, in the case when the flow is driven by the gravity force.
  - Assumptions that $T_{\rm low}$ (the temperature at the lower plane), $T_{\rm upp}$ (the temperature at the upper plane) are constant, $T_{\rm upp} < T_{\rm low}$.
  - Assumption that and the so called Rayleigh number $R := g\beta(T_{\rm low} - T_{\rm upp})h^3/(16\kappa\nu)$ is “sufficiently close” to some of the eigenvalues of a certain linearized problem associated with the original nonlinear problem.
  - The question of non–uniqueness of solutions is studied by means of bifurcations in dependence of the varying Rayleigh number.

- **H. Morimoto (1991, 2007, 2010)**
  - Domain Ω is supposed to be smooth and bounded.
  - Inhomogeneous boundary conditions for velocity and temperature.
  - Existence of a weak solution.

In contrast to Morimoto,

- we consider the heat convection in a domain with corners,
- the fact that Ω is two–dimensional and its special shape enable us to obtain other estimates of a solution than in the papers by Morimoto,
- we prove the existence of a steady weak solution of the problem (P_2) for any function $\mathbf{f} \in \mathbf{L}^{\alpha}(\Omega)$ ($\alpha > 1$),
- we show that every weak solution of the problem (P_2) is in fact a strong solution.

- We denote by $V(\Omega)$ the space of functions from $W^{1,2}(\Omega)$ whose traces on AB and CD are zero.
- $\mathcal{X} \;:=\; \mathbf{W}^{1,2}_{0,\sigma}(\Omega) \times V(\Omega)$

The weak formulation of problem (P$_2$):

$\mathbf{f} \in \mathbf{L}^{\alpha}(\Omega)$ (for some $\alpha > 1$)

We look for $(\mathbf{u}, \theta) \in \mathcal{X}$ such that the integral identities

$$\int_\Omega \big[\nu \nabla \mathbf{u} : \nabla \mathbf{w} + (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{w}\big] \, \mathrm{d}\mathbf{x} \;=\; \int_\Omega \big[1 - \beta \left(T_{\mathrm{ext}} + \theta - T_0\right)\big] \, \mathbf{f} \cdot \mathbf{w} \, \mathrm{d}\mathbf{x}, \qquad (15)$$

$$\int_\Omega \big[\kappa \nabla \theta \cdot \nabla \vartheta + (\mathbf{u} \cdot \nabla \theta)\, \vartheta\big] \, \mathrm{d}\mathbf{x} \;=\; - \int_\Omega \big[\kappa \nabla T_{\mathrm{ext}} \cdot \nabla \vartheta + (\mathbf{u} \cdot \nabla T_{\mathrm{ext}})\, \vartheta\big] \, \mathrm{d}\mathbf{x}. \qquad (16)$$

hold for all $(\mathbf{w}, \vartheta) \in \mathcal{X}$.

Theorem 1 (existence of a strong solution of problem (P_2)).

Let functions
$\mathbf{f} \in \mathbf{L}^\alpha(\Omega)$ (for some $\alpha > 1$) and T_{AB}, $T_{CD} \in C^3([0,l])$, satisfying (8), be given.

Then problem (P_2) has at least one weak solution $(\mathbf{u}, \theta)$.

Function $\mathbf{u}$ belongs to $\mathbf{W}^{1,2}_{0,\sigma}(\Omega) \cap \mathbf{W}^{2,\alpha_0}(\Omega)$ (where $\alpha_0 := \min\{2; \alpha\}$).

Function θ is in $V(\Omega) \cap W^{3,q}(\Omega)$ for each $q \in (1,\infty)$ (if $\alpha \geq 2$) or
$q \in (1, 2\alpha/(2-\alpha))$ (if $1 < \alpha < 2$).

Consequently, there exists $p \in W^{1,\alpha_0}(\Omega)$ so that the triplet $(\mathbf{u}, p, \theta)$ represents a strong solution of problem (P_2).

Remark.

Since function p enters problem (P_2) only through the gradient ∇p, it can always be chosen so that $p \in W^{1,\alpha_0}_{\mathrm{mv}}(\Omega)$. Then p is uniquely given by $\mathbf{f}$, $\mathbf{u}$ and θ.

Principle of the proof:

We assume that $(\mathbf{u}, \theta)$ is a weak solution of problem (P_2). We use two **basic tools:**

I. Solutions of the Stokes problem

$$\begin{aligned} -\nu \Delta \mathbf{u} + \nabla P &= \mathbf{F} && \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 && \text{in } \Omega, \\ \mathbf{v} &= \mathbf{0} && \text{on } \partial\Omega \end{aligned}$$

in a 2D polygonal domain Ω satisfy the estimate

$$\|\mathbf{u}\|_{2,s} + \|\nabla P\|_s \leq C \|\mathbf{F}\|_s \tag{17}$$

for $1 < s \leq 2$. See **Kellog and Osborn (1976)**, **Grisvard (1979)**, **Girault and Raviart (1986)**.

Assuming that $(\mathbf{u}, \theta)$ is a weak solution of problem (P_2), we show that

$$\mathbf{F} := (\mathbf{u} \cdot \nabla)\mathbf{u} + \left[1 - \beta\,(T_{\text{ext}} + \theta - T_0)\right] \mathbf{f} \in \mathbf{L}^s(\Omega) \qquad \text{for } 1 < s < 2.$$

So the first application of (17) yields the estimate of $\|\mathbf{u}\|_{2,s}$ for $1 < s < 2$.

II. Solution z os the Poisson equation $\Delta z = G$ the homogeneous boundary condition in a smooth domain $\widehat{\Omega}$ satisfies the estimate

$$\|z\|_{k,s;\,\widehat{\Omega}} \;\le\; C\,\|G\|_{k-2,s;\,\widehat{\Omega}}.$$

This estimate cannot be directly applied to the equation

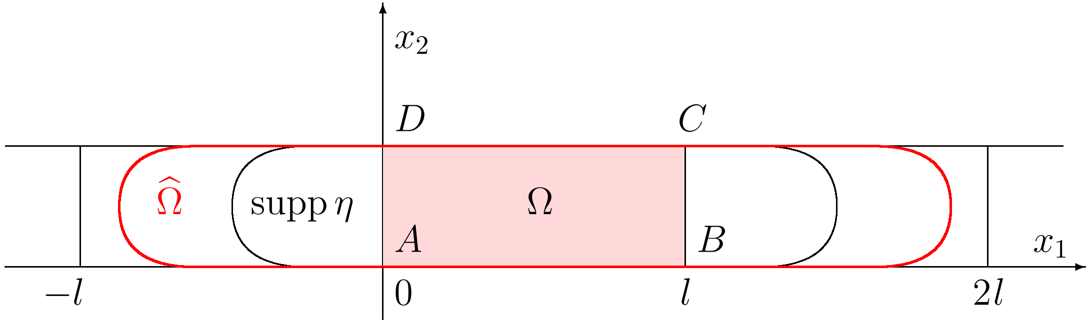
$$-\kappa\Delta\theta + \mathbf{u}\cdot\nabla\theta \;=\; \left[\kappa\Delta T_{\rm ext} - \mathbf{u}\cdot\nabla T_{\rm ext}\right], \tag{12}$$

because this equation is fulfilled in the non–smooth domain Ω. This is why we extend

- u_1 as a $2d$–periodic odd function
- u_2 as a $2d$–periodic even function
- θ as a $2d$–periodic even function

in variable x_1.

We show that the extended function $\widehat{\theta}$ is a weak solution of equation (12) in the larger domain $\widehat{\Omega}$, see the next picture.



We apply Grisvard's estimates to the boundary–value problem for the function $\eta\widehat{\theta}$ (which satisfies the homogeneous Dirichlet boundary condition on $\partial\widehat{\Omega}$), and using the bootstrapping argument, we prove the lemma.

We are limited by the fact that the extended function $\widehat{\mathbf{u}}$ is in $\mathbf{W}^{1,k}(\widehat{\Omega})$ for each $1 < k < 2s/(2-s)$, but it is not in $\mathbf{W}^{2,s}(\widehat{\Omega})$.

We successively derive the estimates

$$\|\mathbf{u}\|_{2,\alpha_0} + \|\nabla p\|_{\alpha_0} \;\le\; C\big(\|\nabla\mathbf{u}\|_2,\, \|\nabla\theta\|_2,\, \|\mathbf{f}\|_{\alpha_0}\big), \tag{18}$$

$$\|\theta\|_{3,q} \;\leq\; C\big(\|\mathbf{u}\|_{2,s}\big), \tag{19}$$

$$-(T_{\max} - T_{\min}) \;\leq\; \theta(\mathbf{x}) \;\leq\; T_{\max} - T_{\min}, \tag{20}$$

$$T_{\min} \leq\; T(\mathbf{x}) \;\leq\; T_{\max}, \tag{21}$$

$$\|\nabla\mathbf{u}\|_2 \;\leq\; C\left[1 + \beta\left(T_0 - T_{\min}\right)\right] \|\mathbf{f}\|_\alpha, \tag{22}$$

$$\|\nabla\theta\|_2 \;\leq\; \left(C d^2 + \kappa\sqrt{d}\right) \|\nabla T_{\text{ext}}\|_\infty, \tag{23}$$

in Ω, where

$$T_{\min} \;:=\; \min_{0\leq t\leq l}\, \min\{T_{AB}(t),\, T_{CD}(t)\},$$

$$T_{\max} \;:=\; \max_{0\leq t\leq l}\, \max\{T_{AB}(t),\, T_{CD}(t)\}.$$

Existence of a solution of problem (P_2) can be finally proven e.g. by means of the Leray–Schauder fixed point principle. □

3. Structure of the solution set of problems (P_1) and (P_2)

We further assume, for convenience, that $\mathbf{f} \in \mathbf{L}^2(\Omega)$ (i.e. $\alpha = 2$). Then $\mathbf{u} \in \mathbf{W}^{1,2}_{0,\sigma}(\Omega) \cap \mathbf{W}^{2,2}(\Omega)$, $\theta \in V(\Omega) \cap W^{3,q}(\Omega)$ for all $q \in (1, \infty)$ and $p \in W^{1,2}_{\mathrm{mv}}(\Omega)$.

Function θ is uniquely determined by $\mathbf{u}$. Hence $\theta = \theta(\mathbf{u})$ and we may consider only $(\mathbf{u}, p)$ to be the strong solution of (P_2).

$$\begin{aligned} \mathfrak{X} &:= \mathbf{W}^{1,2}_{0,\sigma}(\Omega) \cap \mathbf{W}^{2,2}(\Omega), \\ \mathfrak{G}(\mathbf{u}, p) &:= -\nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla \frac{p}{\rho_0}, \\ \mathfrak{F}(\mathbf{u}, p) &:= \Big(1 - \beta \left[T(\mathbf{u}) - T_0)\right]\Big)^{-1} \mathfrak{G}(\mathbf{u}, p). \end{aligned} \tag{24}$$

We can suppose without loss of generality that the reference temperature T_0 is chosen so that they satisfy the condition

$$1 + \beta T_0 \; > \; \beta T_{\max}. \tag{25}$$

Then, using inequalities (21), we observe that $T(\mathbf{u})$ satisfies

$$1-\beta\big[T(\mathbf{u})-T_0\big] \;>\; 1-\beta(T_{\max}-T_0) \;>\; 0 \qquad \text{in } \Omega. \tag{26}$$

We denote by $\mathcal{S}(\mathbf{f})$ the solution set of problem (P_1) (or (P_2)).

The inclusion $(\mathbf{u},p) \in \mathcal{S}(\mathbf{f})$ can be equivalently written:

$$\boxed{\mathfrak{F}(\mathbf{u},p) \;=\; \mathbf{f}} \tag{27}$$

We can successively prove the lemmas:

Lemma 1 *The operator $\mathbf{u} \mapsto T(\mathbf{u})$ is a C^2–mapping from $\mathfrak{X}$ to $W^{2,r}(\Omega)$ (for any $r \in (1,2)$).*

Lemma 2 *Operator $\mathfrak{F}$ is a C^2–mapping from $\mathfrak{X} \times W^{1,2}_{\mathrm{mv}}(\Omega)$ into $\mathbf{L}^2(\Omega)$.*

Lemma 3 *Operator $\mathfrak{F}$ is a proper mapping from $\mathfrak{X} \times W^{1,2}_{\mathrm{mv}}(\Omega)$ to $\mathbf{L}^2(\Omega)$.*

(Operator $\mathfrak{F}$ is said to be **proper** if, for any compact set $K \subset \mathbf{L}^2(\Omega)$, the pre–image $\mathfrak{F}^{-1}(K)$ is compact in $\mathfrak{X} \times W^{1,2}_{\mathrm{mv}}(\Omega)$. For closed operators, the properness is equivalent to the property "$\mathcal{S}(\mathbf{f})$ is compact for all $\mathbf{f}$".)

Lemma 4 *Operator $\mathfrak{F}$ is a Fredholm mapping from $\mathfrak{X} \times W^{1,2}_{\mathrm{mv}}(\Omega)$ to $\mathbf{L}^2(\Omega)$ of index* 0*.*

A closed linear operator L from $\mathfrak{X} \times W^{1,2}_{\mathrm{mv}}(\Omega)$ into $\mathbf{L}^2(\Omega)$ is called a **Fredholm operator** if its range $R(L)$ is closed and both the numbers $\mathrm{nul}\, L$ (the nullity of L, i.e. the dimension of $\mathrm{Ker}(L)$) and $\mathrm{def}\, L$ (the deficiency of L, i.e. the dimension of the quotienr space $\mathbf{L}^2(\Omega)|_{R(L)}$) are finite.

The nonlinear operator $\mathfrak{F}$ is said to be a **Fredholm mapping** if the Fréchet differential $\big[\mathfrak{F}'(\mathbf{u}, p)\big](\mathbf{u}^*, p^*)$ is a linear Fredholm operator (in dependence on $\mathbf{u}^*$, p^*) from $\mathfrak{X} \times W^{1,2}_{\mathrm{mv}}(\Omega)$ into $\mathbf{L}^2(\Omega)$ for all $(\mathbf{u}, p) \in \mathfrak{X} \times W^{1,2}(\Omega)$.

In this case, $\mathrm{ind}\, \mathfrak{F}'(\mathbf{u}, p) := \mathrm{nul}\, \mathfrak{F}'(\mathbf{u}, p) - \mathrm{def}\, \mathfrak{F}'(\mathbf{u}, p)$ is independent of $(\mathbf{u}, p)$ and it is called the **index of operator $\mathfrak{F}$.**

Recall that the so called **singular values** of $\mathfrak{F}$ are images of singular points, i.e. the points $(\mathbf{u}, p) \in \mathfrak{X} \times W^{1,2}_{\mathrm{mv}}(\Omega)$ where $\mathfrak{F}'(\mathbf{u}, p)$ is not surjective. All other points in $\mathbf{L}^2(\Omega)$ are said to be the **regular values** of $\mathfrak{F}$. Thus, if $\mathbf{f} \in \mathbf{L}^2(\Omega)$ is a regular value of $\mathfrak{F}$, operator $\mathfrak{F}'(\mathbf{u}, p)$ is surjective for all $(\mathbf{u}, p) \in \mathcal{S}(\mathbf{f})$.

Since $\mathfrak{F} : \mathfrak{X} \times W^{1,2}_{\mathrm{mv}}(\Omega) \to \mathbf{L}^2(\Omega)$ is a proper C^2–Fredholm mapping of index 0, the set $\mathcal{O}$ of regular values of $\mathfrak{F}$ is open and dense in $\mathbf{L}^2(\Omega)$ (by to the Sard–Smale theorem).

Due to the Preimage Theorem (see e.g. the book by Zeidler), $S(\mathbf{f})$ is a C^2–Banach manifold in $\mathfrak{X} \times W^{1,2}_{\mathrm{mv}}(\Omega)$ for each $\mathbf{f} \in \mathcal{O}$.

Furthermore, $\mathcal{S}(\mathbf{0})$ reduces to just one point $(\mathbf{0}, 0)$. Expressing explicitly $\mathfrak{F}'(\mathbf{0}, 0)$, one can show that $\dim \mathrm{Ker} \mathfrak{F}'(\mathbf{0}, 0) = 0$. Connecting now homotopically point $\mathbf{f} \in \mathcal{O}$ with $\mathbf{0}$, we conclude that $\mathcal{S}(\mathbf{f})$ is finite.

Thus, we arrive at the theorem:

Theorem 2.

There exists an open dense subset $\mathcal{O} \subset \mathbf{L}^2(\Omega)$ with the properties:

1) For every $\mathbf{f} \in \mathcal{O}$ the set $\mathcal{S}(\mathbf{f})$ is finite.
2) The number of elements of $\mathcal{S}(\mathbf{f})$, for $\mathbf{f}$ in every connected component of $\mathcal{O}$, is constant.
3) Each element of $\mathcal{S}(\mathbf{f})$ is a C^2–function of $\mathbf{f}$ for $\mathbf{f}$ in every connected component of $\mathcal{O}$.

4. The case when the body force is essentially the gravity and angle φ varies

$$\mathbf{f} \equiv \mathbf{f}_\varphi := -g\mathbf{e}_\varphi + \mathbf{h}, \tag{28}$$

where $\mathbf{e}_\varphi = (\sin\varphi, \cos\varphi)$.

The case $\mathrm{h} = \mathbf{0}$, T_{AB}, T_{CD} constant, and small Rayleigh number.

In this case, it has been observed in experiments that the velocity $\mathbf{u}$ is also "small" if angle φ is "small". Particularly, $\mathbf{u} = \mathbf{0}$ if $\varphi = 0$.

Using the assumptions that $\mathbf{h} = \mathbf{0}$ and T_{AB} and T_{CD} constant, multiplying this equation by $\mathbf{u}$, the equation for thermal convection/conduction by θ, integrating by parts and using inequalities (20) and the condition

$$g\beta d^3 \left|T_{AB} - T_{CD}\right| < \kappa\nu, \tag{29}$$

we obtain:

$$\|\nabla \mathbf{u}\|_2 \;\le\; \frac{g\beta\kappa\,|T_{CD}-T_{AB}|\,d\,(\sin\varphi)\,\sqrt{ld}}{\nu\kappa - g\beta d^3\,|T_{CD}-T_{AB}|} \;=:\; c_1\,\sin\varphi, \tag{30}$$

$$\|\nabla \theta\|_2 \;\le\; \frac{g\beta\,|T_{CD}-T_{AB}|^2\,d^2\,(\sin\varphi)\,\sqrt{ld}}{\nu\kappa - g\beta d^3\,|T_{CD}-T_{AB}|} \;=:\; c_2\,\sin\varphi. \tag{31}$$

Some observations.

These estimates show that $\mathbf{u} \to \mathbf{0}$ in $\mathbf{W}^{1,2}_{0,\sigma}(\Omega)$ and $\theta \to 0$ in $V(\Omega)$ for $\sin\varphi \to 0$ (i.e. for $\varphi \to 0$ or $\varphi \to \pi$).

Particularly, if $\varphi = 0$ or $\varphi = \pi$ then the only possible solution is $\mathbf{u} = \mathbf{0}$, $\theta = 0$.

However, $\mathbf{u} = \mathbf{0}$, $\theta = 0$ is not the trivial solution in the case $0 < \varphi < \pi$.

(P_2) does not generally have a trivial solution if functions T_{AB}, T_{CD} are not constant.

The case of varying angle φ.

For each fixed considered perturbation $\mathbf{h} \in \mathbf{L}^2(\Omega)$ in formula (28), the family $\{\mathbf{f}_\varphi\}_{0\leq\varphi\leq\pi}$ forms a continuous curve in the space $\mathbf{L}^2(\Omega)$.

Question: *Does this curve lie in set $\mathcal{O}$ for "most values" of φ ?*

We write equation (27) in the form

$$\mathfrak{A}(\mathbf{u}, p, \varphi) \;:=\; \mathfrak{F}(\mathbf{u}, p) + g\mathbf{e}_\varphi \;=\; \mathbf{h} \tag{32}$$

and we consider the operator $\mathfrak{A}$ on the left hand side to be the mapping from $\mathfrak{X} \times W^{1,2}_{\mathrm{mv}}(\Omega) \times (0, \pi)$ to $\mathbf{L}^2(\Omega)$.

One can verify that $\mathfrak{A}$ is a proper C^2–Fredholm mapping of $\mathfrak{X} \times W^{1,2}_{\mathrm{mv}}(\Omega) \times (0, \pi)$ of index 1.

Applying similar considerations and tools as before, we prove the theorem:

Theorem 3.

There exists an open dense subset $\mathcal{M} \subset \mathbf{L}^2(\Omega)$ such that to every $\mathbf{h} \in \mathcal{M}$ there is an open dense set $\Phi = \Phi(\mathbf{h}) \in (0, \pi)$ with the property that for φ varying in each connected component of Φ, the set of pairs $(\mathbf{u}, p)$ such that $(\mathbf{u}, p, \varphi)$ satisfies equation (32) forms a system of finitely many 1–dimensional C^2–manifolds (i.e. C^2–curves) in $\mathfrak{X} \times W^{1,2}_{\rm mv}(\Omega)$.

Since set $\mathcal{M}$ is open and dense in $\mathbf{L}^2(\Omega)$, the information provided by Theorem 3 is **"generic" with respect to the choice of function** h on the right hand side of (32), or, in other words, it holds for "almost all" $\mathbf{h} \in \mathbf{L}^2(\Omega)$.

Thank you for the attention.