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Weak lower semicontinuity of integral functionals and applications

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"Nothing takes place in the world whose meaning is not that of some maximum or minimum."
LEONHARD PAUL EULER (1707–1783)

Abstract. In 1965, N.G. Meyers [110] significantly extended weak lower semicontinuity results for integral functionals depending on maps and their gradients available at that time. We recapitulate the development on this topic from that time on. Special attention is paid to signed integrands and to applications to continuum mechanics of solids. In particular, we review existing results for polyconvex simple as well as nonsimple materials and related statements about sequential weak continuity of minors. These are non-coercive and belong precisely to the class of integrands studied by Meyers in his seminal work. Besides, we emphasize some recent progress in lower semicontinuity of functionals along sequences satisfying differential and algebraic constraints which have applications in continuum mechanics of solids to ensure injectivity and orientation-preservation of elastic deformations. Finally, we outline generalization of these results to more general first-order partial differential operators and make some suggestions for further reading.

1. Introduction. The observation that continuous functions attain extreme values on compact sets goes back to Bernard Bolzano who proved it in his work "Function Theory" in 1830. This result is called the *Extreme Value Theorem*. Later on, it was independently shown by Karl Weierstrass around 1860. The main ingredient of the proof, namely the fact that one can extract a convergent subsequence from a closed bounded interval of reals, is nowadays known as the Bolzano-Weierstrass theorem. While Riesz and Hilbert already used the weak topology on Hilbert spaces from the beginning of the 20th century, Stefan Banach defined it on other normed spaces around 1929 [124, 156] and opened the possibility to extend Bolzano's Extreme Value Theorem to more general situations and, in particular, to the calculus of variations.

Calculus of variations has in its background minimization problems of the type

$$y \mapsto \int_a^b v(x, y, y')\,\mathrm{d}x \to \inf \;\text{ with } y(a) = y_a \text{ and } y(b) = y_b \;.$$

It includes, for example, the *brachistochrone problem*, i.e., the problem of finding curves with a minimum time of descent in a gravitational field. Foundations of the calculus of variations were laid down in the 18th century by L.P. Euler and J.L. Lagrange who also realized its important connections to physics and to mechanics.

Lower semicontinuity of functional (cf. Definition 1.1 below) plays a fundamental role in the *direct method of the calculus of variations*, an algorithm, proposed by David Hilbert around 1900, to show (in a non-constructive way) the existence of a solution to the minimization problem

$$\text{find minimum of } I \text{ on } \mathcal{Y} \;.$$

It consists of three steps: First, we find a minimizing sequence along which I converges to its infimum on $\mathcal{Y}$. The second step is to show that a subsequence of the minimizing sequence converges to an element of $\mathcal{Y}$ in some topology τ. Finally, it remains to prove that this limit element is a minimizer. This is easily done if I is (sequentially) lower semicontinuous with respect to the topology τ. In the most typical situation, the topology τ is either the weak or the weak* one; thus, we shall also limit our view to this case.

DEFINITION 1.1. *Let $\mathcal{Y}$ be a subset of a Banach space. We say that the functional $I : \mathcal{Y} \to \mathbb{R}$ is (sequentially) weakly/weakly* lower-semicontinuous on $\mathcal{Y}$ if for any sequence $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{Y}$ converging weakly/weakly* to $u \in \mathcal{Y}$, we have that*

$$I(u) \le \liminf_{k\to\infty} I(u_k).$$

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While the first two steps of the direct method can be satisfied by assuming coercivity of I and by choosing a sufficiently weak topology on $\mathcal{Y}$, the last step essentially relies on fine properties of I as convexity, for instance.

Let us point out that earlier studies of minimizers of integral functionals of the form

$$y \mapsto \int_a^b v(x, y, y') \,\mathrm{d}x \tag{1.1}$$

naturally relied on smoothness properties of v when calculating variations of this integral; see for example the book by Bolza [29]. On the other hand, the direct method is not based on calculating derivatives and thus it is natural to expect that it will cope also with non-smooth and possibly also partially discontinuous integrands in (1.1). This expectation is indeed true and relaxing smoothness/continuity assumptions of v will be a re-occurring theme throughout this review.

As indicated above, many phenomena in nature are successfully modeled by solving a minimization problem for a suitably chosen (energy) functional. A prominent example is found in continuum mechanics of solid media, where minimization of the stored energy

$$\mathcal{E}(y) := \int_\Omega W(\nabla y(x)) \,\mathrm{d}x \ , \tag{1.2}$$

determines stable states of a hyperelastic body. Here, W is the stored energy density and the map $y : \Omega \to \mathbb{R}^3$, with Ω a bounded domain representing the undeformed material, is the deformation of the modeled medium.

Naturally, the question arises under what conditions on W minimizers of (1.2) exist on a suitable function space $\mathcal{Y}$. In view of the direct method described above, this particularly includes the study of *weak lower semicontinuity* of functionals $\mathcal{E}$.

Although the study of weak lower semicontinuity is motivated by understanding minimization problems, it has become an independent subject in mathematical literature that has been studied for its own right. In 1920, Tonelli [152] showed that if $v : \Omega \times \mathbb{R} \times \mathbb{R}$ is a Caratheóodory integrand,[3] and

$$I(u) := \int_\Omega v(x, u(x), \nabla u(x)) \,\mathrm{d}x \ , \tag{1.3}$$

with $\Omega = (a, b)$ and $u \in W^{1,\infty}((a, b); \mathbb{R})$ then I is weakly lower semicontinuous if and only if v is convex in its last variable, i.e., in the derivative $\nabla u = u'$. Later, several authors generalized this result to functions in $W^{1,\infty}(\Omega; \mathbb{R})$ with $\Omega \subset \mathbb{R}^n$ and $n > 1$; see for example Serrin [139], where differentiability properties of v were removed from assumptions, and Marcellini and Sbordone [108]. On the other hand, if we allow the function u to be vector-valued, i.e., $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ with $\Omega \subset \mathbb{R}^n$ and $n > 1$ as well as $m > 1$, then the convexity hypothesis turns out to be sufficient but unnecessary. A suitable condition, termed quasiconvexity, was introduced by Morrey [114].

Definition 1.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. We say that a function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is quasiconvex if for any $A \in \mathbb{R}^{m\times n}$ and any $\varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^m)$*

$$f(A)\mathcal{L}^n(\Omega) \le \int_\Omega f(A + \nabla\varphi(x)) \,\mathrm{d}x \ . \tag{1.4}$$

Morrey showed, under strong regularity assumptions on v, that I from (1.3) is weakly lower semicontinuous in $W^{1,\infty}(\Omega; \mathbb{R}^m)$ if and only if v is quasiconvex in the last variable (i.e. in the gradient). All the above characterizations turn the problem of sequential weak lower semicontinuity of I into conditions on the integrand. Obviously, this is a much more *explicit* as it does not need to deal with weakly convergent sequences.

[3] i.e. $v(x, \cdot, \cdot)$ is continuous for almost all $x \in \Omega$ and $v(\cdot, s, A)$ is measurable for all $(s, A) \in \mathbb{R} \times \mathbb{R}$

These results were generalized more than fifty years ago, in 1965, by Norman G. Meyers in his seminal paper [110]. There he investigated $W^{k,p}$-weak (weak* if $p = +\infty$) lower semicontinuity of integral functionals of the form

$$I(u) := \int_\Omega v(x, u(x), \nabla u(x), \ldots, \nabla^k u(x))\,\mathrm{d}x \ , \tag{1.5}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $u : \Omega \to \mathbb{R}^m$ is a mapping possessing (weak) derivatives up to the order $k \in \mathbb{N}$. The function v was supposed to be continuous in all its arguments. Since now higher gradients than the first ones are considered, the definition of quasiconvexity also needs to be generalized accordingly (see Section 2 for the notation).

Definition 1.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. We say that a function $f : X(n,m,k) \to \mathbb{R}$ is k-quasiconvex*[4] *if for any $A \in X(n,m,k)$ and any $\varphi \in W_0^{k,\infty}(\Omega;\mathbb{R}^m)$*

$$f(A)\mathcal{L}^n(\Omega) \le \int_\Omega f(A + \nabla^k\varphi(x))\,\mathrm{d}x \ . \tag{1.6}$$

Thus, more precisely, k-quasiconvexity of v (i.e. quasiconvexity with respect to the k-th gradient) means that $A^k \mapsto v(x, A^{[k-1]}, A^k)$ is quasiconvex for all fixed $(x, A^{[k-1]}) \in \Omega \times Y(m,n,k-1)$; here, we already used the notation which will be introduced in Section 2, however, $X(\cdot)$ and $Y(\cdot)$ are simply Euclidean spaces hosting pointwise values of corresponding maps.

Remark 1.1. *In fact, it was shown in [45] that if $k = 2$ and if f satisfies a (slightly) stronger version of 2-quasiconvexity then 2-quasiconvexity coincides with 1-quasiconvexity. See [34] for an analogous result with general k.*

However, more generally than in Morrey's work, the function v is not necessarily bounded from bellow in [110]. From this, additional difficulties arise and, in fact, quasiconvexity is no longer a sufficient condition for weak lower semicontinuity (cf. Section 3). Moreover, the regularity assumptions on the integrand in (1.5) were weakened in Meyers' work.

The motivation for studying functionals of the type (1.5) is twofold: from the point of view of applications in continuum mechanics it is reasonable to let v depend also on higher-order gradients since their appearance in the energy usually models interfacial energies or multipolar elastic materials [71]. Another reason might be to consider deformation-gradient dependent surface loads [13]. On the other hand, not assuming a constant lower bound on v is important to consider for mathematical completeness. Additionally, integrands of the type $v(A) := \det A$, which are unbounded from below, are of crucial importance in continuum mechanics.

Meyers' main results are necessary and sufficient conditions on v so that I is weakly lower semicontinuous on $W^{k,p}(\Omega;\mathbb{R}^m)$. We review these results in Section 3. He first discusses the problem $p = +\infty$, where quasiconvexity in the highest-order gradient (cf. Theorem 3.1) turns out to be a necessary and sufficient condition for weak*-lower semicontinuity. For the case $1 \le p < +\infty$, the situation is, however, much more subtle and an additional condition (cf. Theorem 3.3 and Section 3.1) is needed.

Since the appearance of Meyers' work, significant progress has been achieved with respect to the characterization of weak lower semicontinuity of functionals of the type (1.2) or (1.5). In particular, for $k = 1$ in (1.5) the additional condition for sequential weak lower semicontinuity was characterized more explicitly and results relaxing Meyer's continuity assumptions were obtained for functionals bounded from below; cf. Section 3.

Moreover, it has been identified for which functions v the functional I in (1.5) is even weakly continuous (see Section 4) – these functions are the so-called null Lagrangians – and this knowledge led to the notion of polyconvexity (see Section 6) that is sufficient for weak lower semicontinuity and of particular importance in mathematical elasticity. In fact, quasiconvexity, which is, for a large class of integrands, the necessary and sufficient condition for weak lower semicontinuity is not well-suited for elasticity. We explain this issue in Section 7 and review some recent progress in this field. Null Lagrangians have also been identified for

[4] In the original paper [110], quasiconvexity with respect to the k-th gradient is also referred to as quasiconvexity.

functional defined on the boundary (see Section 5). Finally, we review weak lower semi-continuity results for functionals depending on maps that satisfy general differential constraints in Section 8 and we conclude with some suggestions for further reading in Section 9.

2. Notation. In this section, we summarize the notation that shall be used throughout the paper. It largely coincides with the one used in [13]. In what follows, $\Omega \subset \mathbb{R}^n$ is a bounded domain whose boundary is Lipschitz or smoother. This domain is mapped to a set in $\mathbb{R}^m$ by means of a mapping $u : \Omega \to \mathbb{R}^m$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. If $J := (j_1, \dots, j_n) \in \mathbb{N}_0^n$ and $K := (k_1, \dots, k_n) \in \mathbb{N}_0^n$ are two multiindices we define $J \pm K := (j_1 \pm k_1, \dots, j_n \pm k_n)$, further $|J| = \sum_{i=1}^n j_i$, $J! := \Pi_{i=1}^n j_i!$, and we say that $J \le K$ if $j_i \le k_i$ for all i. Then we also define $\binom{J}{K} := J!/K!/(J-K)!$, $\partial u_K^j := \frac{\partial^{k_1} \dots \partial^{k_n}}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} u^j$, $x_K = x^K := x_1^{k_1} \dots x_n^{k_n}$, and $(-D)^K := \frac{(-\partial)^{k_1} \dots (-\partial)^{k_n}}{\partial x_1^{k_1} \dots \partial x_n^{k_n}}$.

We will work with the space of matrices $X = X(n,m,k)$ with the dimension $m\binom{n+k-1}{k}$. This is the space of matrices $M = (M_K^i)$ for $1 \le i \le m$ and $|K| = k$. Similarly, $Y = Y(n,m,k)$ is a space of matrices $M = (M_K^i)$ for $1 \le i \le m$ and $|K| \le k$. Its dimension is $m\binom{n+k}{k}$. We denote the elements of $X(n,m,k)$ by A^k while the $A^{[k]} = (A, A^2, \dots, A^k)$ is an element of $Y(n,m,k)$. We use an analogous notation also for gradients; thus, if $x \in \Omega$, then $\nabla^k u(x) \in X(n,m,k)$ while $\nabla^{[k]} u(x) \in Y(n,m,k)$.

We denote by $B(x_0, r)$ the ball of origin x_0 with the radius r while $D_\rho(x_0, r)$ is the half-ball with ρ being the normal of the planar component of its boundary; i.e.

$$D_\rho(x_0, r) := \{x \in B(x_0, r) : (x - x_0) \cdot \varrho < 0\},$$

and we write $D_\rho := D_\rho(0,1)$.

We shall use the standard notation for the Lebesgue spaces $L^p(\Omega; \mathbb{R}^m)$ and Sobolev spaces $W^{k,p}(\Omega; \mathbb{R}^m)$. Moreover, $\mathrm{BV}(\Omega; \mathbb{R}^m)$ is the space of functions of a bounded variation. If $m = 1$, we may omit the target space. If Ω is a bounded open domain we denote $\mathcal{M}(\Omega)$ the space of Radon measures on Ω and $\mathcal{L}^n$ denotes the n-dimensional Lebesgue measure; cf. [76]. Moreover, $\mathcal{D}(\Omega)$ is the space of infinitely differentiable functions with compact support in Ω and its dual $\mathcal{D}'(\Omega)$ is the space of distributions.

If $n = m = 3$ and $F \in \mathbb{R}^{3\times3}$ the cofactor matrix $\mathrm{Cof}F \in \mathbb{R}^{3\times3}$ is a matrix whose entries are signed subdeterminants of 2×2 submatrices of F. More precisely, $[\mathrm{Cof}F]_{ij} := (-1)^{i+j} \det F'_{ij}$ where F'_{ij} for $i, j \in \{1,2,3\}$ is a submatrix of F obtained by removing the i-th row and j-th column. If F is invertible, we have $\mathrm{Cof}F = (\det F) F^{-\top}$. Rotation matrices with determinants equal one are denoted $\mathrm{SO}(n)$ while orthogonal matrices with determinants ±1 are denote by $\mathrm{O}(n)$.

3. A review of Meyers' results. Meyers studies in [110] weak lower semicontinuity of (1.5) on a fairly general class of integrands. In particular, for weak lower semicontinuity on $W^{k,p}(\Omega, \mathbb{R}^m)$ with $1 \le p < +\infty$ he introduces the class $\mathcal{F}_p(\Omega)$ (cf. [110, Def. 4] and Definition 3.2 below). On $W^{k,\infty}(\Omega, \mathbb{R}^m)$, any continuous integrand is admitted and Meyers proves an analogous result to the one found in the original work of Morrey for $k = 1$ [114]:

Theorem 3.1. *Let Ω be a bounded domain and v a continuous function. Then I from* (1.5) *is weakly* lower semicontinuous on $W^{k,\infty}(\Omega; \mathbb{R}^m)$ if and only if it is k-quasiconvex.*

Nevertheless, when it comes to the case of $W^{k,p}(\Omega; \mathbb{R}^m)$ with $p \ge 1$ finite the situation is substantially more involved; in particular, because the considered integrands are not bounded from below. This can be seen from the definition of the class $\mathcal{F}_p(\Omega)$.

Definition 3.2 (Class $\mathcal{F}_p(\Omega)$). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. A continuous integrand $v : \Omega \times Y(n,m,k) \to \mathbb{R}$ is said to be in the class $\mathcal{F}_p(\Omega)$ for $1 \le p < +\infty$ if ($C > 0$ is a constant depending only on v)*

(i) $v(x, A^{[k]}) \le C\big(1 + |A^{[k]}|\big)^p$,

(ii) $|v(x, A^{[k]} + B^{[k]}) - v(x, A^{[k]})| \le C\big(1 + |A^{[k]}| + |B^{[k]}|\big)^{p-\gamma} |B^{[k]}|^\gamma$, *where* $0 < \gamma \le 1$,

(iii) $|v(x+y, A^{[k]}) - v(x, A^{[k]})| \le (1 + |A^{[k]}|)^p \eta(|y|)$ *with* $\eta : [0; +\infty) \to [0; +\infty)$ *continuous, increasing and vanishing at zero.*

Indeed, when setting $A^{[k]} = 0$ in (ii) we get that $|v(x, B^{[k]})| \le C(1 + |B^{[k]}|)^p$ and thus the class $\mathcal{F}_p(\Omega)$ contains also noncoercive integrands and, in particular, those which decay as $-|\cdot|^p$.

This decay is problematic with respect to weak lower semicontinuity, because then, along concentrating sequences of gradients[5], energy may be *gained* and hence the lower semicontinuity is destroyed. On the boundary of the domain this effect cannot be excluded by quasiconvexity as the following example shows.

Example 3.1 (following [97], [7]). *Choose $\Omega = (0,1)$ and define a sequence on* $\mathrm{BV}((0,1))$ *defined through $u_n := \chi_{(0,\frac{1}{n})}$, i.e. the characteristic function of $(0,\frac{1}{n})$, so that $Du_n = -\delta_{\frac{1}{n}}$. Further let us choose and $v(x,s,A) := A$; i.e. v is a linear function and so quasiconvex. Then the functional*

$$I(u) = \int_\Omega v(x, u(x), A)\, \mathrm{d}Du(x),$$

which is a BV*-equivalent* (1.5) *with $k = 1$, fulfills $I(u_n) = -1$ for all n, but $u_n \overset{*}{\rightharpoonup} 0$ in* $\mathrm{BV}((0,1))$ *and $I(0) = 0 > -1$.*

The example illustrates the above mentioned effect that a sequence concentrating on the boundary (such as u_n) may actually lead to an energy gain in the limit. While the above example is in $\mathrm{BV}((0,1))$*, because this allows us to take a linear, and thus a particularly easy, integrand in* (1.5) *appropriate nonlinear integrands lead to the same effect in $W^{k,p}(\Omega;\mathbb{R}^m)$ with $p > 1$; cf. Example 3.2 below.*

Meyers hence introduced an additional condition to ensure sequential weak lower semicontinuity and proved the following.

Theorem 3.3. *Let Ω be a bounded domain and $v \in \mathcal{F}_p(\Omega)$. Then I from* (1.5) *is weakly lower semicontinuous on $W^{k,p}(\Omega;\mathbb{R}^m)$ with $1 \le p < \infty$ if and only if the following two conditions hold simultaneously:*

(i) $v(x, A^{[k-1]}, \cdot)$ is k-quasiconvex for all values of $(x, A^{[k-1]})$,

(ii) $\liminf_{j\to\infty} I(u_j, \Omega') \ge -\mu(\mathcal{L}^n(\Omega'))$ for every subdomain $\Omega' \subset \Omega$ and every sequence $\{u_j\}_{j\in\mathbb{N}} \subset W^{k,p}(\Omega;\mathbb{R}^m)$ such that $u_j = u$ on $\Omega \setminus \Omega'$ and $u_j \rightharpoonup u$ in $W^{k,p}(\Omega;\mathbb{R}^m)$. Here μ is an increasing continuous function with $\mu(0) = 0$ which only depends on u and on $\limsup_{j\to\infty} \|u_j\|_{W^{k,p}(\Omega;\mathbb{R}^m}$.

Above, $I(\cdot, \Omega')$ denotes the functional I when the integration domain Ω is replaced by Ω'.

We immediately see that condition (ii) is satisfied, for example, if $v \ge 0$. To see why this condition excludes the effect of concentrations on the boundary, take a sequence of "rings" Ω'_k around the boundary of Ω. The measure of such rings converges to zero and so, also $\mu(\mathcal{L}^n(\Omega'_k))$ tends to zero as $k \to \infty$. But if $\{|\nabla^k u_j|^p\}$ is a concentrating sequence which converges to a measure supported on $\partial\Omega$ then $I(u_j, \Omega'_k)$ may take a fixed negative value and thus it *violates* condition (ii) from Theorem 3.3.

Since condition (ii) in Theorem 3.3 is connected with concentrations on the boundary, Meyers conjectured [110, p. 146] that it can be dropped if $\partial\Omega$ is "smooth enough" or a "smooth enough" function is prescribed on the boundary *as the datum*. The second part of the conjecture turned out to be true in the following special cases: if $k = 1$ in (1.5)(see [110, Thm. 5] and Thm. 3.7) or if the integrand in (1.5) depends just on the highest gradient (see end of Section 8). However, the general case is still open:

Open problem 3.4. *Is the functional* (1.5) *weakly lower semicontinuous along sequences with fixed Dirichlet boundary data if v is a general function in the class $\mathcal{F}_p(\Omega)$?*

The first part of the conjecture of Meyers turned out *not* to hold as the following example illustrates:

Example 3.2 (See [18]). *Let $n = m = p = 2$, $0 < a < 1$, $\Omega := (0,a)^2$ and for $x \in \Omega$ define*

$$u_j(x_1, x_2) = \frac{1}{\sqrt{j}}(1 - |x_2|)^j (\sin jx_1, \cos jx_1).$$

We see that $\{u_j\}_{j\in\mathbb{N}}$ converges weakly in $W^{1,2}(\Omega;\mathbb{R}^2)$ as well as pointwise to zero. Moreover, we calculate for $j \to \infty$

$$\int_0^a \int_0^a \det \nabla u_j(x)\, \mathrm{d}x \to \frac{-a}{2} < 0\ .$$

[5]We say that a sequence bounded in L^1 is concentrating if it converges weak* in measures but not weakly in L^1.

Hence, we see that $I(u) := \int_\Omega \det \nabla u(x)\,\mathrm{d}x$ *is not weakly lower semicontinuous in* $W^{1,2}(\Omega;\mathbb{R}^2)$. *This example can be generalized to arbitrary dimensions* $m = n \geq 2$. *Indeed, take* $u \in W_0^{1,n}(B(0,1);\mathbb{R}^n)$ *and extend* u *by zero to the whole* $\mathbb{R}^n$. *We get that* $\int_{B(0,1)} \det \nabla u(x)\,\mathrm{d}x = 0$ *because of the zero Dirichlet boundary conditions on* $\partial B(0,1)$. *Take* $\varrho \in \mathbb{R}^n$, *a unit vector, such that* $\int_{D_\varrho} \det \nabla u(x)\,\mathrm{d}x < 0$. *Notice that this condition can be fulfilled, if we take* u *suitably.*

Denote $u_j(x) := u(jx)$ *for all* $j \in \mathbb{N}$; *then* $u_j \rightharpoonup 0$ *in* $W^{1,n}(B(0,1);\mathbb{R}^n)$ *(even in measure) but also* $\int_{D_\varrho} \det \nabla u_j(x)\,\mathrm{d}x \to \int_{D_\varrho} \det \nabla u(x)\,\mathrm{d}x < 0$ *by our construction. The same conclusion can be drawn if we take* $\Omega \subset \mathbb{R}^n$ *with arbitrarily smooth boundary and such that* $0 \in \partial\Omega$. *Let* ϱ *be the outer unit normal to* $\partial\Omega$ *at zero. Then we have for the same sequence as before*

$$\lim_{j\to\infty} \int_\Omega \det \nabla u_j(x)\,\mathrm{d}x = \lim_{j\to\infty} \int_{B(0,1)\cap\Omega} \det \nabla u_j(x)\,\mathrm{d}x$$
$$= \lim_{j\to\infty} \int_{B(0,1)\cap\Omega} j^n \det \nabla (u(jx))\,\mathrm{d}x = \int_{D_\varrho} \det \nabla u(y)\,\mathrm{d}y < 0\ .$$

3.1. Understanding condition (ii) in Theorem 3.3. Condition (ii) in Theorem 3.3 is rather implicit and thus hard to verify. Nevertheless, as Examples 3.1 and 3.2 show, it should be linked to concentrations on the boundary of the domain. To our best knowledge, this link has been fully drawn only in the case $k = 1$ and for integrands $v(x,u,\nabla u) := v(x,\nabla u)$ in (1.5).

First, we present a result showing that indeed concentrations are the key issue.

Theorem 3.5 (adapted from [85]). *Let* $v \in C(\bar\Omega \times \mathbb{R}^{m\times n})$, $|v| \leq C(1+|\cdot|^p)$, $C > 0$, $v(x,\cdot)$ *quasiconvex for all* $x \in \bar\Omega$, *and* $1 < p < +\infty$. *Then the functional*

$$I(u) := \int_\Omega v(x,\nabla u(x))\,\mathrm{d}x$$

is sequentially weakly lower semicontinuous on $W^{1,p}(\Omega;\mathbb{R}^m)$ *if and only if for any bounded sequence* $\{u_j\} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ *such that* $\nabla u_j \to 0$ *in measure we have* $\liminf_{j\to\infty} I(u_j) \geq I(0)$.

Recall that two effects may cause a sequence $\{u_j\} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ to converge weakly but not strongly to some limit function u: *oscillations* and *concentrations*. The above theorem then states that a functional with a quasiconvex integrand is lower semicontinuous along any weakly converging sequences if it is so along *purely concentrating* ones. Indeed, realize that a purely concentrating sequence converges to zero in measure.

The proof of Theorem 3.5 relies on two main ingredients. The first one is the so-called p-Lipschitz continuity of quasiconvex functions. It asserts that if $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is quasiconvex and $|f| \leq C(1+|\cdot|^p)$ for some $C > 0$, and $1 \leq p < +\infty$ then there is another constant $\alpha \geq 0$ such that for all $A, B \in \mathbb{R}^{m\times n}$

$$|f(A) - f(B)| \leq \alpha(1 + |A|^{p-1} + |B|^{p-1})|A - B|\ . \tag{3.1}$$

Remark 3.3. *The p-Lipschitz continuity holds even if* v *is only separately convex, i.e. convex along Cartesian axes in* $\mathbb{R}^{m\times n}$. *Various variants are proven e.g. in [66, 107] and in [42]. It shows that quasiconvex functions satisfying the mentioned bound are locally Lipschitz.*

The second ingredient is the decomposition lemma due to Kristensen [95] and Fonseca, Müller, and Pedregal [61].

Lemma 3.6 (Decomposition lemma). *Let* $1 < p < +\infty$ *and* $\Omega \subset \mathbb{R}^n$ *be an open bounded set and let* $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ *be bounded. Then there is a subsequence* $\{u_j\}_{j\in\mathbb{N}}$ *and a sequence* $\{z_j\}_{j\in\mathbb{N}} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ *such that*

$$\lim_{j\to\infty} \mathcal{L}^n(\{x \in \Omega;\ z_j(x) \neq u_j(x) \text{ or } \nabla z_j(x) \neq \nabla u_j(x)\}) = 0 \tag{3.2}$$

and $\{|\nabla z_j|^p\}_{j\in\mathbb{N}}$ *is relatively weakly compact in* $L^1(\Omega)$.

This lemma allows us to find, for a general sequence bounded in $W^{1,p}(\Omega;\mathbb{R}^m)$, another one, called $\{z_j\}\subset W^{1,p}(\Omega;\mathbb{R}^m)$, whose gradients are p-equiintegrable, i.e., for which $\{|\nabla z_j|^p\}$ is relatively weakly compact in $L^1(\Omega)$ and so it is a *purely oscillating* sequence. Thus, we decompose $u_j = z_j + w_j$, and $\{|\nabla w_j|^p\}_{j\in\mathbb{N}}$ tends to zero in measure for $j\to\infty$; i.e., it is a *purely concentrating* sequence. Roughly speaking, this means that for any weakly converging sequence in $W^{1,p}(\Omega;\mathbb{R}^m)$, $p>1$, we can be decompose the sequence of gradients into a purely oscillating and a purely concentrating one. Note, however, that due to (3.2), this decomposition is very special. Notice that Lemma 3.6 inherited its name exactly from this decomposition.

Hence, denoting R_j the set appearing in (3.2) we get

$$\begin{aligned}
&\left|\int_\Omega v(x,\nabla w_j(x))\,\mathrm{d}x - \int_\Omega (v(x,\nabla u_j(x)) - v(x,\nabla z_j(x)))\,\mathrm{d}x\right|\\
&\le \left(\int_{R_j} |v(x,\nabla u_j(x)-\nabla z_j(x)) - v(x,\nabla u_j(x))|\,\mathrm{d}x + \int_{R_j} |v(x,\nabla z_j(x))|\,\mathrm{d}x\right) \qquad (3.3)\\
&\le \alpha\int_{R_j} \left[(1+|\nabla u_j(x)-\nabla z_j(x)|^{p-1}+|\nabla u_j|^{p-1})|\nabla z_j(x)| + (1+|\nabla z_j|^p)\right]\,\mathrm{d}x\\
&\le c\left(\left(\int_{R_j}|\nabla z_j(x)|^p\,\mathrm{d}x\right)^{1/p} + \int_{R_j} 1+|\nabla z_j(x)|^p\,\mathrm{d}x + \int_{R_j}|\nabla z_j(x)|\,\mathrm{d}x\right)
\end{aligned}$$

for a constant $c>0$ (which may depend also on $\sup_j\|\nabla u_j\|_{L^p(\Omega)}$ and $\sup_j\|\nabla z_j\|_{L^p(\Omega)}$). The last term goes to zero as $j\to\infty$ because $\{|\nabla z_j|^p\}$ is relatively weakly compact in $L^1(\Omega)$ and $\mathcal{L}^n(R_j)\to 0$ as $j\to\infty$. This calculation shows that for v quasiconvex we can separate oscillation and concentration effects of $\{\nabla u_j\}$. Thus, we get for (non-relabeled) subsequences

$$\begin{aligned}
&\lim_{j\to\infty}\int_\Omega v(x,\nabla u_j(x))\,\mathrm{d}x = \lim_{j\to\infty}\int_\Omega v(x,\nabla z_j(x))\,\mathrm{d}x + \lim_{j\to\infty}\int_\Omega v(x,\nabla w_j(x))\,\mathrm{d}x\\
&\ge \int_\Omega v(x,\nabla u(x))\,\mathrm{d}x + \lim_{j\to\infty}\int_\Omega v(x,\nabla w_j(x))\,\mathrm{d}x\ . \qquad (3.4)
\end{aligned}$$

The inequality follows from Theorem 3.3 because $\{|\nabla z_j|^p\}_{j\in\mathbb{N}}$ is uniformly integrable, hence (ii) holds automatically. Therefore, sequential weak lower semicontinuity of the functional is equivalent to

$$\lim_{j\to\infty}\int_\Omega v(x,\nabla w_j(x))\,\mathrm{d}x \ge \int_\Omega v(x,0)\,\mathrm{d}x = I(0)\ .$$

We now indicate why quasiconvexity is capable of preventing concentrations *in the domain* Ω from breaking weak lower semicontinuity. Indeed, let $\zeta\in\mathcal{D}(\Omega)$, $0\le\zeta\le 1$ and take a quasiconvex function $v:\mathbb{R}^{m\times n}\to\mathbb{R}$ such that $|v(A)|\le C(1+|A|^p)$ for some $C>0$ and all $A\in\mathbb{R}^{m\times n}$ with $p>1$. We calculate using Definition 1.2 for $A:=0$, the sequence $\{w_j\}$ from the Decomposition Lemma 3.6, and (3.1)

$$\begin{aligned}
&|\Omega|v(0)\le\int_\Omega v(\nabla(\zeta(x)w_j(x)))\,\mathrm{d}x = \int_\Omega v(\zeta(x)\nabla w_j(x) + w_j(x)\otimes\nabla\zeta(x))\,\mathrm{d}x\\
&\le \int_\Omega v(\zeta(x)\nabla w_j(x))\,\mathrm{d}x + \alpha\int_\Omega (1+|\zeta(x)\nabla w_j(x)+w_j(x)\otimes\nabla\zeta(x)|^{p-1})|w_j(x)\otimes\nabla\zeta(x)|\,\mathrm{d}x\\
&+\alpha\int_\Omega(|\zeta(x)\nabla w_j(x)|^{p-1})|w_j(x)\otimes\nabla\zeta(x)|\,\mathrm{d}x \le \int_\Omega v(\zeta(x)\nabla w_j(x))\,\mathrm{d}x\\
&+\alpha\int_\Omega(1+2^{p-1})|\zeta(x)\nabla w_j(x)|^{p-1})|w_j(x)\otimes\nabla\zeta(x)|\,\mathrm{d}x\\
&+\alpha\int_\Omega(2^{p-1}|w_j(x)\otimes\nabla\zeta(x)|^{p-1})|w_j(x)\otimes\nabla\zeta(x)|\,\mathrm{d}x\\
&\le\int_\Omega v(\zeta(x)\nabla w_j(x))\,\mathrm{d}x + \alpha(1+2^{p-1})\|\zeta\nabla w_j\|^{p-1}_{L^p(\Omega;\mathbb{R}^{m\times n})}\|w_j\otimes\nabla\zeta\|_{L^p(\Omega;\mathbb{R}^m)}\\
&+2^{p-1}\alpha\|w_j\otimes\nabla\zeta\|^p_{L^p(\Omega;\mathbb{R}^n)}\ .
\end{aligned}$$

Since $w_j \to 0$ strongly in $L^p(\Omega;\mathbb{R}^n)$ and $\{\nabla w_j\}_{k\in\mathbb{N}}$ is bounded in $L^p(\Omega;\mathbb{R}^{m\times n})$ the last two terms tend to zero if $j\to\infty$. Therefore, we have

$$\mathcal{L}^n(\Omega)v(0) \le \liminf_{j\to\infty}\int_\Omega v(\zeta(x)\nabla w_j(x))\,\mathrm{d}x \ . \tag{3.5}$$

Let $|\nabla w_j|^p \overset{*}{\rightharpoonup} \sigma$ in $\mathcal{M}(\bar\Omega)$ for a (non-relabeled) subsequence. Assume that $\sigma(\partial\Omega)=0$. We continue with the following estimate

$$\begin{aligned}
&\lim_{j\to\infty}\int_\Omega v(\zeta(x)\nabla w_j(x))\,\mathrm{d}x \le \lim_{j\to\infty}\int_\Omega v(\nabla w_j(x))\,\mathrm{d}x\\
&+\alpha\lim_{j\to\infty}\int_\Omega (1-\zeta(x))(1+\zeta^{p-1}(x))|\nabla w_j(x)|^p\,\mathrm{d}x+\alpha\lim_{j\to\infty}\int_\Omega(1-\zeta(x))|\nabla w_j(x)|\,\mathrm{d}x\\
&=\lim_{j\to\infty}\int_\Omega v(\nabla w_j(x))\,\mathrm{d}x+\alpha\int_\Omega(1-\zeta(x))(1+\zeta^{p-1}(x))\sigma(\mathrm{d}x)\ .
\end{aligned}\tag{3.6}$$

Now, we construct a sequence $\{\zeta_j\}_{j\in\mathbb{N}}\subset\mathcal{D}(\Omega)$, satisfying $0\le\zeta_j\le 1$ that pointwise tends to the characteristic function of Ω, χ_Ω, σ-a.e. Taking into account (3.5) and (3.6), we have by the Lebesgue's dominated convergence theorem

$$\mathcal{L}^n(\Omega)v(0)\le\lim_{j\to\infty}\int_\Omega v(\nabla w_j(x))\,\mathrm{d}x\ .$$

Hence, weak lower semicontinuity is preserved. This reasoning, however, clearly breaks if $\partial\Omega$ is not a σ-null set. Nevertheless, not every boundary concentration is fatal for weak lower semicontinuity. If Ω is a Lipschitz domain, we can extend $\{w_j\}_{j\in\mathbb{N}}$ to a larger domain $\tilde\Omega\supset\Omega$, and denote this extension $\{\tilde w_j\}\subset W^{1,p}(\tilde\Omega;\mathbb{R}^m)$, in such a way that $\partial\Omega$ is now in the interior of $\tilde\Omega$ and the extension satisfies zero Dirichlet boundary conditions on $\partial\tilde\Omega$. Then the above calculation holds if we replace Ω by $\tilde\Omega$ and $\{w_j\}$ by $\{\tilde w_j\}$. Arguing heuristically, concentrations at $\partial\Omega$ are influenced by interior concentrations coming from Ω and exterior ones arriving from $\tilde\Omega\setminus\Omega$. If added to each other, they are *harmless* for weak lower semicontinuity. For instance, if $\tilde w_j=0$ for all $j\in\mathbb{N}$ outside Ω then exterior concentrations do not exist at all. Hence, the interior one cannot spoil weak lower semicontinuity. That is, roughly speaking, why Dirichlet boundary conditions suffice to ensure (ii) in Theorem 3.3 at least if $k=1$. More generally, Dirichlet boundary conditions can be replaced by the requirement that $\{|\tilde w_j|^p\}_{j\in\mathbb{N}}$ is equiintegrable in $L^1(\tilde\Omega\setminus\Omega)$.

The next theorem shows that weak lower semicontinuity of (1.5) for quasiconvex v can be proved, for example, if the negative part of v has sub-critical growth or Dirichlet boundary conditions are fixed.

THEOREM 3.7 (taken from [85]). *Let the assumptions of Theorem 3.5 hold. Let further $\{u_j\}\subset W^{1,p}(\Omega;\mathbb{R}^m)$, $u_j\rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$ and at least one of the following conditions be satisfied:*

(i) for any subsequence of $\{u_j\}$ (not relabeled) such that $|\nabla u_j|^p\overset{*}{\rightharpoonup}\sigma$ in $\mathcal{M}(\bar\Omega)$ it holds that $\sigma(\partial\Omega)=0$,

(ii) $\lim_{|A|\to\infty}\frac{v^-(x,A)}{1+|A|^p}=0$ for all $x\in\bar\Omega$ where $v^-:=\max\{0,-v\}$,

(iii) $u_j=u$ on $\partial\Omega$ for any $j\in\mathbb{N}$ and Ω is Lipschitz.
Then $I(u)\le\liminf_{j\to\infty}I(u_j)$.

Notice that *(ii)* is satisfied for example, if $v\ge 0$ or if $v^-\le C(1+|\cdot|^q)$ for some $1\le q<p$ in which case $-C(1+|A|^q)\le v(s)\le C(1+|A|^p)$, $C>0$. This result can be found e.g. in [42].

In 1990, Ball and Zhang [20] considered the following bound on a Carathéodory integrand

$$|v(x,s,A)|\le a(x)+C(|s|^p+|A|^p)\ , \tag{3.7}$$

where $C>0$ and $a\in L^1(\Omega)$. Notice that contrary to the above considered results, (3.7) allows the integrand v to also depend on u not just the gradient of u. Under (3.7), we cannot expect weak lower semicontinuity of I along generic sequences. Indeed, they proved the following weaker result.

THEOREM 3.8 (Ball and Zhang [20]). *Let $1 \le p < +\infty$, $u_k \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$, $v(x,s,\cdot)$ be quasiconvex for all $s \in \mathbb{R}^m$ and almost all $x \in \Omega$, and let (3.7) hold. Then there exist a sequence of sets $\{\Omega_j\}_{j\in\mathbb{N}} \subset \Omega$ such that $\Omega_{j+1} \subseteq \Omega_j$ for all $j \ge 1$, and $\lim_{j\to\infty} \mathcal{L}^n(\Omega_j) = 0$ such that for all $j \ge 1$*

$$\int_{\Omega\setminus\Omega_j} v(x,u(x),\nabla u(x))\,\mathrm{d}x \le \liminf_{k\to\infty} \int_{\Omega\setminus\Omega_j} v(x,u_k(x),\nabla u_k(x))\,\mathrm{d}x \ . \tag{3.8}$$

We immediately see that if $v \ge 0$ then the statement holds for $\Omega_1 = \Omega_j = \emptyset$, i.e., that weak lower semicontinuity is recovered. The sets $\{\Omega_j\}$ that must be removed (or bitten) from Ω are the sets where possible concentration effects of the bounded sequence $\{v(x,u_k,\nabla u_k)\}_{k\in\mathbb{N}} \subset L^1(\Omega)$ take place. Thus, $\{\Omega_j\}$ depends on the sequence $\{u_k\}$ itself and Ω_j are not known a-priori. The main tool of the proof is the Biting Lemma due to Chacon [33, 19].

LEMMA 3.9 (Biting lemma). *Let $\Omega \subset \mathbb{R}^n$ be a bounded measurable set. Let $\{z_k\} \subset L^1(\Omega;\mathbb{R}^m)$ be bounded. Then there is a (non-relabeled) subsequence of $\{z_k\}$'s, $z \in L^1(\Omega;\mathbb{R}^m)$ and a nonincreasing sequence of sets $\{\Omega_j\}_{j\in\mathbb{N}} \subset \Omega$ with $\mathcal{L}^n(\Omega_j) \to 0$ for $j \to \infty$ such that $z_k \rightharpoonup z$ in $L^1(\Omega \setminus \Omega_j;\mathbb{R}^m)$ for $k \to \infty$ and any $j \in \mathbb{N}$.*

Let us return to the issue of understanding better condition (ii) in Theorem 3.3. It has been identified in [98] that a suitable *growth* from below of the whole functional in (1.5) (which does not necessarily imply a lower bound on the integrand v itself) equivalently replaces this condition. First, let us illustrate that some form of boundedness from below is indeed necessary for weak lower semicontinuity.

EXAMPLE 3.4. *Take $u \in W^{1,p}_0(B(0,1);\mathbb{R}^m)$ $(1 < p < \infty)$ and extend it by zero to the whole of $\mathbb{R}^n$. Define for $x \in \mathbb{R}^n$ and $j \in \mathbb{N}$ $u_j(x) = j^{\frac{n-p}{p}} u(jx)$, i.e., $u_j \rightharpoonup 0$ in $W^{1,p}(B(0,1);\mathbb{R}^m)$ and consider a smooth domain $\Omega \in \mathbb{R}^n$ such that $0 \in \partial\Omega$; denote by ρ the outer unit normal to $\partial\Omega$ at 0. Moreover, take a function $v : \mathbb{R}^{m\times n} \to \mathbb{R}$ that is positively p-homogeneous, i.e., $v(\alpha\xi) = \alpha^p v(\xi)$ for all $\alpha \ge 0$. If*

$$I(u) = \int_\Omega v(\nabla u(x))\,\mathrm{d}x$$

is weakly lower semicontinuous on $W^{1,p}(\Omega;\mathbb{R}^m)$ then

$$\begin{aligned} 0 = I(0) &\le \liminf_{j\to\infty} \int_\Omega v(\nabla u_j(x))\,\mathrm{d}x = \liminf_{j\to\infty} \int_{B(0,1/j)\cap\Omega} v(\nabla u_j(x))\,\mathrm{d}x \\ &= \liminf_{j\to\infty} \int_{B(0,1/j)\cap\Omega} j^n v(\nabla u(jx))\,\mathrm{d}x = \int_{D_\rho} v(\nabla u(y))\,\mathrm{d}y \ . \end{aligned} \tag{3.9}$$

Thus, we see that

$$0 \le \int_{D_\rho} v(\nabla u(y))\,\mathrm{d}y \tag{3.10}$$

for all $u \in W^{1,p}_0(B(0,1);\mathbb{R}^m)$ forms a necessary condition for weak lower semicontinuity of I whenever v is positively p-homogeneous.*

For functions that are not p-homogeneous, S. Krömer [98] generalized (3.10) as follows.

DEFINITION 3.10 (following [98][6]). *Assume that $\Omega \subset \mathbb{R}^n$ has a smooth boundary and let ρ be a unit outer normal to $\partial\Omega$ at x_0. We say that a function $v : \Omega \times \mathbb{R}^{m\times n} \to \mathbb{R}$ is of* p-quasi-subcritical growth from below *(p-qsb) if for every $x_0 \in \partial\Omega$*

for every $\varepsilon > 0$, there exists $C_\varepsilon \ge 0$ such that

$$\int_{D_\rho(x_0,1)} v(x_0,\nabla u(x))\mathrm{d}x \ge -\varepsilon \int_{D_\rho(x_0,1)} |\nabla u(x)|^p \mathrm{d}x - C_\varepsilon \quad \text{for all } u \in W^{1,p}_0(B(0,1);\mathbb{R}^m), \tag{3.11}$$

[6] In [98] this condition is actually not referred to as p-quasi-subcritical growth from below but is introduced in Theorem 1.6 (ii).

It has been proved in [98] that the p-quasi-subcritical growth from below of the function $v := v(x, \nabla u)$ equivalently replaces (ii) in Theorem 3.3.

Notice that (3.11) is expressed only in terms of v and that it is local in x. It also again shows that, at least in the case when v does depend only on the first gradient of u but not on u itself, only *concentrations at the boundary* may interfere with weak lower semicontinuity of functionals involving quasiconvex functions. This means that concentrations inside of the domain Ω are already "taken care of" by the quasiconvexity itself.

REMARK 3.5. *Let us realize that* (3.11) *implies* (3.10) *if v is positively p-homogeneous and independent of x. To this end, we use, for $t \geq 0$, $u = t\tilde{u}$ in* (3.11) *to see that*

$$0 \leq \frac{1}{t^p}\left(\int_{D_\varrho(x_0,1)} v(t\nabla\tilde{u}(x))\,dx + \varepsilon|t\nabla\tilde{u}(x)|^p \mathrm{d}x + C_\varepsilon\right).$$

Letting now $t \to \infty$ gives that $C_\varepsilon = 0$. Then, we may also send $\varepsilon \to 0$ to get (3.10).

Since only concentration effects play a role for (ii) in Theorem 3.3, it is natural to expect that weak lower semicontinuity can be linked to properties of the so-called *recession function* of the function v, if it admits one. Recall, that we say that the functions $v_\infty : \overline{\Omega} \times \mathbb{R}^{m\times n} \to \mathbb{R}$ is a recession function for $v : \bar{\Omega} \times \mathbb{R}^{m\times n} \to \mathbb{R}$ if for all $x \in \Omega$

$$\lim_{|A|\to\infty} \frac{v(x,A) - v_\infty(x,A)}{|A|^p} = 0.$$

Thus, informally speaking, the recession function describes the behavior of v at "infinitely large matrices". Note that v_∞ is necessarily positively p-homogeneous; i.e. $v_\infty(x, \lambda A) = \lambda^p v_\infty(x, A)$ for all $\lambda \geq 0$, all $x \in \bar{\Omega}$, and all $A \in \mathbb{R}^{m\times n}$.

It follows from Remark 3.9 in [98] that if v admits a recession function, then quasi-subcritical growth from below is equivalent to (3.10) for v_∞.

Since weak lower semicontinuity is connected to quasiconvexity and to condition (ii) in Theorem 3.3 which is connected to effects at the boundary, it is reasonable to ask whether the two ingredients can be combined. Indeed, so-called *quasiconvexity at the boundary* was introduced in [15] to study necessary conditions satisfied by local minimizers of variational problems – we also refer to [144, 112, 141] where this condition is analyzed, too. In order to define quasiconvexity at the boundary, we put for $1 \leq p \leq +\infty$

$$W^{1,p}_{\Gamma_\rho}(D_\rho; \mathbb{R}^m) := \{u \in W^{1,p}(D_\rho; \mathbb{R}^m);\ u = 0 \text{ on } \partial D_\rho \setminus \Gamma_\rho\}\ , \tag{3.12}$$

where Γ_ρ is the planar part of ∂D_ρ.

DEFINITION 3.11 (taken from [112]). [7] *Let $\varrho \in \mathbb{R}^n$ be a unit vector. A function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is called quasiconvex at the boundary at the point $A \in \mathbb{R}^{m\times n}$ with respect to ρ if there is $q \in \mathbb{R}^m$ such that for all $\varphi \in W^{1,\infty}_{\Gamma_\rho}(D_\rho; \mathbb{R}^m)$ it holds*

$$\int_{\Gamma_\rho} q \cdot \varphi(x)\,\mathrm{d}S + f(A)\mathcal{L}^n(D_\varrho) \leq \int_{D_\rho} f(A + \nabla\varphi(x))\,\mathrm{d}x\ . \tag{3.13}$$

Let us remark that, analogously to quasiconvexity, we may generalize quasiconvexity at the boundary to $W^{1,p}$-quasiconvexity at the boundary (for $1 < p < \infty$) by using all $u \in W^{1,p}_{\Gamma_\varrho}(D_\rho; \mathbb{R}^m)$ as test functions in (3.13). For functions with p-growth these two notions coincide.

REMARK 3.6. *Let us give an intuition on the above definition. Take a smooth convex function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ and $\varphi \in W^{1,\infty}_{\Gamma_\varrho}(D_\rho; \mathbb{R}^m)$. Then we know that*

$$f(A + \nabla\varphi(x)) \geq f(A) + \frac{\partial f}{\partial A}(A) : \nabla\varphi(x);$$

[7] The original definition in [15] considers the case $q := 0$.

integrating this expression over Ω then gives

$$\int_\Omega f(A+\nabla\varphi(x))\mathrm{d}x \geq \int_\Omega f(A) + \frac{\partial f}{\partial A}(A):\nabla\varphi \mathrm{d}x = \mathcal{L}^n(\Omega)f(A) + \int_{\partial\Omega}\Big(\frac{\partial f}{\partial A}(A)\rho\Big)\cdot\varphi\mathrm{d}S,$$

where ρ is the outer normal to $\partial\Omega$. Now when setting $q := \frac{\partial f}{\partial A}(A)$ we obtained the definition of the quasi-convexity at the boundary. Notice also that if φ is zero at the whole boundary we recover the definition of classical quasiconvexity, too.

Remark 3.7. *It is possible to work with more general domains that half-balls in Definition 3.11; namely with so-called* standard boundary domains. *We say that $\tilde D_\rho$ is a standard boundary domain with the normal ρ if there is $a \in \mathbb{R}^n$ such that $\tilde D_\rho \subset H_{a,\rho} := \{x \in \mathbb{R}^n;\ \rho\cdot x < a\}$ and the $(n-1)$- dimensional interior Γ_ρ of $\partial\tilde D_\rho \cap \partial H_{a,\rho}$ is nonempty. Roughly speaking, this means that the boundary of $\tilde D_\rho$ should contain a planar part.*

As with standard quasiconvexity, if (3.13) holds for one standard boundary domain it holds for other standard boundary domains, too.

Remark 3.8. *If $p > 1$, $v : \mathbb{R}^{m\times n} \to \mathbb{R}$ is positively p-homogeneous, continuous, and $W^{1,p}$-quasiconvex at the boundary at $(0,\rho)$ then $q = 0$ in (3.13). Indeed, we have $v(0) = 0$ and suppose, by contradiction, that $\int_{D_\rho} v(\nabla u(x))\,\mathrm{d}x < 0$ for some $u \in W^{1,\infty}_{\Gamma_\rho}(D_\rho;\mathbb{R}^m)$. By (3.13), we must have for all $\lambda > 0$*

$$0 \leq \lambda^p \int_{D_\rho} v(\nabla u(x))\,\mathrm{d}x - \lambda\int_{\Gamma_\rho} q\cdot u(x)\,\mathrm{d}S\ .$$

However, this is not possible for $\lambda > 0$ large enough and therefore for all $\varphi \in W^{1,\infty}_{\Gamma_\rho}(D_\rho;\mathbb{R}^m)$ it holds that $\int_{D_\rho} v(\nabla\varphi(x))\,\mathrm{d}x \geq 0$. Thus, we can take $q = 0$.

From the above remark and from (3.10), we have the following lemma:

Lemma 3.12. *If a function $v : \mathbb{R}^{m\times n} \to \mathbb{R}$ is $W^{1,p}$-quasiconvex at the boundary at zero and every $\varrho \in \mathbb{R}^n$, a unit normal vector to $\partial\Omega$, then it is also of p-subcritical growth from below. The two notions become equivalent if v is also positively p-homogeneous. Here Ω must have a smooth boundary.*

3.2. Integrands bounded from below. As already mentioned, condition (ii) in Theorem 3.3 is automatically satisfied if the integrand in (1.5) is bounded from below. Moreover, in this case, the continuity assumptions stated in Definition 3.2 can be considerably weakened. In fact, the Carathéodory property is sufficient in case $k = 1$ in (1.5) as the following famous result due to E. Acerbi and N. Fusco [1] shows.

Theorem 3.13 (Acerbi and Fusco [1]). *Let $k = 1$, $\Omega \subset \mathbb{R}^n$ be an open, bounded set, and let $v : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to [0;+\infty)$ be a Carathéodory integrand, i.e., $v(\cdot, s, A)$ is measurable for all $(s,A) \in \mathbb{R}^m \times \mathbb{R}^{m\times n}$ and $v(x,\cdot,\cdot)$ is continuous for almost all $x \in \Omega$. Let further $v(x, s, \cdot)$ be quasiconvex for almost all $x \in \Omega$ and all $s \in \mathbb{R}^m$, and suppose that for some $C > 0$, $1 \leq p < +\infty$, and $a \in L^1(\Omega)$ we have that*[8]

$$0 \leq v(x,s,A) \leq a(x) + C(|s|^p + |A|^p)\ . \tag{3.14}$$

Then $I : W^{1,p}(\Omega;\mathbb{R}^m) \to [0;+\infty)$ given in (1.5) is weakly lower semicontinuous on $W^{1,p}(\Omega;\mathbb{R}^m)$.

Interestingly, the paper by Acerbi and Fusco [1] already implicitly contains a version of Decomposition lemma 3.6.

Marcellini [107] proved, by a different technique of constructing a suitable non-decreasing sequence of approximations, a very similar result to Theorem 3.13 allowing also for a slightly more general growth

$$-c_1|A|^r - c_2|s|^t - c_3(x) \leq v(x,s,A) \leq g(x,s)\big(1+|A|^p\big), \tag{3.15}$$

[8]This bound is often called "natural growth conditions".

where $c_1, c_2 \geq 0$, $c_3 \in L^1(\Omega)$; g is Carathéodory but otherwise arbitrary and for the exponents we have that $p \geq 1$, $1 \leq r < p$ (but $r = 1$ if $p = 1$) and if $p < n$ $1 \leq t < np/(n-p)$, otherwise $t \geq 1$.

Note that the growth condition (3.15) actually allows for integrands unbounded from below but the exponent r determining this growth is strictly smaller than p. Such integrands are of *sub-critical growth* and for integrand of the class $\mathcal{F}_p(\Omega)$ weak lower semicontinuity under this growth follows also from Theorem 3.7(ii).

Acerbi and Fusco [1, p. 127] remarked that "*...using more complicated notations as in [13], [110], our results can be extended to the case of functionals of the type* (1.5)".This extension has been considered by Fusco [66] for the case $p = 1$ an later by Guidorzi and Poggilioni [73] who rewrote functional (1.5) as (using the notation from Section 2)

$$I(u) = \int_\Omega v(x, \nabla^{[k-1]}u(x), \nabla^k u(x))\mathrm{d}x \tag{3.16}$$

and proved the following.

Proposition 3.14 ([73]). *Let $v : \Omega \times Y(n,m,k-1) \times X(n,m,k) \to \mathbb{R}$ be a Carathéodory k-quasiconvex function satisfying for all $H \in Y(n,m,k-1)$ and all $A \in X(n,m,k)$*

$$0 \leq v(x,H,A) \leq g(x,H)(1+|A|)^p$$
$$|v(x,H,A) - v(x,H,B)| \leq C(1+|A|^{p-1}+|B|^{p-1})|A-B|$$

where g is a Carathéodory function and $C \geq 0$. Then the functional from (3.16) *is weakly lower semicontinuous in $W^{k,p}(\Omega;\mathbb{R}^n)$ for $1 \leq p < \infty$ and $k \in \mathbb{N}$.*

Note that in this result the continuity of the integrand in the space variable x could be omitted, which is, roughly speaking, due to the fact that quasiconvexity is enough to handle the concentration effects. On the other hand, the continuity assumption from Definition 3.2(ii) still remains present (with $\gamma = 1$). A similar result can be drawn from the more general setting of $\mathcal{A}$-quasiconvexity (which we review in Section 8 below) considered in [31].

While the above results handle also weak lower semicontinuity on $W^{k,1}(\Omega;\mathbb{R}^m)$ with respect to the standard weak convergence in this space, it is more suitable to investigate lower semicontinuity with respect to the strong convergence in $W^{k-1,1}(\Omega;\mathbb{R}^m)$. This is due to the fact that $W^{k,1}(\Omega;\mathbb{R}^m)$ is not reflexive and therefore coercivity of (1.5) does not allow us to select a minimizing sequence that would be weakly convergent in $W^{k,1}(\Omega;\mathbb{R}^m)$ but the strong convergence in $W^{k-1,1}(\Omega;\mathbb{R}^m)$ can be assured.

The case for $k = 1$ was treated by Fonseca and Müller [59] who considered continuous integrands under mild growth conditions. The result was later generalized by Fonseca, Leoni, Malý, and Paroni [58] not only with respect to the continuity of the integrand that could be partially dropped, but also to arbitrary k. We give the result in Theorem 3.15.

Theorem 3.15 (taken from [58]). *Let v in* (1.5) *be a Borel integrand that is moreover continuous in the following sense: For all $\varepsilon > 0$ and $(x_0, H_0) \in \Omega \times Y(n,m,k-1)$ there exist $\delta > 0$ and a modulus of continuity ω with the property that, for some $C > 0$, $\omega(s) \leq C(1+s), s > 0$ such that*

$$v(x_0,H_0,A) - v(x,H,A) \leq \varepsilon(1+v(x,H,A)) + \omega(|H_0 - H|),$$

for all $x \in \Omega$ satisfying $|x - x_0| \leq \delta$ and for all $H \in Y(n,m,k-1)$ and all $A \in X(n,m,k)$. Suppose further that v is k-quasiconvex and satisfies[9]

$$\frac{1}{c}|A| - c \leq v(x_0,H_0,A) \leq c(1+|A|),$$

[9]If $k = 1$ the growth condition can be relaxed to

$$0 \leq v(x_0,s,A) \leq c(1+|A|) \qquad \forall A \in \mathbb{R}^{m\times n}$$

and it can be even omitted if v is convex in its last variable.

for some $c > 0$ and all $A \in X(n, m, k)$.

Then, v in (1.5) *is lower semicontinuous with respect to the strong convergence in $W^{k-1,1}(\Omega; \mathbb{R}^m)$.*

For the functions $v : X(m, n, k) \to \mathbb{R}$, i.e. those depending only on the highest gradient, an analogous result has been obtained in [4].

4. Null Lagrangians. In this section, we study under which conditions on v the functional (1.5) is not only weakly lower semicontinuous but actually *weakly continuous* in $W^{k,p}(\Omega; \mathbb{R}^m)$. This question is tightly connected (cf. Theorem 4.3 below) to the study of so-called null Lagrangians. We start the discussion by presenting definitions of null Lagrangians of the first and higher order.

Definition 4.1. *We say that a continuous map $L : \mathbb{R}^{m\times n} \to \mathbb{R}$ is a null Lagrangian of the first order, if for every $u \in C^1(\bar{\Omega}; \mathbb{R}^m)$ and every $\varphi \in C^1_0(\Omega; \mathbb{R}^m)$ it holds that*

$$\int_\Omega L(\nabla(u(x) + \varphi(x)))\,\mathrm{d}x = \int_\Omega L(\nabla u(x))\,\mathrm{d}x\ . \tag{4.1}$$

Notice that the definition is independent of the particular Lipschitz domain Ω. In fact, if (4.1) holds for one domain Ω it also holds for all other (Lipschitz) domains.

Remark 4.1. *The name "null Lagrangians" comes from the fact that, if L is even smooth so that the Gateaux derivative of $J(u) := \int_\Omega L(\nabla u(x))\,\mathrm{d}x$ can be evaluated, it easily follows from* (4.1) *that J satisfies $J'(u) = 0$ for all $u \in C^1(\bar{\Omega}; \mathbb{R}^m)$. In other words, the Euler-Lagrange equations of J are fulfilled identically in the sense of distributions.*

Remark 4.2. *Let us notice that, if L is a null Lagrangian, the value of $J(u) = \int_\Omega L(\nabla u(x))\,\mathrm{d}x$ is only dependent on the boundary values of $u(x)$. This can be seen from* (4.1) *as the value remains unchanged even if we add arbitrary functions vanishing on the boundary.*

It is straightforward to generalize (4.1) also to higher order problems.

Definition 4.2. *Let $k \geq 2$. We say that $L : X \to \mathbb{R}$ is a (higher-order) null Lagrangian if*

$$\int_\Omega L(\nabla^k(u(x) + \varphi(x))\,\mathrm{d}x = \int_\Omega L(\nabla^k(u(x))\,\mathrm{d}x \tag{4.2}$$

for all $u \in C^k(\bar{\Omega}; \mathbb{R}^m)$ and all $\varphi \in C^k_0(\Omega; \mathbb{R}^m)$.

Similarly as in the first-order gradient case, the definition is independent of the particular (Lipschitz) domain Ω. In the same way as in the first order case, it follows that Euler-Lagrange equations

$$\sum_{|K|\leq l} (-D)^K \frac{\partial L}{\partial u^i_I}(\nabla^l u) = 0 \tag{4.3}$$

are satisfied in the sense of distributions for arbitrary $u \in C^k(\bar{\Omega}; \mathbb{R}^m)$.

Remark 4.3. *It is natural to generalize the notion of null Lagrangians to functionals of the type* (1.5)*, i.e. those depending also on lower order gradients, in the following way: We say that the function $L : \Omega \times Y(n, m, k) \to \mathbb{R}$ is a null Lagrangian for the functional* (1.5) *if for all $u \in C^k(\Omega; \mathbb{R}^m)$ and all $\varphi \in C^k_0(\Omega; \mathbb{R}^m)$ it holds that*

$$J(u + \varphi) = J(u) \qquad \text{and} \qquad J(u) = \int_\Omega L(x, u(x), \nabla u(x), \ldots, \nabla^k u(x))\,\mathrm{d}x.$$

We shall see in the end of the section that null Lagrangians for these types of functionals are actually determined by higher order null Lagrangians at least if $k = 1$.

The following result characterizes null Lagrangians (of first and higher order) by means of a few equivalent statements. In particular, it shows that null Lagrangians are the only integrands along which $\int_\Omega v(\nabla^k(u(x))\,\mathrm{d}x$ is weakly continuous. It is taken from [13].

Theorem 4.3 (Characterization of (higher-order) null Lagrangians). *Let $L : X(n, m, k) \to \mathbb{R}$ be continuous. Then the following statements are mutually equivalent:*

(i) L *is a null Lagrangian,*
(ii) $\int_\Omega L(A+\nabla^k\varphi(x))\,\mathrm{d}x=\int_\Omega L(A)\,\mathrm{d}x$ *for every* $\varphi\in C_0^\infty(\Omega;\mathbb{R}^m)$ *and every* $A\in X(n,m,k)$ *and every open subset* $\Omega\subset\mathbb{R}^n$*,*
(iii) L *is continuously differentiable and* (4.3) *holds in the sense of distributions,*
(iv) *The map* $u\mapsto L(\nabla^k u)$ *is sequentially weakly* continuous from* $W^{k,\infty}(\Omega;\mathbb{R}^m)$ *to* $L^\infty(\Omega)$*. This means that if* $u_j\overset{*}{\rightharpoonup}u$ *in* $W^{k,\infty}(\Omega;\mathbb{R}^m)$ *as* $j\to\infty$ *then* $L(\nabla^k u_j)\overset{*}{\rightharpoonup}L(\nabla^k u)$ *in* $L^\infty(\Omega)$*,*
(v) L *is a polynomial of degree* p *and the map* $u\mapsto L(\nabla^k u)$ *is sequentially weakly* continuous from* $W^{k,p}(\Omega;\mathbb{R}^m)$ *to* $\mathcal{D}'(\Omega)$*. This means that if* $u_j\rightharpoonup u$ *in* $W^{k,p}(\Omega;\mathbb{R}^m)$ *as* $j\to\infty$ *then* $L(\nabla^k u_j)\rightharpoonup L(\nabla^k u)$ *in* $\mathcal{D}'(\Omega)$*.*

While Theorem 4.3 provides us with very useful properties of null Lagrangians it is interesting to note that that they are known *explicitly* in the first as well as in the higher order. In fact, null Lagrangians are formed by minors or sub-determinants of the gradient entering the integrand in J.

4.1. Explicit characterization of null Lagrangians of the first order. Let us start with the first order case: If $A\in\mathbb{R}^{m\times n}$ we denote by $\mathbb{T}_i(A)$ the vector of all subdeterminants of A of order i for $1\le i\le\min(m,n)$. Notice that the dimension of $\mathbb{T}_i(A)$ is $d(i):=\binom{m}{i}\binom{n}{i}$, hence the the number of all subdeterminants of A is $\sigma:=\binom{m+n}{n}-1$. Finally, we write $\mathbb{T}:=(\mathbb{T}_1,\ldots,\mathbb{T}_{\min(m,n)})$. For example, if $m=1$ or $n=1$ then $\mathbb{T}(A)$ consists only of entries of A, if $m=n=2$ then $\mathbb{T}(A)=(A,\det A)$ and for $m=n=3$ we obtain $\mathbb{T}(A)=(A,\mathrm{Cof}A,\det A)$.

Clearly, linear maps are weakly continuous. Yet, it has been known at least since [114, 130, 8] that also minors have this property (see Theorem 4.4 below). This result, usually called *(sequential) weak continuity of minors* is unexpected because if $i>1$ then $A\mapsto\mathbb{T}_i(A)$ is a nonlinear polynomial of the i-th order. As it is well-known, weak convergence generically does not commute with nonlinear mappings.

Theorem 4.4 (Weak continuity of minors (see e.g. [42])). *Let* $\Omega\subset\mathbb{R}^n$ *be a bounded Lipschitz domain. Let* $1\le i\le\min(m,n)$*. Let* $\{u_k\}_{k\in\mathbb{N}}\subset W^{1,p}(\Omega;\mathbb{R}^m)$ *be such that* $u_k\rightharpoonup u$ *in* $W^{1,p}(\Omega;\mathbb{R}^m)$ *for* $p>i$*. Then* $\mathbb{T}_i(\nabla u_k)\rightharpoonup\mathbb{T}_i(\nabla u)$ *in* $L^{p/i}(\Omega;\mathbb{R}^{d(i)})$*.*

It follows from Theorem 4.7 below (see also [42]) that minors are the only mappings depending exclusively on ∇u which have this property. Thus in view of Theorem 4.3, any null Lagrangian can be written as an affine combination of elements of $\mathbb{T}$, i.e., for any $A\in\mathbb{R}^{m\times n}$

$$L(A)=c_0+c\cdot\mathbb{T}(A)\ , \tag{4.4}$$

where $c_0\in\mathbb{R}$ and $c\in\mathbb{R}^\sigma$ are arbitrary constants. Let us note however, that it has been realized independently in e.g. [52, 53] that minors are the only maps for which the Euler-Lagrange equation of $J(u)=\int_\Omega L(\nabla u)\mathrm{d}x$ is satisfied identically.

As we saw in Example 3.2, Theorem 4.4 fails if $p=i$. Nevertheless, the results can be much improved if we additionally assume that, for every $k\in\mathbb{N}$, $\mathbb{T}_i(\nabla u_k)\ge0$ almost everywhere in Ω. Indeed, Müller [117] proved the following result.

Proposition 4.5 (Higher integrability of determinant). *Assume that* $\omega\subset\Omega\subset\mathbb{R}^n$ *is compact,* $u\in W^{1,n}(\Omega;\mathbb{R}^n)$*, and that* $\det\nabla u\ge0$ *almost everywhere in* Ω*. Then*

$$\|(\det\nabla u)\ln(2+\det\nabla u)\|_{L^1(\omega)}\le C(\omega,\|u\|_{W^{1,n}(\Omega;\mathbb{R}^n)}) \tag{4.5}$$

for some $C(\omega,\|u\|_{W^{1,n}(\Omega;\mathbb{R}^n)})>0$ *a constant depending only on* ω *and the Sobolev norm of* u *in* Ω*.*

This proposition results in the following Corollary:

Corollary 4.6 (Uniform integrability of determinant). *If* $\{u_k\}_{k\in\mathbb{N}}\subset W^{1,n}(\Omega;\mathbb{R}^n)$ *is bounded and* $\det\nabla u_k\ge0$ *almost everywhere in* Ω *for all* $k\in\mathbb{N}$ *then* $\det\nabla u_k\rightharpoonup\det\nabla u$ *in* $L^1(\omega)$ *for every compact set* $\omega\subset\Omega$*.*

A related statement was achieved by Kinderlehrer and Pedregal in [87]. It says that under the assumptions of Corollary 4.6 and if $u_k=u$ on $\partial\Omega$ for all $k\in\mathbb{N}$ the claim of Corollary 4.6 holds for $\omega:=\Omega$.

4.2. Explicit characterization of null Lagrangians of higher order. Null-Lagrangians of higher order are of the same structure as those of the first order. Indeed, they also correspond to minors. In order to make the statement more precise, we assume that $K := (k_1, \ldots, k_r)$ is such that $1 \le k_i \le n$ and denote by $\alpha := (\nu_1, J_1; \nu_2, J_2; \ldots; J_r, \nu_r)$ with $|J_i| = k-1$ and where $1 \le \nu_i \le m$. We define the k-th order Jacobian determinant $J_K^\alpha : X \to \mathbb{R}$ by the formula

$$J_K^\alpha(\nabla u) = \frac{\partial(\partial u_{J_1}^{\nu_1}, \ldots, \partial u_{J_r}^{\nu_r})}{\partial(x^{k_1}, \ldots, x^{k_r})} = \det\left(\frac{\partial u_{J_i}^{\nu_i}}{\partial x^{k_j}}\right) .$$

Then any null Lagrangian of higher order is just an affine combination of J_K^α, i.e.,

Theorem 4.7 (See [13]). *Let $L \in C(X(n,m,k))$. Then L is a null Lagrangian if and only if it is an affine combination of k-th order Jacobian determinant, i.e.,*

$$L = C_0 + \sum_{\alpha,K} C_K^\alpha J_K^\alpha$$

for some constants C_0 and C_K^α.

Remark 4.4. *The maximum degree of nonzero $J_K^\alpha(\nabla^k y)$ is denoted by R. It can be shown that $R = \min(m,n)$ if $k = 1$ and $R := n$ for $k > 1$.*

4.3. Null Lagrangians with lower order terms. As pointed out in Remark 4.3, the notion of null Lagrangians can be generalized also to functionals of the type (1.5); i.e. those containing also lower order terms. A characterization of these null Lagrangians is due to Olver and Sivaloganathan [122] who considered the first order case; i.e., null Lagrangians for those functionals which can also depend on x and u.

Based on Olver's results [121], they showed in [122] that such null Lagrangians are given by the formula

$$\tilde{L}(x, u, \nabla u) = C_0(x,u) + \sum_i C_i(x,u)\mathbb{T}_i(\nabla u) ,$$

where C_0 is a real-valued C^1-function and C_i are C^1-functions of its arguments for $1 \le i \le \min(m,n)$ with values in $R^{d(i)}$, $i > 0$. This means that they are determined by the already known null Lagrangians of the first order. Let us remark, that it is noted in [122] that the result generalizes analogously to the higher order case.

5. Null Lagrangians at the boundary. We have seen that null Lagrangians of the first order are exactly those functions that fulfill (1.4) in the definition of quasiconvexity with an equality. This, of course, assures that null-Lagrangians are weakly* continuous with respect to the $W^{1,\infty}(\Omega;\mathbb{R}^m)$ weak* topology; in addition, due to Theorem 4.4, the are weakly continuous with respect to the $W^{1,p}(\Omega;\mathbb{R}^m)$ weak topology if $p > \min(m,n)$ with $\Omega \subset \mathbb{R}^n$.

However, in the critical case when $p = \min(m,n)$ the weak continuity fails. In fact, as we have seen in Example 3.2 for $n = m = p = 2$ the functional (1.5) with $k = 1$ and $v(x, u, \nabla u) = \det(\nabla u)$ is not even weakly lower semicontinuous, even though the determinant itself is definitely a null-Lagrangian. Once again, the reason for the failure of weak continuity are concentrations on the boundary combined with the fact that null-Lagrangians are unbounded from below.

Nevertheless, as we have seen in Section 3.1, at least for p-homogeneous functions, weak lower semicontinuity can be assured for for functionals with integrands that are quasiconvex at the boundary; i.e., fulfill (3.13). Thus, a proper equivalent of null Lagrangians in this case are those functions that fulfill (3.13) with an equality—these functions are referred to as *null Lagrangians at the boundary.* We study these functions in this section.

Clearly, null Lagrangians at the boundary form a subset of null Lagrangians of the first order. Moreover, they have exactly the sought properties: If $\mathcal{N}$ is a null Lagrangian at the boundary then it is a polynomial of order p, say. If, additionally $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ converges weakly to $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ then $\{\mathcal{N}(\nabla u_k)\}_{k\in\mathbb{N}} \subset L^1(\Omega)$ weak* converges to $\mathcal{N}(\nabla u)$ in $\mathcal{M}(\bar{\Omega})$, i.e., in measures on the closure of the domain. This means that the L^1-bounded sequence $\{\mathcal{N}(\nabla u_k)\}$ converges to a Radon measure whose singular part vanishes. Thus, functionals with integrands that are null-Lagrangians at the boundary are weakly continuous even in the critical case. Null Lagrangians at the boundary can be also used to construct functions quasiconvex at the boundary; cf. Definition 3.11.

We first give a formal definition of null Lagrangians at the boundary.

Definition 5.1. *Let $\varrho \in \mathbb{R}^n$ be a unit vector and let $\mathcal{N} : \mathbb{R}^{m\times n} \to \mathbb{R}$ be a given function.*

(i) $\mathcal{N}$ is called a null Lagrangian at the boundary at given $A \in \mathbb{R}^{m\times n}$ if both $\mathcal{N}$ and $-\mathcal{N}$ are quasiconvex at the boundary at A in the sense of Definition 3.11; cf. [144]. This means that there is $q \in \mathbb{R}^m$ such that for all $\varphi \in W^{1,\infty}_{\Gamma_\rho}(D_\rho;\mathbb{R}^m)$ it holds

$$\int_{\Gamma_\rho} q\cdot\varphi(x)\,\mathrm{d}S + \mathcal{N}(A)\mathcal{L}^n(D_\varrho) = \int_{D_\rho} \mathcal{N}(A+\nabla\varphi(x))\,\mathrm{d}x\ . \tag{5.1}$$

(ii) If $\mathcal{N}$ is a null Lagrangian at the boundary at every $F \in \mathbb{R}^{m\times n}$, we call it a null Lagrangian at the boundary.

The following theorem explicitly characterizes all possible null Lagrangians at the boundary. It was first proved by P. Sprenger in his thesis [141, Satz 1.27]. Later on, the proof was slightly simplified in [84]. Before stating the result we recall that $\mathrm{SO}(n) := \{R \in \mathbb{R}^{n\times n};\ R^\top R = RR^\top = \mathbb{I}\ ,\ \det R = 1\}$ denotes the set of orientation-preserving rotations and if we write $A = (B|\varrho)$ for some $B \in \mathbb{R}^{n\times(n-1)}$ and $\varrho \in \mathbb{R}^n$ then $A \in \mathbb{R}^{n\times n}$, its last column is ϱ and $A_{ij} = B_{ij}$ for $1 \le i \le n$ and $1 \le j \le n-1$. We remind also that $\mathbb{T}_i(A)$ denotes the vector of all subdeterminants of A of order i.

Theorem 5.2. *Let $\varrho \in \mathbb{R}^n$ be a unit vector and let $\mathcal{N} : \mathbb{R}^{m\times n} \to \mathbb{R}$ be a given continuous function. Then the following three statements are equivalent.*

(i) $\mathcal{N}$ satisfies (5.1) for every $F \in \mathbb{R}^{m\times n}$;

(ii) $\mathcal{N}$ satisfies (5.1) for $F = 0$,

(iii) There are constants $\tilde\beta_s \in \mathbb{R}^{\binom{m}{s}\times\binom{n-1}{s}}$, $1 \le s \le \min(m, n-1)$, such that for all $H \in \mathbb{R}^{m\times n}$,

$$\mathcal{N}(H) = \mathcal{N}(0) + \sum_{i=1}^{\min(m,n-1)} \tilde\beta_i \cdot \mathbb{T}_i(H\tilde R), \tag{5.2}$$

where $\tilde R \in \mathbb{R}^{n\times(n-1)}$ is a matrix such that $R = (\tilde R|\varrho)$ belongs to $\mathrm{SO}(n)$;

(iv) $\mathcal{N}(F + a\otimes\varrho) = \mathcal{N}(F)$ for every $F \in \mathbb{R}^{m\times n}$ and every $a \in \mathbb{R}^m$.

If $m = n = 3$ the only nonlinear null Lagrangian at the boundary with the normal ϱ is

$$\mathcal{N}(F) = \mathrm{Cof}\,F\cdot(a\otimes\varrho) = a\cdot\mathrm{Cof}\,F\varrho$$

where $a \in \mathbb{R}^3$ is some fixed vector; see [144].

In the following theorem, we let ϱ freely move along the boundary which introduces an x-dependence to the problem. Then the vector a may depend on x as well.

Theorem 5.3 (due to [100]). *Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain. Let $\{u_k\} \subset W^{1,2}(\Omega;\mathbb{R}^3)$ be such that $u_k \rightharpoonup u$ in $W^{1,2}(\Omega;\mathbb{R}^3)$. Let $\mathcal{N}(x,F) := \mathrm{Cof}\,F\cdot(a(x)\otimes\varrho(x))$, where $a,\varrho \in C(\bar\Omega;\mathbb{R}^3)$, ϱ coincides at $\partial\Omega$ with the outer unit normal to $\partial\Omega$. Then for all $g \in C(\bar\Omega)$*

$$\lim_{k\to\infty}\int_\Omega g(x)\mathcal{N}(x,\nabla u_k(x))\,\mathrm{d}x = \int_\Omega g(x)\mathcal{N}(x,\nabla u(x))\,\mathrm{d}x\ . \tag{5.3}$$

If, moreover, for all $k \in \mathbb{N}$ $\mathcal{N}(\cdot,\nabla u_k) \ge 0$ almost everywhere in Ω then $\mathcal{N}(\cdot,\nabla u_k) \rightharpoonup h(\cdot,\nabla u)$ in $L^1(\Omega)$.

Notice that even though $\{\mathcal{N}(\cdot,\nabla u_k)\}_{k\in\mathbb{N}}$ is bounded merely in $L^1(\Omega)$ its weak* limit in measures is $h(\cdot,\nabla u) \in L^1(\Omega)$, i.e., a measure which is absolutely continuous with respect to the Lebesgue measure on Ω. This holds independently of $\{\nabla u_k\}$. Therefore, the fact that $\mathcal{N}$ is a null Lagrangian at the boundary automatically improved regularity of the limit measure, namely its singular part vanishes. In order to understand why this happens, denote $\mathbb{P}(x) := \mathbb{I} - \varrho(x) \otimes \varrho(x)$ the orthogonal projector on the plane with the normal $\varrho(x)$, i.e., a tangent plane to $\partial\Omega$ at $x \in \partial\Omega$. Then

$$\mathrm{Cof}(F\mathbb{P}) = \mathrm{Cof}F\mathrm{Cof}\mathbb{P} = (\mathrm{Cof}F)(\varrho \otimes \varrho)\ .$$

Consequently,

$$\mathrm{Cof}(F\mathbb{P})\varrho = (\mathrm{Cof}F)\varrho\ ,$$

and if F is a placeholder for ∇u we see that $\mathcal{N}(x,\cdot)$ only depends on the surface gradient of u. In other words, concentrations in the sequence of normal derivatives, $\{\nabla u_k(\varrho\otimes\varrho)\}_{k\in\mathbb{N}}$, are filtered out. The following two statements describing weak sequential continuity of null Lagrangians at the boundary can be found in [84]. They apply to cases in which the condition (ii) from Theorem 3.3 is always satisfied.

THEOREM 5.4 (see [84]). *Let $m, n \in \mathbb{N}$ with $n \geq 2$, let $\Omega \subset \mathbb{R}^n$ be open and bounded with a boundary of class C^1, and let $\mathcal{N} : \overline{\Omega} \times \mathbb{R}^{m\times n} \to \mathbb{R}$ be a continuous function. In addition, suppose that for every $x \in \Omega$, $\mathcal{N}(x,\cdot)$ is a null Lagrangian and for every $x \in \partial\Omega$, $\mathcal{N}(x,\cdot)$ is a null Lagrangian at the boundary with respect to $\varrho(x)$, the outer normal to $\partial\Omega$ at x. Hence, by Theorem 5.2, $\mathcal{N}(x,\cdot)$ is a polynomial, the degree of which we denote by $d_{\mathcal{N}}(x)$. Finally, let $p \in (1,\infty)$ with $p \geq d_v(x)$ for every $x \in \overline{\Omega}$ and let $\{u_k\} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ be a sequence such that $u_k \rightharpoonup u$ in $W^{1,p}$. If*

$$\mathcal{N}(x, \nabla u_k(x)) \geq 0 \quad \textit{for every } k \in \mathbb{N} \textit{ and a.e. } x \in \Omega,$$

then $\mathcal{N}(\cdot,\nabla u_n) \rightharpoonup \mathcal{N}(\cdot,\nabla u)$ weakly in $L^1(\Omega)$.

THEOREM 5.5 (see [84]). *Let $h : \Omega \times \mathbb{R} \to \mathbb{R}\cup\{+\infty\}$ be such that $h(\cdot,s)$ is measurable for all $s \in \mathbb{R}$ and $h(x,\cdot)$ is convex for almost all $x \in \Omega$. Let $\mathcal{N}$ and $d_{\mathcal{N}}$ be as in Theorem 5.4. Then $\int_\Omega h(x,\mathcal{N}(x,\nabla u(x)))\,\mathrm{d}x$ is weakly lower semicontinuous on the set $\{u \in W^{1,p}(\Omega;\mathbb{R}^m); \mathcal{N}(\cdot,\nabla u) \geq 0$ in $\Omega\}$.*

Let us finally point out that $A \mapsto h(\mathcal{N}(A))$ for a convex function h is quasiconvex at the boundary [15].

6. Polyconvexity and applications to hyperelasticity. We saw that, at least for integrands bounded from below and satisfying (i) in Definition 3.2, quasiconvexity is an equivalent condition for weak lower semicontinuity. This presents an *explicit* characterization of the latter since it is not necessary to examine all weakly converging sequences. Nevertheless, in practice quasiconvexity is almost impossible to verify since, in a sense, its verification calls for solving a minimization problem itself. Therefore, it is desirable to find at least *sufficient* conditions for weak lower semicontinuity that can be easily verified. Such a notion, called *polyconvexity* introduced by J.M. Ball, can be designed by employing the null Lagrangians introduced in the last section.

We start with the definition of polyconvexity for first order functionals $I(u) = \int_\Omega v(\nabla u(x))\mathrm{d}x$.

DEFINITION 6.1 (Due to [8]). *We say that $v : \mathbb{R}^{m\times n} \to \mathbb{R}\cup\{+\infty\}$ is polyconvex if there exists a convex function $h : \mathbb{R}^\sigma \to \mathbb{R}\cup\{+\infty\}$ such that $v(A) = h(\mathbb{T}(A))$[10] for all $A \in \mathbb{R}^{m\times n}$.*

REMARK 6.1. *Interestingly, already Morrey in [114, Thm. 5.3] proved that one-homogeneous convex functions depending on minors are quasiconvex.*

If h is affine in the above definition, we call v polyaffine. In this case, $v(A)$ is a linear combination of all minors of A plus a real constant. Consequently, any polyconvex function is bounded from below by a polyaffine function. Similarly, as in the convex case, a polyconvex function is found by forming the supremum of all polyaffine functions lying below it see e.g. [42, Rem. 6.7]; i.e., we have the following lemma.

LEMMA 6.2. *The function $v : \mathbb{R}^{m\times n} \to \mathbb{R}$ is polyconvex if and only if*

$$v(A) = \sup\{\varphi(A); \varphi \textit{ polyaffine and } \varphi \leq v\}.$$

[10] Recall that $\mathbb{T}(A)$ denotes the vector of all minors of A.

It is a straightforward idea to generalize polyconvexity to higher-order variational problems, i.e., those that depend on higher-order gradients of a mapping. The attractiveness of such problems for applications is clear. Suitably chosen terms depending on higher-order gradients allow for compactness of a minimizing sequence in some stronger topology than the weak one on $W^{1,p}$ which enable us to pass to a limit in lower-order terms without restrictive assumptions on their convexity properties. Thus, for example, models of shape memory alloys (see Section 7) can be treated by this approach; cf. e.g. [118, 119].

Thus, we extend the notion of polyconvexity to higher order problems (1.5) and it employs the notion of null Lagrangians of higher order and is due to to Ball, Currie, and Olver [13].

Definition 6.3 (Higher-order polyconvexity). *Let $1 \le r \le R$ where R is defined in Remark 4.4. Let $U \subset X(n,m,k)$ be open. A function $G : U \to \mathbb{R}$ is r-polyconvex if there exists a convex function $h : \mathrm{Co}(J^{[r]}(U)) \to \mathbb{R}$ such that $v(A) = h(J^{[r]}(A))$ for all $A \in U$; here $\mathrm{Co}(J^{[r]}(U))$ is the convex hull of $J^{[r]}(U)$. G is polyconvex if it is R-polyconvex. Here $J^r(H) := (J^{r,1}(H), \ldots, J^{r,N_r}(H))$ is a N_r-tuple with the property that any Jacobian determinant of degree r can be written as a linear combination of elements of J^r. Consequently, $J^{[r]} := (J^1, \ldots, J^r)$. If h is affine then we call v r-polyaffine.*

Since polyconvexity implies quasiconvexity, we may deduce by the results in Section 3 that polyconvex functions in the class $\mathcal{F}_p(\Omega)$ (from Definition 3.2) are weakly lower semicontinuous. Yet, weak lower semicontinuity can be proved for wider class of polyconvex functions than those in $\mathcal{F}_p(\Omega)$; in particular, the functions do not have to be of p-growth. This is of great importance in elasticity as explained later in this section.

The proof of weak lower semicontinuity of polyconvex functions is actually based on *convexity* and weak continuity of null Lagrangians. Thus, since weak lower semicontinuity can be shown for arbitrarily growing convex functions, it generalizes to polyconvex ones, too. The following result for convex functions can be found in [13, Thm. 5.4] and is based on results by Eisen [51] who proved this theorem for $\Phi < +\infty$.

Theorem 6.4 (weak lower semicontinuity). *Let $\Phi : \Omega \times \mathbb{R}^s \times \mathbb{R}^\sigma \to \mathbb{R} \cup \{+\infty\}$ satisfy the following properties*

(i) $\Phi(\cdot, z, a) : \Omega \to \mathbb{R} \cup \{+\infty\}$ is measurable for all $(z, a) \in \mathbb{R}^s \times \mathbb{R}^\sigma$,
(ii) $\Phi(x, \cdot, \cdot) : \mathbb{R}^s \times \mathbb{R}^\sigma \to \mathbb{R} \cup \{+\infty\}$ is continuous for almost every $x \in \Omega$,
(iii) $\Phi(x, z, \cdot) : \mathbb{R}^\sigma \to \mathbb{R} \cup \{+\infty\}$ is convex.

Assume further that for all $(z, a) \in \mathbb{R}^s \times \mathbb{R}^\sigma$ $\Phi(\cdot, z, a) \ge \phi$ for some $\phi \in L^1(\Omega)$. Let $z_k \to z$ almost everywhere in Ω and let $a_k \rightharpoonup a$ in $L^1(\Omega; R^\sigma)$. Then

$$\int_\Omega \Phi(x, z(x), a(x))\,\mathrm{d}x \le \liminf_{k\to\infty} \int_\Omega \Phi(x, z_k(x), a_k(x))\,\mathrm{d}x\ .$$

Using this theorem, we may easily deduce weak lower semicontinuity of polyconvex functions. For the sake of clarity, let us start with first order problems. Then, consider $u_k \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^m)$ as $k \to \infty$ where $p > \min(m, n)$. Then $u_k \to u$ in $L^p(\Omega; \mathbb{R}^m)$, so, for a (non-relabeled) subsequence, even $u_k \to u$ almost everywhere in Ω. Hence, we can apply Theorem 6.4 with $z_k := u_k$, $a_k := \mathbb{T}(\nabla u_k)$ and $v(x, y, \nabla y) := \Phi(x, y, \mathbb{T}(\nabla y))$ to obtain the following corollary:

Corollary 6.5. *Let $v : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R} \cup \{+\infty\}$ satisfy the following properties*

(i) $v(\cdot, z, A) : \Omega \to \mathbb{R} \cup \{+\infty\}$ is measurable for all $(z, A) \in R^m \times \mathbb{R}^{m\times n}$,
(ii) $v(x, \cdot, \cdot) : \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R} \cup \{+\infty\}$ is continuous for almost every $x \in \Omega$,
(iii) $v(x, z, A) = \Phi(x, z, \mathbb{T}(A))$ where Φ satisfies (i)–(iii) from Theorem 6.4.

If $u_k \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^m)$ as $k \to \infty$ where $p > \min(m, n)$ then

$$\int_\Omega v(x, u(x), \nabla u(x))\,\mathrm{d}x \le \liminf_{k\to\infty} \int_\Omega v(x, u_k(x), \nabla u_k(x))\,\mathrm{d}x\ .$$

Similarly as in the case of first order problems, we can exploit (v) of Theorem 4.3 and Theorem 6.4 to show the existence of minimizers to energy functionals (1.5). Let us present the result just for functionals (1.5) with $k = 2$; generalizations for higher k are straightforward and can be found in [13].

Corollary 6.6 (after [13]). *Assume that* $\Omega \subset \mathbb{R}^n$ *is a bounded smooth domain and that* $1 \le r \le R$. *Let* $v : \Omega \times Y(n,m,2) \to \mathbb{R} \cup \{+\infty\}$ *in*

$$I(u) = \int_\Omega v(x,u,\nabla u,\nabla^2 u)\mathrm{d}x$$

satisfy the following assumptions:

(i) $v(x,H,A) = h(x,H,J^{[r]}(A))$, *where* $h(x,\cdot,\cdot) : (\mathbb{R}^m \times \mathbb{R}^{m\times n}) \times J^{[r]}(X(n,m,2)) \to \mathbb{R} \cup \{+\infty\}$ *is continuous for almost every* $x \in \Omega$,

(ii) $h(\cdot,H,J^{[r]}(A)) : \Omega \to \mathbb{R}\cup\{+\infty\}$ *is measurable for all* $(H,J^{[r]}(A)) \in (\mathbb{R}^m \times \mathbb{R}^{m\times n}) \times J^{[r]}(X(n,m,2))$,

(iii) $h(x,H,\cdot) : J^{[r]}(X) \to \mathbb{R} \cup \{+\infty\}$ *is convex for almost all* $x \in \Omega$ *and all* $F \in \mathbb{R}^{m\times n}$,

(iv) $v(x,H,A) \ge C(-1+|A|^p)$ *for some* $C > 0$, $p > n$, *almost all* $x \in \Omega$ *and all* $A \in \mathbb{R}^{m\times n}$,

Let further for some $u_0, u_1 \in W^{2,p}(\Omega;\mathbb{R}^m)$

$$\mathcal{Y} := \{u \in W^{2,p}(\Omega;\mathbb{R}^m) : \ u = u_0 \ on\ \Gamma_\mathrm{D}\ ,\ \nabla u = \nabla u_1 \ on\ \Gamma_\mathrm{D}\} \neq \emptyset$$

and such that $\inf_{\mathcal{Y}} I(u) < +\infty$. *Then there is a minimum of* $I(u)$ *on* $\mathcal{Y}$.

It is important to realize that main strength of polyconvexity consists in the fact that convexity in subdeterminants can be advantageously combined with the Mazur lemma to show weak lower semicontinuity in similar a way like in the proof for mere convex and lower semicontinuous integrands. This contrasts with proofs available for quasiconvex integrands where manipulations with boundary conditions are usually needed to prove the result. This is already clearly visible in Meyers paper [110]. These manipulations typically destroy any pointwise constraints on the determinant of ∇y, which, however, are crucial in elasticity. We shall return to this issue in Section 7.

6.1. Rank-1 convexity. Since polyconvexity is an explicit sufficient condition for quasiconvexity, we may ask if similarly a simpler necessary condition can be found. This is indeed so, the sought notion of convexity is *rank-1 convexity*:

Definition 6.7 (Due to [115]). *We say that* $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ *is rank-1 convex if*

$$f(\lambda A_1 + (1-\lambda)A_2) \le \lambda f(A_1) + (1-\lambda) f(A_2). \tag{6.1}$$

for all $\lambda \in [0,1]$ *and all* A_1, A_2 *such that* $\mathrm{rank}(A_1 - A_2) \le 1$.

The relations among the introduced notions of convexity are as follows:

convexity ⇒ polyconvexity ⇒ quasiconvexity ⇒ rank-1 convexity;

however, none of the converse implications holds if $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ and $m > 2$ and $n \ge 2$. To see that polyconvexity does not imply convexity (even for $m, m > 1$) just consider the function $v(F) := \det(F)$ which is even polyaffine but not convex. Also quasiconvexity does not imply polyconvexity even for $m, n > 1$ as was shown in e.g. [3, 151]. Šverák's important counter example [148] is a construction of a function that is rank-1 convex, but not quasiconvex and holds for $m \ge 3$ and $n \ge 2$. For $m = 2$ and $n \ge 2$ the question of equivalence between quasiconvexity and rank-1 convexity is still unsolved. On the other hand, it was shown in [155] that rank-one convexity coincides with quasiconvexity for quadratic forms in any dimension.

Open problem 6.8. *Let* $m = 2$ *and* $n \ge 2$. *Does rank-1 convexity imply quasiconvexity for* $v : \mathbb{R}^{m\times n} \to \mathbb{R}$.

Notice that, if $m = 1$ or $n = 1$ all the generalized notions of convexity trivially coincide with standard convexity itself.

An equivalent to rank-1 convexity can also be defined for higher-order problems–the corresponding notion is called Λ-convexity. Following [13], we define a nonconvex cone $\Lambda \subset X(n,m,k)$ as $\Lambda := \{a \otimes^l b : a \in \mathbb{R}^m,\ b \in \mathbb{R}^n\}$ where $(a \otimes^l b)^i_K = a^i b_K$.

Definition 6.9. *A function* $f : X \to \mathbb{R}$ *is called* Λ*-convex if* $t \mapsto f(A + tB) : \mathbb{R} \to \mathbb{R}$ *is convex for any* $A \in X(n,m,k)$ *and any* $B \in \Lambda$.

Notice that for $l = 1$ Λ-convexity coincides with rank-1 convexity. If f is twice continuously differentiable then Λ-convexity is equivalent to the Legendre-Hadamard condition

$$\sum_{j,k=1}^{m} \sum_{|J|=|K|=l} \frac{\partial^2 f(A)}{\partial A_J^j \partial A_K^k} a^j a^k b_J b_K \geq 0$$

for all $A \in X(n, m, k)$, $a \in \mathbb{R}^m$, and $b \in \mathbb{R}^n$.

PROPOSITION 6.10 (see [13]). *Continuous and k-quasiconvex functions* $f : X(n, m, k) \to \mathbb{R}$ *are* Λ-*convex.*

Hence, Λ-convexity forms a necessary condition for (k-)quasiconvexity. This proposition was first proved by Meyers [110, Thm. 7] for smooth functions and then generalized in [13] to the continuous case. The opposite assertion does not hold. Indeed, if $n = l = 2$ and $m = 3$ then we have the following example due to Ball, Currie, and Olver for $f : X \to \mathbb{R}$

$$v(\nabla^2 u) = \sum_{i,j,k=1}^{3} \varepsilon_{ijk} \frac{\partial^2 u^i}{\partial x_1^2} \frac{\partial^2 u^j}{\partial x_1 \partial x_2} \frac{\partial^2 u^k}{\partial x_3^2} \, .$$

This function is even Λ-affine (i.e., both $\pm v$ are Λ-convex) but not a null Lagrangian and it is not quasiconvex. As Λ-convexity replaces rank-one convexity in the current setting we see, that this example is a reminiscent of Šverák's example mentioned above.

6.2. Applications to hyperelasticity in the first order setting. In elasticity, one is interested in modeling the response of a rubber-like material to the action of applied outer forces. This response is obtained by solving a minimization problem; to be more specific, we are to minimize the free energy of the material. We will see that polyconvexity is perfectly fitted to the setting in elasticity and that existence of minimizers can be assured for polyconvex energies. We give a short introduction to this matter in this section and refer the reader e.g. to the monographs [74, 75, 144] for more details on the physical modeling.

Take a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ which, for $n = 3$, plays a role of a reference configuration of an elastic material. For given applied loads, we search for a mapping $y : \Omega \to \mathbb{R}^m$, the *deformation* of the material, which describes the new "shape" $y(\Omega)$ of the body. The mapping y is found by solving the following system of equations

$$-\operatorname{div} S = f \quad \text{in } \Omega, \tag{6.2}$$

$$S\nu = g \quad \text{on } \Gamma_{\mathrm{N}}, \tag{6.3}$$

$$y = y_0 \quad \text{on } \Gamma_{\mathrm{D}}. \tag{6.4}$$

Here, (6.2) is the reduced version of Newton's law of motion for the (quasi)static case, f is the applied volume force. Further, (6.3) represents the action of applied surface forces g (ν denotes the outer unit normal vector to Γ_{N}) and (6.4) models that the body may be clamped at some part of the boundary to a prescribed shape y_0. We shall require that $\Gamma_{\mathrm{D}} \subset \partial\Omega$ is disjoint from Γ_{N} and of positive $(n-1)$-dimensional Lebesgue measure.

The material properties of the specimen are encoded in the first Piola-Kirchhoff stress tensor $S : \Omega \to \mathbb{R}^{m\times n}$ in (6.2) and (6.3). The form of the Piola-Kirchhoff stress tensor cannot be deduced from first principles within continuum mechanics[11] but has to be prescribed phenomenologically. The prescription for S is called the *constitutive relation* of the given material. In the easiest case, we assume the form $S(x) = \hat{S}(x, \nabla y(x))$ for some given $\hat{S}$. Materials for which this assumption is adequate are sometimes referred to a *simple* materials as opposed to non-simple materials for which $\hat{S}$ may depend also higher gradients of y. Later, in subsection 6.3, we will consider also these sophisticated constitutive relations.

[11] Though it may be deduced from first principles when, e.g., working on a lattice of atoms and sending the number of atoms to infinity. In some cases, one may perform this *discrete-to-continuum* transition rigorously by means of a so-called Γ-limit; cf. e.g. [2, 32] or [102] for details

Hyperelasticity is a part of elasticity where an *additional assumption* is made; namely, that S has a potential $W : \mathbb{R}^{n\times n} \to [0;+\infty]$ such that

$$S_{ij}(x) = \frac{\partial W(F)}{\partial F_{ij}}|_{F=\nabla y(x)} \ .$$

This assumption emphasizes the idea that there are no energy losses in elasticity and all work, made by external forces and/or Dirichlet boundary conditions, stored in the material can be fully exploited.

In the following, let us restrict our attention to *deformations of bulks*, i.e. we do not treat plates and rods, and set thus $m = n$. In order to fulfill the basic physical requirements, W has to satisfy the following relations:

$$W(RF) = W(F) \text{ for all } F \in \mathbb{R}^{n\times n} \text{ and for all } R \in \mathrm{SO}(n) \tag{6.5}$$

$$W(F) = +\infty \text{ if } \det F \leq 0, \text{ and} \tag{6.6}$$

$$W(F) \to +\infty \quad \text{if } \det F \to 0_+. \tag{6.7}$$

Indeed, assumption (6.5) is a consequence of the *axiom of frame indifference* [37]; in other words the assumptions assures that material properties are independent of the position of the observer. Conditions (6.6) and (6.7) ensure, respectively, that the material does not locally penetrate itself and that compression of a finite volume of the specimen into zero volume is not possible. These conditions, however, do not yet assure that the body does not penetrate through itself, which is also natural to assume from a physical point of view. Nevertheless, we shall see in the end of this section that with additional assumptions on the growth of the energy and, e.g., the boundary conditions even complete non-interpenetration can be assured.

The assumptions (6.5)-(6.7) rule out that $W(x,\cdot)$ can be convex. Moreover, due to (6.6)-(6.7) even if $W(x,\cdot)$ was quasiconvex, we could not apply the theorems in Section 3 since W cannot be an element of the class $\mathcal{F}_p(\Omega)$. Nevertheless, *polyconvexity* is fully compatible with these assumptions.

The mechanical model is that stable states of the system are found by minimizing the overall free energy

$$\mathcal{E}(y) = \int_\Omega W(\nabla y(x))\,\mathrm{d}x, \tag{6.8}$$

subject to (6.4). Smooth minimizers fulfill the balance equations (6.2)-(6.3); however, even in the smooth case there might exist solutions to (6.2)-(6.3) which are not minimizers of (6.8). Nevertheless, such solutions are thought to be metastable and hence left after a small perturbation. Thus, minimizing (6.8) is the proper way to find indeed stable states.

Remark 6.2. *Let us note that, since the minimizers of* (6.8) *might be non-smooth, it is not guaranteed that they will satisfy the Euler-Lagrange equations either in strong or weak form. Indeed, in [16] even one-dimensional examples of smooth W were given such that the minimizer does not fulfill the Euler-Lagrange equation.*

One of the reasons why deducing the Euler-Lagrange equation might be difficult is that even the calculation of the variation of $\mathcal{E}$ itself can pose difficulties. Indeed, due to (6.6)*, the minimizer y might be such that $\mathcal{E}(y+t\varphi)$ is infinite for all small enough $t > 0$ and a large class of φ. Let us refer to [16] for explicit examples in which this situation occurs.*

Remark 6.3. *Let us notice that the condition* (6.6) *is really necessarily to be stated explicitly. Namely, from the physical point of view, the frame-indifference* (6.5) *requires that $W(F) := \tilde{W}(C)$ where $C := F^\top F$ is the so-called right Cauchy-Green strain tensor. Note that $F^\top Q^\top QF = F^\top F$ for any orthogonal matrix Q. Hence, pointwise minimizers the energy density W contain the set $\{QF_0 : Q \in \mathrm{O}(n)\}$ for some given matrix F_0 with $\det F_0 > 0$ which is a pointwise minimizer itself. Besides the physically acceptable energy wells $\{RF_0 : R \in SO(n)\}$ other minimizers live on a "dark" wells $\{RF_0 : R \in \mathrm{O}(n) \setminus \mathrm{SO}(n)\}$ which is not mechanically admissible. Those wells are excluded by* (6.6).

In order to prove existence of stable states, that is minimizers of (6.8), we assume suitable growth of the energy density:

$$W(F) \geq C(-1 + |F|^p) \text{ for all } F \in \mathbb{R}^{n\times n} \text{ and for some } C > 0, \tag{6.9}$$

The existence theorem follows then directly from Corollary 6.5.

THEOREM 6.11. *Let* $\Omega \subset \mathbb{R}^3$ *be a Lipschitz bounded domain,* $p > 3$*,* $y_0 \in W^{1,p}(\Omega;\mathbb{R}^3)$*, and* $\Gamma_{\rm D} \subset \partial\Omega$ *have a finite two-dimensional Lebesgue measure. Let* W *satisfy (i)-(iii) from Corollary 6.5 with* $m = n = 3$*. Let further* (6.5)–(6.7) *and* (6.9) *hold. If*

$$\mathcal{Y} := \{y \in W^{1,p}(\Omega;\mathbb{R}^3) : \, y = y_0 \text{ on } \Gamma_{\rm D}\}$$

is such that $\inf_{\mathcal{Y}} I < +\infty$ *then there is a minimizer of* $\mathcal{E}$ *on* $\mathcal{Y}$*.*

This result can be generalized for different growth conditions like the one considered in (6.13) below. Even more general settings can be found in [37] where various additional requirements on minimizers, as e.g. conditions ensuring a friction-less contact (Signorini problem); are included, too.

Let us mention a few important examples of polyconvex stored energy densities. Contrary to nontrivial examples of quasiconvex functions, it is relatively easy to design a polyconvex function. To ease our notation we only define the densities for matrices of positive determinant. Otherwise, it is implicitly extended by infinity. We refer to [136, 137, 138] for for examples of polyconvex functions with various special symmetries.

EXAMPLE 6.4 (*Compressible Mooney-Rivlin material.*). *This material has a stored energy of the form*

$$W(F) = a|F|^2 + b|{\rm Cof}F|^2 + \gamma(\det F)\,, \tag{6.10}$$

where $a, b > 0$ *and* $\gamma(\delta) = c_1\delta^2 - c_2 \log\delta$*,* $c_1, c_2 > 0$*.*

It can be shown that for $n = 3$

$$W(F) = \frac{\lambda}{2}({\rm tr}E)^2 + \mu|E|^2 + \mathcal{O}(|E|^3)\ ,\ E = (C - \mathbb{I})/2$$

where λ *and* μ *are the usual Lamé constants, and* $\mathbb{I}$ *denotes the identity matrix. Indeed, it is a matter of a tedious computation to show that, given* λ, μ*, the following equations must be fulfilled by* a, b, c_1, c_2*:* $c_2 := (\lambda + 2\mu)/2$ $2a + 2b = \mu$*, and* $4b + 4c_1 = \lambda$*.*

EXAMPLE 6.5 (Compressible neo-Hookean material.). *This material has a stored energy of the form*

$$W(F) = a|F|^2 + \gamma(\det F) \tag{6.11}$$

with the same constants as for the compressible Mooney-Rivlin materials.

EXAMPLE 6.6 (Ogden material.). *This material has a stored energy of the form (recall that* $C = F^\top F$*)*

$$W(F) = \sum_{i=1}^{M} a_i {\rm tr}\, C^{\gamma_i/2} + \sum_{i=1}^{N} b_i {\rm tr}({\rm Cof}\, C)^{\delta_i/2} + \gamma(\det F) \tag{6.12}$$

and $a_i, b_i > 0$, $\lim_{\delta\to 0_+} \gamma(\delta) = +\infty$ *for* $\gamma : \mathbb{R}_+ \to \mathbb{R}$ *convex growing suitably at infinity.*

If W satisfies conditions (6.6)-(6.7) then any $y \in C^1(\Omega, \mathbb{R}^3)$ for which $\mathcal{E}(y)$ from (6.8) is finite is also locally invertible. This follows from the standard inverse function theorem. Nevertheless, what is actually desired for a physical deformation is that it is *injective* [37]. Indeed, non-injectivity of the deformation would mean that two material points from the reference configuration would be mapped to just one in the deformed configuration which means that the specimen penetrated through itself. Thus, additional assumptions to (6.6)-(6.7) on W are needed to assure *global invertibility of* y. Preferably, these assumptions should be compatible with polyconvexity and weak lower semicontinuity.

Take a diffeomorphism $y : \Omega \to y(\Omega)$ with $\det\nabla y > 0$ on Ω. Then, we have by the change of variables formula for $p > 1$

$$\int_{y(\Omega)} |\nabla y^{-1}(w)|^p {\rm d}w = \int_\Omega |\nabla y^{-1}(y(x))|^p \det\nabla y(x){\rm d}x = \int_\Omega |(\nabla y(x))^{-1}|^p \det\nabla y(x){\rm d}x = \int_\Omega \frac{|{\rm Cof}^\top \nabla y(x)|^p}{(\det\nabla y(x))^{p-1}}{\rm d}x$$

where we used that $\nabla y^{-1}(y(x)) = (\nabla y(x))^{-1}$ for all x in Ω and that for any invertible matrix the relation $A^{-1} = \frac{\mathrm{Cof}^\top A}{\det A}$ holds.

Therefore, for energies satisfying a stricter growth condition than (6.9) in the form of

$$W(F) \geq C\left(-1 + |F|^p + \frac{|\mathrm{Cof} F^\top|^p}{(\det F)^{p-1}}\right) \text{ for a.a. } x \in \Omega \text{ and for some } C > 0, \tag{6.13}$$

one could rather expect that deformations on which $\mathcal{E}(y)$ is finite are invertible. This is indeed so, as the Theorem 6.12 (below) shows.

Nevertheless, before proceeding to the theorem, let us point out that the new growth condition (6.13) is *fully compatible with polyconvexity.* Indeed, since the function $g(x,y) = \frac{x^p}{y^{p-1}}$ is convex for $p > 1$ on the set $\{(x,y) \in \mathbb{R}^2; y > 0\}$, $\frac{|\mathrm{cof}^\top A|^p}{(\det A)^{p-1}}$ is polyconvex on the set of matrices having a positive determinant.

Theorem 6.12 (Taken from [9]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let $y_0 : \overline{\Omega} \to \mathbb{R}^n$ be continuous in $\overline{\Omega}$ and one-to-one in Ω such that $y_0(\Omega)$ is also bounded and Lipschitz. Let $y \in W^{1,p}(\Omega; \mathbb{R}^n)$ for some $p > n$, $y(x) = y_0(x)$ for all $x \in \partial\Omega$, and let* $\det \nabla y > 0$ *a.e. in Ω. Finally, assume that for some $q > n$*

$$\int_\Omega |(\nabla y(x))^{-1}|^q \det \nabla y(x)\,\mathrm{d}x < +\infty\ . \tag{6.14}$$

Then $y(\overline{\Omega}) = y_0(\overline{\Omega})$ and y is a homeomorphism of Ω onto $y_0(\Omega)$. Moreover, the inverse map $y^{-1} \in W^{1,q}(y_0(\Omega); \mathbb{R}^n)$ and $\nabla y^{-1}(w) = (\nabla y(x))^{-1}$ for $w = y(x)$ and a.a. $x \in \Omega$.

Let us note that the Sobolev regularity needed in the theorem has been weakened later in [147]. Indeed, in this work it was shown that an inverse to deformation can be defined even for $p > n-1$ and $q \geq \frac{p}{p-1}$.

Theorem 6.12 assures injectivity of y under the growth (6.13) if a up-to-the-boundary injective Dirichlet condition is prescribed. This, however, has the disadvantage that we could not model situations in which hard loads (Dirichlet boundary conditions) are prescribed only on a part on the boundary.

One possible remedy is to minimize $\mathcal{E}$ along with the so-called *Ciarlet-Nečas* condition

$$\int_\Omega \det\nabla y(x)\mathrm{d}x \leq \mathcal{L}^n(y(\Omega)), \tag{6.15}$$

that was introduced in [38] (for $n = 3$) in order to assure *global injectivity* of deformations. It was shown in [38] that C^1-functions satisfying (6.15) and that $\det\nabla y > 0$ are actually injective. The result generalizes to $W^{1,p}$-functions as well, but injectivity is obtained only almost everywhere in the deformed configuration; i.e., almost every point in the deformed configuration has only one pre-image.

Remark 6.7. *Maps that are injective almost everywhere in the deformed configuration still include rather nonphysical situations. For example a dense, countable set of points could be mapped to one point. This can be prevented if the deformation is* injective everywhere.

Using condition (6.15)*, this can be achieved for finite deformations of the energy $\mathcal{E}$ with a density W satisfying* (6.13) *for $p = r = n = 2$. This setting is the most explored one due to its relations to quasiconformal maps (see Section 7). Such deformations are open (that is they maps open sets to open sets) and discrete (the set of pre-images for any point does not accumulate) and, moreover, satisfy the Lusin N-condition (i.e. they map sets of zero measure again to sets of zero measure); cf. e.g. [77]).*

Then, we have by the area formula

$$\int_\Omega \det\nabla y\mathrm{d}x = \int_{\mathbb{R}^n} N(y,\Omega,z)\mathrm{d}z = \int_{y(\Omega)} N(y,\Omega,z)\mathrm{d}z$$

where $N(y,\Omega,z)$ is defined as the number of pre-images of $z \in y(\Omega)$ in Ω. So the Ciarlet-Nečas condition is satisfied if and only if $N(y,\Omega,z) = 1$ almost everywhere on $y(\Omega)$. Also we can immediately see that the reverse inequality to (6.15) always holds.

Further, if there existed $z \in y(\Omega)$ that had at least to two pre-images x_1 and x_2 then we could find an $\varepsilon > 0$ such that $B(x_1, \varepsilon) \cap B(x_2, \varepsilon) = \emptyset$ and $B(x_j, \varepsilon) \subset \Omega$ for $j = 1, 2$. On the other hand, for the images we have that $y(B(x_1, \varepsilon) \cap y(B(x_2, \varepsilon)) \neq \emptyset$. In fact, $y(B(x_1, \varepsilon)) \cap y(B(x_2, \varepsilon))$ is of positive measure since both $y(B(x_j, \varepsilon))$ are open. Therefore, there exists a set of positive measure where $N(y, \Omega, z)$ is at least two; a contradiction to (6.15).

6.3. Applications to hyperelasticity in the higher order setting. Let us now turn our attention to models of hyperelastic materials depending on higher-order gradients. Such materials are called non-simple of grade N, where N refers to the highest derivatives appearing in the stored energy density. The concept of such materials has been developing for long time, since the work by R.A. Toupin [153], under various names as non-simple materials as e.g. in [65, 90, 128, 143] or multipolar materials (in particular fluids).

Here, we will consider only second-grade non-simple materials, i.e., those for which second-order deformation gradients (first-order strain gradients) are involved. The main mathematical advantage of nonsimple materials is that higher-order deformation gradients bring additional regularity of deformations and, possibly, also compactness of the set of admissible deformations in a stronger topology. Moreover, there the stored energy can be even convex in the highest derivatives of the deformation which is helpful in proving existence of minimizers. The downside of this approach is that there are not many physically justified models of non-simple materials and material constants are rarely available.

For non-simple materials of grade two, we define an energy functional

$$\mathcal{E}(y) := \int_\Omega W(\nabla y(x), \nabla^2 y(x))\,\mathrm{d}x - \int_\Omega f(x) \cdot y(x)\,\mathrm{d}x - \int_{\Gamma_\mathrm{N}} \left(g(x) \cdot y(x) + \hat{g}_1(x) \cdot \frac{\partial y(x)}{\partial \nu} \right) \mathrm{d}S \ , \tag{6.16}$$

where $\hat{g}_1 : \Gamma_\mathrm{N} \to \mathbb{R}^n$ is the surface density of (hypertraction) forces balancing the *hyperstress*

$$x \mapsto \frac{\partial}{\partial G_{ijk}} W(F, G)|_{F=\nabla y(x),\, G=\nabla^2 y(x)} \ . \tag{6.17}$$

The corresponding first Piola-Kirchhoff stress tensor is constructed as follows.

Denote for $i, j \in \{1, \ldots, n\}$

$$H_{ij}(x, F, G) := \sum_{k=1}^n \frac{\partial}{\partial G_{ijk}} W(F, G) \ .$$

Then for $x \in \Omega$, $F := \nabla y(x)$, and $G := \nabla^2 y(x)$ we evaluate the first Piola-Kirchhof stress tensor as

$$S_{ij}(x) = \frac{\partial W(F, G)}{\partial F_{ij}} - H_{ij}(x, F, G) \ .$$

We will assume that

$$y \mapsto \int_\Omega f(x) \cdot y(x)\,\mathrm{d}x + \int_{\Gamma_\mathrm{N}} (g(x) \cdot y(x) + \hat{g}_1(x) \cdot \frac{\partial y(x)}{\partial \nu})\,\mathrm{d}S \tag{6.18}$$

is a linear functional evaluating the work of external forces on the specimen. The other terms containing f and g are volume and surface forces. Here we, however, assume for simplicity that f, $\hat{g}_1$, and g depend only on $x \in$ and $x \in \Gamma_\mathrm{N}$, respectively.

Notice that existence of minimizers of $\mathcal{E}(y)$ is guaranteed by Corollary 6.6.

Similarly, as in the case of simple materials, it allows for formal derivation of Euler-Lagrange equations for minimizers of I. Again, the approach is far from being rigorous because, in particular, we should compose deformations rather than to add them to each other. Contrary to the simple-material situation, here the smoothness of $\partial\Omega$ is important because the mean curvature κ of the boundary enters the equations. Details on surface differential operators can be found, for example in [127].

7. Weak lower semicontinuity in general hyperelasticity. We have seen in the last section that polyconvexity is relatively easy to be verified and it ensures weak lower semicontinuity of the corresponding energy functional. Nevertheless, there are materials that cannot be modeled by polyconvex energy densities.

A prototypical example are systems featuring phase transition with each phase characterized by some specific deformation of the underlying atomic lattice. This setup is for example found in *shape-memory alloys* (see e.g. the monographs [28, 50, 63, 64, 126], or a recent review [82]). Shape memory alloys are intermetallic materials which have a high-temperature highly symmetric phase called austenite and a low temperature phase called martensite which can, however, exist in several variants. Such systems are (for a suitable temperature range) typically modeled by a multi-well stored energy of the form

$$\begin{cases} W(QU_i) = 0 & \forall i = 1 \dots M, \quad \forall Q \in \mathrm{SO}(n), \\ W(F) > 0 & \forall F \neq QU_i \;\; \forall i = 1 \dots M, \quad \forall Q \in \mathrm{SO}(n), \end{cases} \tag{7.1}$$

where $U_1 \dots U_M$ is a given set of matrices representing the phases found in the material and $\mathrm{SO}(n)$ is the set of rotations in $\mathbb{R}^{n\times n}$. These materials form complicated patterns (microstructures) composed from different variants of martensite cf. Figure 1.

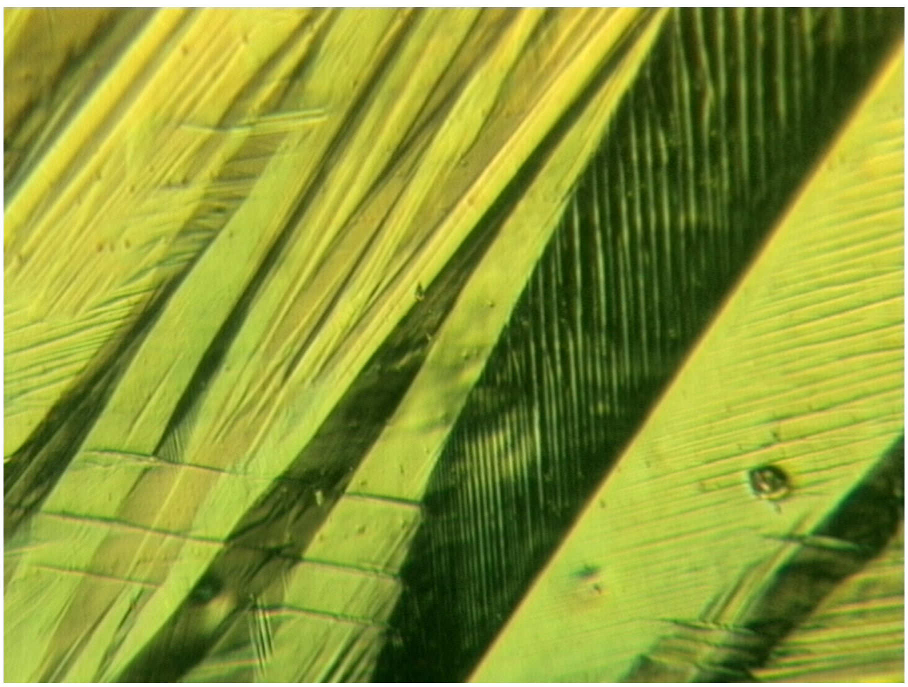

FIGURE 1. *Laminated microstructure in CuAlNi. Courtesy of P. Šittner (Inst. of Physics, CAS, Prague)*

Now, an energy density as given in (7.1) is neither polyconvex nor quasiconvex. and its construction is a modeling issue [158]. Therefore, for constructing an appropriate model one is to find the *weakly lower semicontinuous envelope* of (1.2) with an energy density given by (7.1); in other words, one seeks the supremum of weakly lower semicontinuous functionals lying below the given energy. We refer also to the subsection 7.1 on more details on how this *relaxation* of the problem may be performed.

In order to find the weakly lower semicontinuous envelope of (7.1), a precise characterization of weak lower semicontinuity in terms of convexity conditions on W is needed. We have found these conditions in Section 3; however, only under the growth condition (i) in Definition 3.2. Yet, this is incompatible with the physical assumptions formulated in (6.6)-(6.7).

For such energies it is no longer known that quasiconvexity implies weak lower semicontinuity. Indeed, this is one of the standing problems in elasticity, which was formulated by Ball in the following way:

Open problem 7.1 (Problem 1 in [11]). *"Prove the existence of energy minimizers for elastostatics for quasiconvex stored-energy functions satisfying* (6.7)*."*

Let us remark that these difficulties persist even if we used a geometrically linear description of energy wells; see [36], for instance.

Remark 7.1. *Notice that if* (7.1) *is additively enriched by a convex term of the form* $\varepsilon \int_\Omega |\nabla^2 y|^p \mathrm{d}x$*, which is usually interpreted as some kind of interfacial energy of the microstructure, Corollary 6.6 can be readily applied to show the existence of minimizers for* $\mathcal{E}$*.*

Let us also point out that a different approach has been proposed recently [145, 146]. There, a new notion of interface polyconvexity has been introduced which enables to prove existence of minimizers for simple materials with additional phase field variable.

Remark 7.2. *It has been pointed out in [11, 12] that one of the reasons why this problem is hard to solve is the fact that quasiconvexity possesses no local characterization [96].*

Let us stress that Problem 7.1 is an important attempt towards combining quasiconvexity and elasticity but additional steps are still required. Namely, if $u : \Omega \to \mathbb{R}^m$ entering (1.2) ought to represent a deformation of a physical body, it should be *injective* and *orientation-preserving*. Notice that this is not automatically satisfied for all maps on which the functional (1.2) is finite even if W fulfills (6.6)-(6.7). However, we may rely on Theorem 6.12 to assure this, provided suitable coercivity of the energy.

An alternative (and related approach) is to study directly weak lower semicontinuity along sequences found in a suitable class of mappings that are injective and orientation-preserving. As a first step, one may study classes of functions that fulfill some constraint on the Jacobian, e.g. that $\det\nabla u > 0$.

Even though Problem 7.1 remains widely open to date, it has been approached it from different perspectives recently. We review the results within this section.

In [91, 92], the authors study weak lower semicontinuity along sequences in $\{u_k\} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ with $p < n$ satisfying that $\det\nabla u_k > 0$. They proved that (1.3) with $v = v(x, \nabla u)$ is weak lower semi-continuous along such sequences if and only if it is $W^{1,p}$*-orientation preserving quasiconvex*, i.e.

$$v(x, A) \le \frac{1}{\mathcal{L}^n(\Omega)} \int_\Omega v(x, \nabla\varphi(x))\mathrm{d}x,$$

for all A with $\det(A) > 0$, all $\varphi \in W^{1,p}(\Omega;\mathbb{R}^m)$ satisfying that $\varphi(x) = Ax$ on $\partial\Omega$ and $\det\nabla\varphi(x) > 0$ for a.a. $x \in \Omega$.

However, in [92] the authors also show that, in fact, for $p < n$ *no* $W^{1,p}$-orientation preserving quasiconvex integrands exist that would satisfy the natural coercivity/growth condition

$$\frac{1}{C}\big(|A|^p + \kappa(\det A)\big) \le v(x, A) \le C\big(|A|^p + \kappa(\det A)\big)$$

for almost all $x \in \Omega$. Here, κ is a convex function satisfying that $\lim_{s\to 0}\kappa(s) = \infty$, $\kappa(s) = \infty$ for $s \le 0$ and $\limsup_{s\to\infty} \frac{\kappa(s)}{s^{p/n}} < \infty$. Notice that this growth condition is compatible with (6.6)-(6.7).

The proof in [91] is based on the so-called *convex integration*, a technique for solving differential inclusions. It goes back to Gromov [72] and it found applications in various problems including continuum mechanics and regularity theory; see e.g. [120]. We refer also to the monograph [43] where solutions to partial differential inclusions by means of Baire category methods are introduced, too. Interestingly, convex integration is an approach that found applications also in fluid dynamics [48].

To the best of our knowledge, the only works in which the authors actually considered equivalent characterization of weak lower semicontinuity for *injective maps* are [25] and [26] where the authors studied bi-Lipschitz and quasiconformal maps *in the plane*, respectively.

Here, by bi-Lipschitz maps the following set is meant

$$\begin{aligned} W^{1,\infty,-\infty}_+(\Omega;\mathbb{R}^2) = \Big\{ & y : \Omega \mapsto y(\Omega) \text{ an orientation preserving homeomorphism;} \\ & y \in W^{1,\infty}(\Omega;\mathbb{R}^2) \text{ and } y^{-1} \in W^{1,\infty}(y(\Omega);\mathbb{R}^2) \Big\}, \end{aligned} \tag{7.2}$$

while quasiconformal maps are introduced as follows

$$\mathcal{QC}(\Omega;\mathbb{R}^2) = \Big\{ y \in W^{1,2}(\Omega;\mathbb{R}^2) : \ y \text{ is a homeomorphism and } \exists K \geq 1 \text{ such that}$$
$$|\nabla y|^2 \leq K \det \nabla y \text{ a.e. in } \Omega \Big\}. \tag{7.3}$$

It is natural to expect that weak lower semicontinuity of the functional

$$I(y) = \int_\Omega v(\nabla y) \mathrm{d}x,$$

along sequences in $W^{1,\infty,-\infty}_+(\Omega;\mathbb{R}^2)$ or $\mathcal{QC}(\Omega;\mathbb{R}^2)$ is connected to a suitable notion of quasiconvexity of v. One even expects a *weaker* notion than the one from Definition 1.2 since the set of possible sequences along which semicontinuity is studied is restricted. Indeed, the perfectly fitted notion to this setting seems to be an alternation of Definition 1.2 where only function from $W^{1,\infty,-\infty}_+(\Omega;\mathbb{R}^2)$ or $\mathcal{QC}(\Omega;\mathbb{R}^2)$ enter as test functions. Exactly this result has been achieved in [25] and [26]; we review the result in Proposition 7.3.

First, let us introduce a notion of weak convergence on $W^{1,\infty,-\infty}_+(\Omega;\mathbb{R}^2)$ and $\mathcal{QC}(\Omega;\mathbb{R}^2)$. We say that $y_k \overset{*}{\rightharpoonup} y$ in $W^{1,\infty,-\infty}_+(\Omega;\mathbb{R}^2)$ if the sequence has uniformly bounded bi-Lipschitz constants[12] and $y_k \overset{*}{\rightharpoonup} y$ in $W^{1,\infty}(\Omega;\mathbb{R}^2)$. Note that the weak limit is bi-Lipschitz, too.

For a sequence $\{y_k\}_{k\in\mathbb{N}} \subset \mathcal{QC}(\Omega;\mathbb{R}^2)$, we say that it converges weakly to $y \in W^{1,2}(\Omega;\mathbb{R}^2)$ in $\mathcal{QC}(\Omega;\mathbb{R}^2)$ if $y_k \rightharpoonup y$ in $W^{1,2}(\Omega;\mathbb{R}^2)$, there exists a $K \geq 1$ such that the y_k are all K-quasiconformal and $y(x)$ is non-constant. Here it is important to assume that the limit function is non-constant for otherwise the limit function may not quasiconformal.[13]

Moreover, let us introduce the notions of *bi-quasiconvexity* and *quasiconformal quasiconvexity.*

Definition 7.2. *We say that a Borel measurable and bounded from below function $f : \mathbb{R}^{2\times 2} \to \Omega$ is bi-quasiconvex if*

$$\mathcal{L}^2(\Omega) f(A) \leq \int_\Omega f(\nabla \varphi(x))\,\mathrm{d}x \tag{7.5}$$

for all $\varphi \in W^{1,\infty,-\infty}_+(\Omega;\mathbb{R}^2)$, $\varphi = Ax$ on $\partial\Omega$ and all A with $\det A > 0$.

We say that f is quasiconformally quasiconvex if (7.5) holds for all A with $\det(A) > 0$. and all $\varphi \in \mathcal{QC}(\Omega;\mathbb{R}^2)$ such that $\varphi(x) = Ax$ on $\partial\Omega$.

Then we have the following result:

Proposition 7.3 (from [25] and [26]). *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Let v be continuous on the set of matrices with a positive determinant. Then v is bi-quasiconvex if and only if*

$$y \mapsto I(y) = \int_\Omega v(\nabla y(x))\,\mathrm{d}x$$

is sequentially weakly lower semicontinuous on $W^{1,\infty,-\infty}_+(\Omega;\mathbb{R}^2)$.*

Moreover, let v satisfy

$$0 \leq v(A) \leq c(1+|A|^2) \qquad \text{with } c > 0$$

on the set of matrices with a positive determinant. Then v is quasiconformally quasiconvex if and only if I is weakly lower semicontinuous on $\mathcal{QC}(\Omega;\mathbb{R}^2)$.

---

[12]Notice that a function $y \in W^{1,\infty,-\infty}_+(\Omega;\mathbb{R}^2)$ satisfies for all $x_1, x_2 \in \Omega$

$$\frac{1}{L}|x_1 - x_2| \leq |y(x_1) - y(x_2)| \leq L|x_1 - x_2| \ . \tag{7.4}$$

for some $L \geq 1$. This L is then called the bi-Lipschitz constant of y.

[13]Because a sequence of uniformly K-quasiconformal maps converges locally uniformly either to K-quasiconformal function or a constant and the locally uniform convergence is implied by the notion of weak convergence in $\mathcal{QC}(\Omega;\mathbb{R}^2)$ [5].

At the heart of the proof of Proposition 7.3 is the construction of a suitable cutoff method that is compatible with the bi-Lipschitz or the quasiconformal setting. Notice that the standard cutoff method cannot be used since it relies on convex averaging. Thus, as neither $W_+^{1,\infty,-\infty}(\Omega;\mathbb{R}^2)$ nor $\mathcal{QC}(\Omega;\mathbb{R}^2)$ are convex, we may "fall out" from these sets when relying on the standard cutoff method.

The approach taken in [25] and [26] is based on the characterization of the trace operator on sets $W_+^{1,\infty,-\infty}(\Omega;\mathbb{R}^2)$ as well as $\mathcal{QC}(\Omega;\mathbb{R}^2)$ due to [47, 154] and [27], respectively.

Even though Proposition 7.3 provides us with an weak lower semicontinuity result, this is not yet enough to prove existence of minimizers for functionals with densities from some suitable class. This is so, because bi-Lipschitz as well as quasiconformal maps include a L^∞-type constraint which can be enforced by letting the stored energy density be finite only on a suitable subset of $\mathbb{R}^{2\times 2}$; yet, this subset is usually left when employing cutoff methods—this happens even in the standard cases [42]. Thus letting v being infinite on some set of matrices is incompatible with the proof of Proposition 7.3.

The usual remedy for proving existence of minimizers or relaxation results is to work with L^p-type (with p finite) constraints only. In the setting from above this would mean to work with so-called *bi-Sobolev* classes (see e.g. [79]) for $1 < p < \infty$:

$$W_+^{1,p,-p}(\Omega;\mathbb{R}^2) = \Big\{ y : \Omega \mapsto y(\Omega) \text{ an orientation preserving homeomorphism}; \\ y \in W^{1,p}(\Omega;\mathbb{R}^2) \text{ and } y^{-1} \in W^{1,p}(y(\Omega);\mathbb{R}^2) \Big\}\ .$$

However, for these classes of functions, the approach from [25] and [26] cannot be adopted since a complete characterization of the trace operator on these classes is missing to date. In fact, we have the following

Open problem 7.4. *Characterize the class of functions $\mathcal{X}(\partial\Omega;\mathbb{R}^2)$ such that*

$$\mathrm{Tr} : W_+^{1,p,-p}(\Omega;\mathbb{R}^2) \xrightarrow{onto} \mathcal{X}(\partial\Omega;\mathbb{R}^2)$$

at least for Ω being the unit square.

Let us note that the above problem may play a role also when smooth approximation (by diffemorphisms) of deformations in elasticity is concerned. Indeed, the standard techniques of smoothing Sobolev functions (by a mollification kernel) fail under the injectivity requirement since they essentially rely on convex averaging.

Recently, several results on smoothing even under these constraints appeared [81, 46, 113, 78, 80] using completely different techniques and limiting their scope to planar deformations. In particular, in [81] the authors could prove that a homeomorphism in $W^{1,p}(\Omega;\mathbb{R}^2)$ can be strongly approximated by diffeomorphisms in the $W^{1,p}$-norm for $p > 1$. For $p = 1$ this result has recently been extended in [80].

Nevertheless, in elasticity, one might rather be interested in approximating a function in $W_+^{1,p,-p}(\Omega;\mathbb{R}^2)$ *together with its inverse.* To the authors knowledge, the only result in this direction is the one by [46] who showed that bi-Lipschitz maps can we strongly approximated together with their inverse in the $W^{1,p}$-norm for any finite p. Yet, for functions in $W_+^{1,p,-p}(\Omega;\mathbb{R}^2)$ with $p < \infty$ the problem remains largely open as mentioned also in [81].

To end this section let us remark (by formulating several open problems) that the relation of bi-quasiconvexity to the standard notions of convexity mentioned in this paper is still unexplored. We focus here only on bi-quasiconvexity but similar problems could be formulated also for quasiconformal quasiconvexity, too.

It is clear from the definitions that any function that is quasiconvex on the set of matrices with a positive determinant is also bi-quasiconvex. Moreover, bi-quasiconvexity implies, at least in the plane, rank-1 convexity on the set of matrices with a positive determinant.

Remark 7.3. *To see why bi-quasiconvexity implies rank-1 convexity on the set of matrices with a positive determinant, we proceed as follows. First, notice that the determinant changes affinely on rank-1 lines due to the formula*

$$\det(A + \lambda a \otimes n) = \det A\big(1 + \lambda n\cdot(A^{-1}a)\big), \tag{7.6}$$

where a and n are some arbitrary vectors. Therefore, rank-1 convexity on the set of matrices with a positive determinant is really meaningful, since all matrices on a rank-1 line between two matrices with a positive determinant have this property, too.

Next we mimic the proof from [42, Lemma 3.11 and Theorem 5.3] showing that quasiconvexity implies rank-1 convexity. Without loss of generalization, we suppose that Ω is the unit square and that we want to show rank-1 convexity along the line $A + a \otimes e_1$ with e_1 the unit vector in the first coordinate. Then we consider the following sequence of mappings

$$y_n(x) = y_n(x_1, x_2) = \begin{cases} Ax & \text{for } x_1 \in \left[\frac{k}{n}, \frac{k}{n} + \lambda\frac{1}{n}\right) \text{ for } k = 0 \dots n-1, \\ (A + a \otimes e_1)x & \text{for } x_1 \in \left[\frac{k}{n} + \lambda\frac{1}{n}, \frac{k+1}{n}\right) \text{ for } k = 0 \dots n-1, \end{cases}$$

with some $\lambda \in [0, 1]$. Notice that $\{y_n\}$ are Lipschitz, injective and that $(\nabla y)^{-1}$ is uniformly bounded and $\det(\nabla y)$ *is bounded away from zero. Thus, $\{y_n\}$ is a sequence of uniformly bi-Lipschitz maps that converges weakly to $\lambda Ax + (1 - \lambda)(A + a \otimes e_1)x$. We may therefore use the cut-off technique from [25] to modify the sequence in such a way that it attains exactly the value of the weak limit at the boundary. Then, the same procedure as in [42, Theorem 5.3] gives the rank-1 convexity.*

In summary, we have the following series of implications

$$\text{quasiconvexity on } \mathbb{R}^{2\times 2}_+ \Rightarrow \text{bi-quasiconvexity} \Rightarrow \text{rank-1 convexity on } \mathbb{R}^{2\times 2}_+,$$

where we denoted by $\mathbb{R}^{2\times 2}_+$ the two-times-two matrices with positive determinant. But it is unclear whether some of the converse implications holds, too. We have the following:

Open problem 7.5. *Does rank-1 convexity on $\mathbb{R}^{2\times 2}_+$ imply bi-quasiconvexity?*

Open problem 7.6. *Does bi-quasiconvexity imply quasiconvexity on $\mathbb{R}^{2\times 2}_+$?*

7.1. Relaxation of non(quasi)convex variational problems. As we have already seen, mathematical (hyper)elasticity is the area of analysis where mechanical requirements are above current tools and results available in the calculus of variations. Orientation preservation and injectivity for simple non-polyconvex materials are prominent examples. Resorting to non-simple materials depending on second-order deformation gradients might seem as a way out. What is a physically acceptable form of the higher-order energy density is, however, a largely open problem. We refer e.g. to [17] for a discussion on this topic.

Another approach is to accept the fact that our minimization problem may have *no solution* and to trace out behavior of minimizing sequences driving the elastic energy functional to its infimum on a given set of deformations and to read off some effective material properties out of their patterns. This is the idea of *relaxation* in the variational calculus. We explain main ideas on the example from the introduction. Assume we want to

$$\text{minimize } \mathcal{E}(y) := \int_\Omega W(\nabla y(x))\,\mathrm{d}x \ , \tag{7.7}$$

for $y \in \mathcal{Y}$. Here $\mathcal{Y}$ stands for an admissible set of deformations equipped with some topology. In typical situations, $\mathcal{Y}$ is a subset of a Sobolev space and the topology is the weak one on this space. If no minimizer exists but the infimum is finite we want to find a new functional $\mathcal{E}_R$ defined over $\mathcal{Y}$ such that the following properties hold:

(i) $\min_{\mathcal{Y}} \mathcal{E}_R = \inf_{\mathcal{Y}} \mathcal{E}$,
(ii) if $\{y_k\}_{k\in\mathbb{N}} \subset \mathcal{Y}$ is a minimizing sequence of $\mathcal{E}$ then its convergent subsequences converge to minimizers of $\mathcal{E}_R$ on $\mathcal{Y}$, and
(iii) any minimizer of $\mathcal{E}_R$ is a limit of a minimizing sequence of $\mathcal{E}$.

Notice that it is already implicitly assumed in (i) that minimizers of $\mathcal{E}_R$ do exist on $\mathcal{Y}$. Conditions (ii) and (iii) state that, roughly speaking, there is a "one-to-one" correspondence between minimizing sequences of $\mathcal{E}$ and minimizers of $\mathcal{E}_R$. If (i)-(iii) hold we say that $\mathcal{E}_R$ is the relaxation of $\mathcal{E}$ and that $\mathcal{E}_R$ is the *relaxed functional*. The concept of relaxation is also very closely related to Γ-convergence and Γ-limits introduced by E. de Giorgi. We refer to [30, 44] for a modern exposition.

If $\mathcal{Y} \subset W^{1,p}(\Omega;\mathbb{R}^n)$ and the continuous stored energy $W : \mathbb{R}^{m\times n} \to \mathbb{R}$ fulfills

$$c(-1 + |F|^p) \le W(F) \le C(1 + |F|^p) \tag{7.8}$$

with $C > c > 0$, and $1 < p < +\infty$ then Dacorogna [40] showed[14] that

$$\mathcal{E}_R(y) := \int_\Omega QW(\nabla y(x))\,\mathrm{d}x\ , \tag{7.9}$$

where $QW : \mathbb{R}^{m\times n} \to \mathbb{R}$ is the *quasiconvex envelope* (or quasiconvexification) of W which is the largest quasiconvex function not exceeding W. It can also be evaluated at any $A \in \mathbb{R}^{m\times n}$ as

$$QW(A) := \mathcal{L}^n(\Omega)^{-1} \inf_{\varphi\in W_0^{1,\infty}(\Omega;\mathbb{R}^m)} \int_\Omega W(A + \nabla\varphi(x))\,\mathrm{d}x\ . \tag{7.10}$$

The definition of QW does not depend on a (Lipschitz) domain Ω but as we see, calculation of QW requires to solve again a minimization problem. Not surprisingly, there are only a few cases where QW is known in a closed form. We wish to point out [49] where the authors calculated the quasiconvex envelope of the stored energy density arising in modeling of nematic elastomers in three dimensions, and [103, 104, 129] where the quasiconvex envelope of an isotropic homogeneous Saint-Venant Kirchoff energy density ($m = n = 3$)

$$W(F) := \frac{\mu}{4}\|C - \mathbb{I}\|^2 + \frac{\lambda}{8}(\operatorname{tr} C - 3)^2$$

is derived. Here λ, μ are Lamé constants of the material, and $C = F^\top F$ is the right Cauchy-Green strain tensor. Notice that W is convex in C but it is not even rank-one convex in F.

As to relaxation of multi-variant materials we refer to [89] where a geometrically linear two well-problem is considered such that elastic tensors of both variants are equal. It was later extended in [35] where non-equal moduli are admitted.

Another fairly popular and powerful tool for relaxation of variational problems in elasticity are so-called *Young measures* [157]. They allows us to describe the limit of a weakly converging sequence composed with a nonlinear function. In addition to the information contained in the weak limit (that is averaged patterns of the sequence), Young measures encode much more details. The original result of L.C. Young holds for L^∞-bounded sequences, the theorem below valid for L^p can be found in [10, 135].

Theorem 7.7 (L^p-Young measures). *If $\Omega \subset \mathbb{R}^n$ is bounded and $\{\xi_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^{m\times n})$, $1 \le p < +\infty$ is a bounded sequence then there exists a (non-relabeled) subsequence and a family of parametrized (by $x \in \Omega$) probability measures $\nu = \{\nu_x\}_{x\in\Omega}$ supported on $\mathbb{R}^{m\times n}$ such that for every $f \in C(\mathbb{R}^{m\times n})$, $\lim_{|A|\to\infty} f(A)/|A|^p = 0$ and every $g \in L^\infty(\Omega)$*

$$\lim_{k\to\infty} \int_\Omega f(\xi_k(x))g(x)\,\mathrm{d}x = \int_\Omega \int_{\mathbb{R}^{m\times n}} f(A)\nu_x(\mathrm{d}A)g(x)\,\mathrm{d}x\ . \tag{7.11}$$

If $\{|\xi_k|^p\}_{k\in\mathbb{N}}$ is relatively weakly compact in $L^1(\Omega)$ then (7.11) holds even if $|f(A)| \le C(1 + |A|^p)$ for some $C > 0$ and all $A \in \mathbb{R}^{m\times n}$.

The measure ν from Theorem 7.7 is called an L^p-Young measure generated by $\{\xi_k\}$.

The original energy functional introduced in (7.7) is then extended by continuity to obtain its relaxed version. Indeed, if $\{y_k\} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ is a bounded minimizing sequence for $\mathcal{E}$ we can assume that $\{|\nabla y_k|^p\}_{k\in\mathbb{N}}$ is relatively weakly compact in $L^1(\Omega)$ due to the Decomposition Lemma 3.6. Then applying Theorem 7.7 to $\xi_k := \nabla y_k$ we get

$$\inf \mathcal{E} = \lim_{k\to\infty} \mathcal{E}(y_k) = \int_\Omega \int_{\mathbb{R}^{m\times n}} W(A)\nu_x(\mathrm{d}A)\,\mathrm{d}x\ .$$

[14] In fact, Dacorogna's result is stated for more general integrands, namely $|W(F)| \le C(1 + |F|^p)$ with $QW > -\infty$, In this case, however, fixed Dirichlet boundary conditions must be inevitably assigned on the whole $\partial\Omega$. This is again strongly related to condition (ii) in Meyers ' Theorem 3.3.

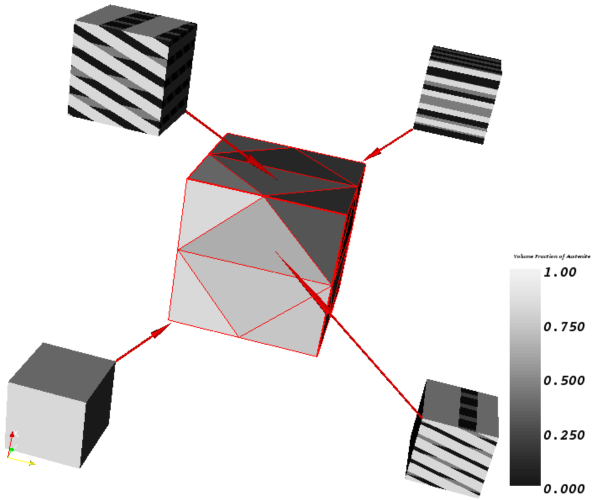

FIGURE 2. *An illustration of the calculation of a relaxed energy of a cube under loading. The cube in the middle is the specimen and on the sides the calculated microstructure in form of laminates is shown in a few elements. The gray scale indicates volume fractions of the phases involved.*

This, however, holds only if $|W| \le C(1+|\cdot|^p)$. In particular, constraints (6.6) and (6.7) cannot be imposed on W. An additional difficulty arises from the fact that created Young measure is generated by $\{\nabla y_k\}$, i.e., by gradients. A characterization of admissible measures, called gradient Young measures, involves quasiconvex functions again [88, 86, 123, 119, 133] which makes the aim of obtaining a closed formula of QW by means of parametrized measures unreachable, too. Nevertheless, subsets (called "laminates") and supersets of gradient Young measures are known and can be advantageously exploited in numerical minimization of (7.7) – see for example the illustration in Figure 2. See [6, 21, 22, 23, 24, 50, 99] for instance, for a numerical treatment of parametrized measures and a review paper by Luskin [106] on different finite-element approaches.

Recently, Conti and Dolzmann [39] proved that (7.9) is the relaxed problem corresponding to (7.7) even if (7.8) is replaced by

$$\begin{cases} c(-1+|F|^p+\theta(\det F)) \le W(F) \le C(1+|F|^p+\theta(\det F) & \text{if } \det F > 0, \\ W(F) = +\infty & \text{otherwise.} \end{cases}$$

Here, $C > c > 0$, $p \ge 1$, and $\theta : (0;+\infty] \to [0;+\infty)$ is a suitable convex function. They, however, require for the result to hold that QW is polyconvex. Needless to say that this assumption is extremely hard to verify. Results applicable to a generic situation are missing, so far.

To summarize, we clearly see that even current cutting-edge tools and weapons of mathematical analysis and calculus of variations are not tailored to fight deep problems in elasticity and new techniques are needed to solve them.

8. $\mathcal{A}$-quasiconvexity. In this section, we summarize results about weak lower semicontinuity of integral functionals along sequence which satisfy a first-order linear differential constraint. Clearly, gradients as curl-free fields are included in this setting and therefore this is really a generalization of (some) previously mentioned results. As emphasized by L. Tartar, besides curl-free fields there are also other PDE important constraints on possible minimizers. Such a setting naturally arises in electromagnetism, linearized elasticity or even higher-order gradients, to name a few. Tartar's program was materialized by Dacorogna in [41] and then studied by many other authors, too.

The problem studied in this section can be formulated as follows: Having a sequence $\{u_k\} \subset L^p(\Omega;\mathbb{R}^m)$, $1 < p < +\infty$ such that each member satisfies a linear differential constraint $\mathcal{A}u_k = 0$ ($\mathcal{A}$-free sequence), or $\mathcal{A}u_k \to 0$ in $W^{-1,p}(\Omega;\mathbb{R}^n)$ (asymptotically $\mathcal{A}$-free sequence), what conditions on v precisely ensure weak lower semicontinuity of integral functionals in the form

$$\mathcal{I}(u) := \int_\Omega v(x, u(x))\,\mathrm{d}x \ . \tag{8.1}$$

Here $\mathcal{A}$ is a first-order linear differential operator.

To the best of our knowledge, the first result of this type was proved in [60] for nonnegative integrands. In this case, the crucial necessary and sufficient condition ensuring weak lower semicontinuity of $\mathcal{I}$ in (8.1) is the so-called $\mathcal{A}$-quasiconvexity; cf. Def. 8.1 below. However, if we refrain from considering only nonnegative integrands, this condition is not necessarily sufficient as we already observed in the case $\mathcal{A}$:=curl.

8.1. The operator $\mathcal{A}$ and $\mathcal{A}$-quasiconvexity. Following [60], we consider linear operators $A^{(i)} : \mathbb{R}^m \to \mathbb{R}^d$, $i = 1, \ldots, n$, and define $\mathcal{A} : L^p(\Omega;\mathbb{R}^m) \to W^{-1,p}(\Omega;\mathbb{R}^d)$ by

$$\mathcal{A}u := \sum_{i=1}^n A^{(i)} \frac{\partial u}{\partial x_i} \ , \text{where} \ \ u : \Omega \to \mathbb{R}^m \ ,$$

i.e., for all $w \in W_0^{1,p'}(\Omega;\mathbb{R}^d)$

$$\langle \mathcal{A}u, w\rangle = -\sum_{i=1}^n \int_\Omega A^{(i)} u(x) \cdot \frac{\partial w(x)}{\partial x_i}\,\mathrm{d}x \ .$$

For $w \in \mathbb{R}^n$ we define the linear map

$$\mathbb{A}(w) := \sum_{i=1}^n w_i A^{(i)} \ : \ \mathbb{R}^m \to \mathbb{R}^d \ .$$

In this review, we assume that there is $r \in \mathbb{N} \cup \{0\}$ such that

$$\operatorname{rank} \mathbb{A}(w) = r \text{ for all } w \in \mathbb{R}^n \ , |w| = 1 \ , \tag{8.2}$$

i.e., $\mathcal{A}$ has the so-called *constant-rank property.* Below we use $\ker \mathcal{A}$ to denote the set of all locally integrable functions u such that $\mathcal{A}u = 0$ in the sense of distributions, i.e., $\int_\Omega u \cdot \mathcal{A}^* w\,\mathrm{d}x = 0$ for all $w \in C^\infty$ compactly supported in Ω. Here, $\mathcal{A}^* = -\sum_{i=1}^n \left(A^{(i)}\right)^T \frac{\partial u}{\partial x_i}$ is the formal adjoint of $\mathcal{A}$. Of course, $\ker \mathcal{A}$ depends on the considered domain Ω, which always should be clear from the context below.

Definition 8.1 (cf. [60, Def. 3.1, 3.2]). *We say that a continuous function $f : \mathbb{R}^m \to \mathbb{R}$, satisfying that $|f(s)| \le C(1+|s|^p)$ for some $C > 0$, is $\mathcal{A}$-quasiconvex if for all $s_0 \in \mathbb{R}^m$ and all $\varphi \in L^p_\#(Q;\mathbb{R}^m) \cap \ker \mathcal{A}$ with $\int_Q \varphi(x)\,\mathrm{d}x = 0$ it holds*

$$f(s_0) \le \int_Q f(s_0 + \varphi(x))\,\mathrm{d}x \ .$$

In the above definition, we used the space of Q-periodic Lebesgue integrable functions:

$$L^p_\#(\mathbb{R}^n;\mathbb{R}^m) := \{u \in L^p_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m) : \ u \text{ is } Q\text{-periodic}\}$$

Here, Q denotes the unit cube $(-1/2, 1/2)^n$ in $\mathbb{R}^n$, and we say that $u : \mathbb{R}^n \to \mathbb{R}^m$ is *Q-periodic* if for all $x \in \mathbb{R}^n$ and all $z \in \mathbb{Z}^n$ it holds that $u(x + z) = u(x)$.

Fonseca and Müller [60] proved the following result linking $\mathcal{A}$-quasiconvexity and weak lower semicontinuity. Notice that the integrand is more general than that one in (8.1).

Theorem 8.2. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and let $v : \Omega \times \mathbb{R}^d \times \mathbb{R}^m \to [0;+\infty)$ be a Carathéodory integrand. Let*

$$0 \le v(x, z, u) \le a(x, z)(1 + |u|^p)$$

for almost every $x \in \Omega$ and all $(z,u) \in \mathbb{R}^d \times \mathbb{R}^m$, $1 < p < +\infty$, and some $0 \le a \in L^\infty_{\rm loc}(\Omega;\mathbb{R}^d)$. Assume that $z_k \to z$ in measure and that $u_k \rightharpoonup u$ in $L^p(\Omega;\mathbb{R}^d)$,$\|\mathcal{A}u_k\|_{W^{-1,p}(\Omega;\mathbb{R}^m)} \to 0$.

Then

$$\liminf_{k\to\infty} \int_\Omega v(x,z_k,u_k)\,\mathrm{d}x \ge \int_\Omega v(x,z,u)\,\mathrm{d}x$$

if and only if $v(x,z,\cdot)$ is $\mathcal{A}$-quasiconvex for almost all $x \in \Omega$ and all $z \in \mathbb{R}^d$.

The following definition is motivated by our discussion above Theorem 3.7. It first appeared in [57].

Definition 8.3. *Let $1 < p < +\infty$ and $\{u_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m) \cap \ker\mathcal{A}$. We say that $\{u_k\}$ has an $\mathcal{A}$-free p-equiintegrable extension if for every domain $\tilde\Omega \subset \mathbb{R}^n$ such that $\Omega \subset \tilde\Omega$, there is a sequence $\{\tilde u_k\}_{k\in\mathbb{N}} \subset L^p(\tilde\Omega;\mathbb{R}^m) \cap \ker\mathcal{A}$ such that*
(i) $\tilde u_k = u_k$ a.e. in Ω for all $k \in \mathbb{N}$,
(ii) $\{|\tilde u_k|^p\}_{k\in\mathbb{N}}$ is equiintegrable on $\tilde\Omega \setminus \Omega$, and
(iii) there is $C > 0$ such that $\|\tilde u_k\|_{L^p(\tilde\Omega;\mathbb{R}^m)} \le C\|u_k\|_{L^p(\Omega;\mathbb{R}^m)}$ for all $k \in \mathbb{N}$.

Then we have the following result proved in [57].

Theorem 8.4. *Let $0 \le g \in C(\bar\Omega)$, let $|v_0| \le C(1+|\cdot|^p)$ be $\mathcal{A}$-quasiconvex, satisfy (3.1), have a recession function, and let $1 < p < +\infty$. Let $\{u_k\} \subset L^p(\Omega;\mathbb{R}^m) \cap \ker\mathcal{A}$, $u_k \rightharpoonup u$ weakly, and assume that $\{u_k\}$ has an $\mathcal{A}$-free p-equiintegrable extension. Then $\mathcal{I}(u) \le \liminf_{k\to\infty} \mathcal{I}(u_k)$, where*

$$\mathcal{I}(u) := \int_\Omega g(x)v_0(u(x))\,\mathrm{d}x. \tag{8.3}$$

Surprisingly, it is shown in [60, p. 1380] that also higher-order gradients can be recast as $\mathcal{A}$-free mappings. They construct $\mathcal{A}$ such that $\mathcal{A}u = 0$ if and only if $u = \nabla^k w$ for some $w \in W^{k,p}(\Omega;\mathbb{R}^m)$. In this situation, $\mathcal{A}$-quasiconvexity coincides with Meyers' k-quasiconvexity. Then, it follows from Theorem 8.4 that taking $w_0 \in W^{k,p}(\Omega;\mathbb{R}^m)$, g, and v as in the theorem then $\mathcal{I}(w) := \int_\Omega g(x)v(\nabla^k w(x))\,\mathrm{d}x$ is weakly lower semicontinuous on

$$\{w \in W^{k,p}(\Omega;\mathbb{R}^m) :\ w = w_0 \text{ on } \partial\Omega\}\ .$$

This result affirmatively answers Meyer's conjecture in this particular setting[15], namely mappings satisfying Dirichlet boundary conditions (belonging to a "Dirichlet class") make (ii) to hold automatically in Theorem 3.3.

9. Suggestions for further reading. The above exposition aims at reflecting developments in weak lower semicontinuity related to Meyers' paper [110] with the emphasize on applications to static problems in continuum mechanics of solids. We dare to hope that it provides a fairly completely picture of the theory starting in 1965 to current trends.

We saw that weak lower semicontinuity serves as a main ingredient of proofs of existence of minimizers to variational integrals and outlined applications in elastostatics. Even in the static case, models of elasticity can be combined with other phenomena, as magnetism, for instance. This leads to magnetoelasticity (magnetostriction), a property of NiMnGa, for instance. We refer e.g. to [54] for a physical background.

Weak lower semicontinuity finds its application in dynamical problems, too. For example, it is the main tool to prove existence of solutions in time-discrete approximations of evolution in various models. We refer e.g. to [111] for many such instances. We also refer to [68] and references therein for further results concerning mathematical treatment of nonlinear elasticity.

Let us finally point out that treatment of $\mathcal{A}$-quasiconvexity for integrands whose negative part growth with the p-th power is a very subtle issue which has recently been treated in [93]. There is a new condition called $\mathcal{A}$-quasiconvexity at the boundary which is introduced in two forms depending whether u can be

[15] In fact, one can consider integrands of the form $v(x,\nabla^k w(x))$ whenever $v(x,\cdot)$ is k-quasiconvex for all $x \in \bar\Omega$, $|v(x,A)| \le C(1+|A|^p)$, $v(\cdot;A)$ is continuous in $\bar\Omega$ for all $A \in X(n,m,k)$, and $v(x,\cdot)$ possesses a recession function.

extended to a larger domain preserving the $\mathcal{A}$-free property or not. This allows us to remove the assumption on the existence of an $\mathcal{A}$-free p-equiintegrable extension from Theorem 8.4.

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