Homogenization of a stationary flow of an electrorheological fluid

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Abstract

We combine two scale convergence, theory of monotone operators and results on approximation of Sobolev functions by Lipschitz functions to prove a homogenization process for a flow of an electrorheological fluid. We avoid the necessity of testing the weak formulation of the initial and homogenized systems by corresponding weak solutions, which allows mild assumptions on lower bound for a growth of the elliptic term. We show that the stress tensor for homogenized problems depends on the symmetric part of the velocity gradient involving the limit of a sequence selected from a family of solutions of initial problems.

Keywords: electrorheological fluid, weak solution, homogenization, Lipschitz truncation method, two scale convergence

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1. Introduction

Electrorheological fluids are special liquids characterized by their ability to change significantly the mechanical properties when an electric field is applied. This behavior has been extensively investigated for the development of smart fluids, which are currently exploited in technological applications, e.g. brakes, clutches or shock absorbers. Results of the ongoing research indicate their possible applications also in electronics. One approach for modeling of the flow of electrorheological fluids is the utilization of a system of partial differential equations derived by Rajagopal and Růžička, for details see [14]. This system in the case of an isothermal, homogeneous (with density equal to one), incompressible electrorheological fluid reads

$$\partial_t \mathbf{u} - \operatorname{div} \mathbf{S} + \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) + \nabla \pi = \mathbf{f}, \quad \operatorname{div} \mathbf{u} = 0, \tag{1}$$

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where $\mathbf{u}$ is the velocity, $\mathbf{S}$ the extra stress tensor, $\operatorname{div}(\mathbf{u}\otimes\mathbf{u})$ is the convective term with $\mathbf{u}\otimes\mathbf{u}$ denoting the tensor product of the vector $\mathbf{u}$ with itself defined as $(u_i u_j)_{i,j=1,\ldots,n}$, π is the pressure and $\mathbf{f}$ the external body force. The stress tensor $\mathbf{S}$ is assumed to depend on the symmetric part $\mathbf{Du}$ of the velocity gradient $\nabla\mathbf{u}$. The presence of an electric field is captured by the supposed dependence of $\mathbf{S}$ on the spatial variable in such a way that the growth of $\mathbf{S}$ corresponds to $|\mathbf{Du}|^{p(\cdot)-1}$ for some variable exponent p.

For this setting assuming additionally a periodic variable exponent with a small period ε, it was shown by Zhikov in [19] that as $\varepsilon\to 0$ a subsequence of solutions of initial problems converges to a solution of the homogenized problem having the extra stress tensor independent of the spatial variable. Zhikov's approach is based on the fact that the regularity of solutions of the initial as well as homogenized problem allows to use these solutions as a test function. In fact, this sufficient regularity is ensured by the value of the lower bound for the variable exponent $p\geq p_0 := \max((d+\sqrt{3d^2+4d})/(d+2), 3d/(d+2))$.

In the seminal article [10] a method of Lipschitz approximation of Sobolev functions is developed that allows to decrease the lower bound for p. In the article [10] the method is applied to the problem of existence of a weak solution to the stationary generalized Navier-Stokes model. The stationary problem with elliptic operator with Orlicz growth is studied in [6]. It took lot of work till the approach was modified in such a way that it is applicable to evolutionary problems. See [7], where the existence of a weak solution to the evolutionary generalized Navier Stokes problem is studied. The method is used to a evolutionary problem in Orlicz setting in the article [4]. The existence of a solutions to the problem (1) can be shown if $p>2d/(d+2)$. It is natural to ask: "Can one proceed with the homogenization process also if the lower bound for p is between p_0 and $2d/(d+2)$?" This paper should be regarded as the first step on the way for the answer to this question. To concentrate on the interplay between method of Lipschitz approximations and two scale convergence we start with the stationary problem first.

Let us introduce the problem, which we deal with. The domain $\Omega\subset\mathbf{R}^d$ is supposed to be bounded and Lipschitz, $Y=(0,1]^d$. For $\varepsilon\in(0,1)$ we consider the following stationary version of the problem (1)

$$
\begin{gathered}
-\operatorname{div}\left(\mathbf{S}\left(\frac{x}{\varepsilon},\mathbf{Du}^\varepsilon\right)-\mathbf{u}^\varepsilon\otimes\mathbf{u}^\varepsilon\right)+\nabla\pi^\varepsilon=-\operatorname{div}\mathbf{F},\quad \operatorname{div}\mathbf{u}^\varepsilon=0\quad \text{in }\Omega,\\
\mathbf{u}^\varepsilon=0\quad\text{on }\partial\Omega,\quad \int_\Omega\pi^\varepsilon=0.
\end{gathered}
\tag{2}
$$

The tensor $\mathbf{S}:\mathbf{R}^d\times\mathbf{R}^{d\times d}_{sym}\to\mathbf{R}^{d\times d}_{sym}$ is given in a special form

$$
\mathbf{S}(y,\boldsymbol{\xi})=\delta|\boldsymbol{\xi}|^{\beta-2}\boldsymbol{\xi}+\mathbf{A}(y,\boldsymbol{\xi}),\tag{3}
$$

where $\delta>0$, $\beta>2d/(d+2)$. There is a Y-periodic function $p:\mathbf{R}^d\to[1,\beta]$ such that $\mathbf{A}$ fulfills:

Assumption 1.1. *1.* $\mathbf{A}$ *is* $Y-$*periodic in the first variable,*

2. $\mathbf{A}(y,\cdot)$ *is continuous,*

3. *for* $\boldsymbol{\xi}_1 \neq \boldsymbol{\xi}_2$ $(\mathbf{A}(y,\boldsymbol{\xi}_1) - \mathbf{A}(y,\boldsymbol{\xi}_2)) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) > 0,$

4. *there are* $c_1, c_2 > 0$: $\mathbf{A}(y,\boldsymbol{\xi}) \cdot \boldsymbol{\xi} \geq c_1(|\boldsymbol{\xi}|^{p(y)} - 1)$, $|\mathbf{A}(y,\boldsymbol{\xi})|^{p'(y)} \leq c_2(|\boldsymbol{\xi}|^{p(y)} + 1)$, *where* $p'(y) = \frac{p(y)}{p(y)-1}$.

For problem (2) we establish a homogenized problem

$$-\operatorname{div}\left(\widehat{\mathbf{S}}(\mathbf{Du}) - \mathbf{u}\otimes\mathbf{u}\right) + \nabla\pi = -\operatorname{div}\mathbf{F} \quad \text{in } \Omega, \quad \operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega,$$
$$\mathbf{u} = 0 \quad \text{on } \partial\Omega, \quad \int_\Omega \pi = 0. \tag{4}$$

Our effort will be spent on determining the homogenized stress tensor $\widehat{\mathbf{S}}$. The situation is similar to the limit passage in the stress tensor in the proof of the existence of weak solutions of generalized Navier-Stokes equations. However, one cannot straightforwardly adopt the methods, which are successfully applied for existence proofs, because of oscillations which occur in the spatial variable of the stress tensor. We prove

Theorem 1.1. *Let* $\Omega \subset \mathbf{R}^d$ *be a bounded Lipschitz domain,* $\beta > 2d/(d+2)$, *the measurable function* $p : \mathbf{R}^d \to [1,\beta]$ *be* Y *periodic,* $\mathbf{S}$ *satisfy Assumption 1.1 and* $\mathbf{F} \in L^{\beta'}(\Omega; \mathbf{R}^{d\times d}_{sym})$. *Let* $\{(\mathbf{u}^\varepsilon, \pi^\varepsilon)\}$ *be a family of weak solutions of the system* (2) *constructed in Theorem 3.1. Then a sequence* $\{\epsilon_k\}$ *exists such that as* $k \to +\infty$

$$\epsilon_k \to 0, \quad \mathbf{u}^{\epsilon_k} \rightharpoonup \mathbf{u} \text{ in } W^{1,\beta}_0(\Omega;\mathbf{R}^d), \quad \pi^{\epsilon_k} \rightharpoonup \pi \text{ in } L^s(\Omega),$$

where s *is determined in* (11) *and* $(\mathbf{u},\pi)$ *is a weak solution of the system* (4) *with* $\hat{\mathbf{S}}$ *given by* (14).

2. Preliminaries

The following function spaces appear further:
$C^\infty_{0,\mathrm{div}}(\Omega) = \left\{\mathbf{u} \in C^\infty_0(\Omega;\mathbf{R}^d) : \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega\right\}$, $C^\infty_{per}(Y) = \{u \in C^\infty(\mathbf{R}^d) : u\ Y\text{-periodic}\}$, $C^\infty_{per,div}(Y) = \{u \in C^\infty_{per}(\mathbf{R}^d) : \operatorname{div}\mathbf{u} = 0 \text{ in } Y\}$, $W^{1,\beta}_{per}(Y,\mathbf{R}^d)$ is a closure of $\{\mathbf{u} \in C^\infty_{per}(Y), \int_Y \mathbf{u} = 0\}$ in the classical Sobolev norm, $\mathcal{D}\left(\Omega; C^\infty_{per}(Y)\right)$ is the space of smooth functions $u : \Omega \times \mathbf{R}^d \to \mathbf{R}$ such that $u(x,\cdot) \in C^\infty_{per}(Y)$ for any $x \in \Omega$ and there is $K \Subset \Omega$ such that for any $x \in \Omega \setminus K$: $u(x,\cdot) = 0$ in $\mathbf{R}^d$.

We introduce a closed subspace of $L^\beta(Y;\mathbf{R}^{d\times d}_{sym})$ and its annihilator, a subspace of $L^{\beta'}(Y;\mathbf{R}^{d\times d}_{sym})$, by

$$G(Y) = \left\{\mathbf{Dw} : \mathbf{w} \in W^{1,\beta}_{per}(Y;\mathbf{R}^d), \operatorname{div}\mathbf{w} = 0 \text{ in } Y\right\},$$

$$G^\perp(Y) = \left\{\mathbf{V}^* \in L^{\beta'}(Y;\mathbf{R}^{d\times d}_{sym}) : \forall \mathbf{V} \in G(Y) \ \int_Y \mathbf{V}^*(y)\cdot\mathbf{V}(y)\,\mathrm{d}y = 0\right\}.$$

Note that $C^\infty_{per,div}(Y)$ is dense in $G(Y)$. If we consider the set $\mathbf{R}^{d\times d}_{sym}$ as a subset of constant functions of $L^\beta(Y;\mathbf{R}^{d\times d}_{sym})$ then $\mathbf{R}^{d\times d}_{sym}\cap G(Y)=\emptyset$.

Proposition 2.1. *We have the following identification for the annihilator of the direct sum* $\mathbf{R}^{d\times d}_{sym}\oplus G(Y)$

$$\left(\mathbf{R}^{d\times d}_{sym}\oplus G(Y)\right)^\perp=\left(G^\perp(Y)\right)_0=\left\{\mathbf{V}^*\in G^\perp;\int_Y\mathbf{V}^*=0\right\}.$$

Proof. Clearly, $\left(G^\perp(Y)\right)_0\subset\left(\mathbf{R}^{d\times d}_{sym}\oplus G(Y)\right)^\perp$. By the definition of the annihilator, $\mathbf{V}^*\in\left(\mathbf{R}^{d\times d}_{sym}\oplus G(Y)\right)^\perp$ means that for any $\boldsymbol{\eta}\in\mathbf{R}^{d\times d}_{sym},\mathbf{V}\in G(Y)$ there holds $\int_Y\mathbf{V}^*(\boldsymbol{\eta}+\mathbf{V})=0$. The choice $\boldsymbol{\eta}=0$ implies $\mathbf{V}^*\in G^\perp(Y)$ and setting $\mathbf{V}=0,\boldsymbol{\eta}=\int_Y\mathbf{V}^*$ we get $\int_Y\mathbf{V}^*=0$. ☐

For the sake of clarity, we recall the meaning of differential operators appearing in the paper. Let us consider $\mathbf{u}:\Omega\times Y\to\mathbf{R}^d$ then

$$\nabla_x\mathbf{u}=\left(\frac{\partial u_i}{\partial x_j}\right)^d_{i,j=1},\ \operatorname{div}_x\mathbf{u}=\sum_{i=1}^d\frac{\partial u_i}{\partial x_i},\ \nabla_y\mathbf{u}=\left(\frac{\partial u_i}{\partial y_j}\right)^d_{i,j=1},\ \operatorname{div}_y\mathbf{u}=\sum_{i=1}^d\frac{\partial u_i}{\partial y_i}.$$

We omit the subscript if the function depends on the variable from one domain only. Throughout the paper the identity matrix is denoted by $\mathbf{I}$, the zero matrix by $\mathbf{O}$. The generic constants are denoted by c. When circumstances require it, we may also include quantities, on which the constant depend, e.g. $c(d)$ for the dependence on the dimension d. If we want to distinguish between different constants in one formula, we utilize subscripts, e.g. c_1,c_2 etc.

Let M,N be open subsets of $\mathbf{R}^d$. $M\Subset N$ means that $M\subset\overline{M}\subset N$, $\overline{M}$ being compact.

2.1. Lebesgue spaces with variable exponents

Definition 2.1. *We introduce the modular*

$$\rho_{p,\Omega}(u)=\int_\Omega|u(x)|^{p(x)}\,\mathrm{d}x$$

for $u\in L^1(\Omega)$*. The variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ *is defined as*

$$L^{p(\cdot)}(\Omega)=\{u\in L^1(\Omega):\rho_{p,\Omega}(u)<\infty\}$$

and endowed with the norm

$$\|u\|_{L^{p(\cdot)}(\Omega)}=\inf\left\{\lambda>0:\rho_{p,\Omega}\left(\frac{u}{\lambda}\right)\le1\right\}.$$

Proposition 2.2. *Let* $\Omega\subset\mathbf{R}^d$ *be bounded. Then the embedding* $L^{p(\cdot)}(\Omega)\hookrightarrow L^{q(\cdot)}(\Omega)$ *holds if and only if* $q\le p$ *almost everywhere in* Ω.

Proof. See [5, Section 3.3]. ☐

2.2. Generalized Orlicz and Sobolev-Orlicz spaces

Definition 2.2 ([12], Section 14.1)**.** *Let φ be a function on a Banach space X. The conjugate function of φ is the function on X^* defined for $\boldsymbol{\zeta} \in X^*$ by*

$$\varphi^*(\boldsymbol{\zeta}) = \sup_{\boldsymbol{\xi} \in X} \{\boldsymbol{\zeta}(\boldsymbol{\xi}) - \varphi(\boldsymbol{\xi})\}.$$

Definition 2.3. *We say that a function $\varphi : \mathbf{R}^n \to \mathbf{R}$ is a generalized $N-$function if φ is convex, even and is superlinear, i.e.*

$$\lim_{|\boldsymbol{\xi}| \to 0} \frac{\varphi(\boldsymbol{\xi})}{|\boldsymbol{\xi}|} = 0, \lim_{|\boldsymbol{\xi}| \to \infty} \frac{\varphi(\boldsymbol{\xi})}{|\boldsymbol{\xi}|} = \infty. \tag{5}$$

We say that the generalized $N-$function φ satisfies Δ_2-condition if there is $C > 0$ such that for all $\boldsymbol{\xi} \in \mathbf{R}^n$ $\varphi(2\boldsymbol{\xi}) \leq C\varphi(\boldsymbol{\xi})$.
We introduce the modular

$$\rho_{\varphi,\Omega}(\mathbf{v}) = \int_\Omega \varphi(\mathbf{v}(x)) \, \mathrm{d}x$$

and if φ satisfies Δ_2-condition we define the Orlicz space

$$L^\varphi(\Omega; \mathbf{R}^n) = \{\mathbf{v} \in L^1(\Omega; \mathbf{R}^n) : \rho_{\varphi,\Omega}(\mathbf{v}) < \infty\},$$

endowed with the norm

$$\|\mathbf{v}\|_{L^\varphi(\Omega)} = \inf \left\{ \lambda > 0 : \rho_{\varphi,\Omega}\left(\frac{\mathbf{v}}{\lambda}\right) \leq 1 \right\}.$$

The notion of generalized $N-$function was introduced in [15]. In [16], Orlicz spaces generated by such functions were introduced and their basic properties were studied.

2.3. Auxiliary tools

Lemma 2.1 (Biting lemma)**.** *Let $E \subset \mathbf{R}^d$ be a bounded domain and $\{v^n\}$ be a sequence of functions bounded in $L^1(E)$. Then there exists a subsequence $\{v^{n_k}\} \subset \{v^n\}$, a function $v \in L^1(E)$ and a sequence of measurable sets $\{E_j\}, E \supseteq E_1 \supseteq E_2 \supseteq \cdots$ with $|E_j| \to 0$ as $j \to +\infty$ such that for each j: $v^{n_k} \rightharpoonup v$ in $L^1(E \setminus E_j)$ as $k \to +\infty$.*

Proof. See [3]. □

Theorem 2.1 (Dunford)**.** *Let $\Sigma \in \mathbf{R}^d$ be a measurable set. A subset M of $L^1(\Sigma)$ is relatively weakly compact if and only if it is bounded and uniformly integrable, i.e. for any $\theta > 0$ there is $\delta > 0$ such that for any $f \in M$ and a measurable $K \subset \Sigma$ with $|K| < \delta$ we have $\int_K |f| < \theta$.*

Proof. See [8, Section III.2 Theorem 15]. □

Lemma 2.2. *Let $\Sigma \subset \mathbf{R}^d$ be an open set, $p, q, r > 1$. Assume*

$$\mathbf{u}^n \rightharpoonup \mathbf{u} \text{ in } L^p(\Sigma; \mathbf{R}^d), \mathbf{v}^n \rightharpoonup \mathbf{v} \text{ in } L^q(\Sigma; \mathbf{R}^d) \text{ as } n \to +\infty \text{ and } \frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1.$$

In addition, let for a certain $s > 1$ $\{\operatorname{div} \mathbf{u}^n\}$ be precompact in $\left(W_0^{1,s}(\Sigma; \mathbf{R}^d)\right)^*$, $\{\operatorname{curl} \mathbf{v}^n\} = \{\nabla(\mathbf{v}^n) - (\nabla(\mathbf{v}^n))^T\}$ be precompact in $\left(W_0^{1,s}(\Sigma; \mathbf{R}^d)^d\right)^*$. Then

$$\mathbf{u}^n \cdot \mathbf{v}^n \rightharpoonup \mathbf{u} \cdot \mathbf{v} \text{ in } L^r(\Sigma).$$

Proof. See [9, Theorem 10.21]. □

Lemma 2.3. *Let $\Sigma \subset \mathbf{R}^d$ be a bounded Lipschitz domain, $q \in (1, \infty)$ and denote $L_0^q(\Sigma) = \{h \in L^q : \int_\Sigma h = 0\}$. There exists a continuous linear operator $\mathcal{B} : L_0^q(\Sigma) \to W_0^{1,q}(\Sigma; \mathbf{R}^d)$ such that $\operatorname{div} \mathcal{B}h = h$ for any $h \in L_0^q(\Sigma)$.*

Proof. See [9, Theorem 10.11]. □

Theorem 2.2. *Let $\Omega \subset \mathbf{R}^d$ be open and $\Phi : Y \times \mathbf{R}^{d\times d}_{sym} \to \mathbf{R}$ satisfy:*

- *(i) Φ is Carathéodory, i.e. for all $\boldsymbol{\xi} \in \mathbf{R}^{d\times d}_{sym}$ $\Phi(\cdot, \boldsymbol{\xi})$ is measurable, for almost all $y \in \Omega$ $\Phi(y, \cdot)$ is continuous,*
- *(ii) for almost all $y \in \Omega$ $\Phi(y, \cdot)$ is convex,*
- *(iii) $\Phi \geq 0$.*

Then for every $\mathbf{U}^\varepsilon, \mathbf{U}^0 \in L^1(\Omega \times Y; \mathbf{R}^{d\times d}_{sym})$ such that $\mathbf{U}^\varepsilon \rightharpoonup \mathbf{U}$ in $L^1(\Omega \times Y; \mathbf{R}^{d\times d}_{sym})$, we have that

$$\liminf_{\varepsilon \to 0} \int_{\Omega \times Y} \Phi\left(y, \mathbf{U}^\varepsilon(x, y)\right) \, \mathrm{d}y \, \mathrm{d}x \geq \int_{\Omega \times Y} \Phi(y, \mathbf{U}(x, y)) \, \mathrm{d}y \, \mathrm{d}x.$$

Proof. See [11, Theorem 4.5], where a more general assertion is proved. □

Theorem 2.3. *Let X be a Banach space, V be a subspace of X, A be a closed convex functional on X and A be continuous at some $x \in X$. Then*

$$\inf_V A + \inf_{V^\perp} A^* = 0.$$

Proof. See [12, Theorem 14.2], where a more general assertion is proved. □

Remark 2.1. *We observe that if g is a functional on a Banach space X and $\eta \in X^*$ then*

$$\forall \xi \in X^* : (g - \eta)(\xi) = \sup_{x \in X} \{\langle \eta + \xi, x \rangle - g(x)\} = g^*(\eta + \xi). \tag{6}$$

Moreover, if g is closed, convex and continuous at some $x \in X$, V is a subspace of X, we obtain combining Theorem 2.3 and (6) *(for $A(x) := (g - \eta)(x)$)*

$$\inf_{x \in V} \{g(x) - \langle \eta, x \rangle\} + \inf_{\xi \in V^\perp} g^*(\eta + \xi) = 0. \tag{7}$$

For $f \in L^1(\mathbf{R}^d)$, we define the Hardy-Littlewood maximal function as

$$(Mf)(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| \, \mathrm{d}y,$$

where $B_r(x)$ stands for a ball having a center at x and radius r.

Theorem 2.4. *Let $\Omega \subset \mathbf{R}^d$ be open and bounded with a Lipschitz boundary and $\alpha \geq 1$. Then there is $c > 0$ such that for any $\mathbf{v} \in W_0^{1,\alpha}(\Omega; \mathbf{R}^d)$ and every $\lambda > 0$ there is $\mathbf{v}^\lambda \in W_0^{1,\infty}(\Omega; \mathbf{R}^d)$ satisfying*

$$\begin{aligned} \|\mathbf{v}^\lambda\|_{W^{1,\infty}(\Omega)} &\leq \lambda, \\ |\{x \in \Omega : \mathbf{v}(x) \neq \mathbf{v}^\lambda(x)\}| &\leq c \frac{\|\mathbf{v}\|^\alpha_{W^{1,\alpha}(\Omega)}}{\lambda^\alpha}. \end{aligned} \tag{8}$$

Proof. The similar assertion, formulated for functions that do not vanish on $\partial\Omega$, appeared in [1]. For our purposes we refer to [6, Theorem 2.3], which for any $\mathbf{v} \in W_0^{1,\alpha}(\Omega; \mathbf{R}^d)$ and any numbers $\theta, \lambda > 0$ ensures the existence of $\mathbf{v}_{\theta,\sigma} \in W_0^{1,\infty}(\Omega; \mathbf{R}^d)$ such that

$$\|\mathbf{v}_{\theta,\sigma}\|_{L^\infty(\Omega)} \leq \theta, \ \ \|\nabla \mathbf{v}_{\theta,\sigma}\|_{L^\infty(\Omega)} \leq c(d, \Omega)\sigma$$

and up to a set of Lebesgue measure zero

$$|\{\mathbf{v}_{\theta,\sigma} \neq \mathbf{v}\}| \subset \Omega \cap (\{M(\mathbf{v}) > \theta\} \cup \{M(\nabla \mathbf{v}) > \sigma\}).$$

We pick $\mathbf{v} \in W_0^{1,\alpha}(\Omega; \mathbf{R}^d)$ and $\lambda > 0$. We apply [6, Theorem 2.3] with $\lambda, \frac{\lambda}{c(d,\Omega)}$ and denote $\mathbf{v}^\lambda = \mathbf{v}_{\lambda, \frac{\lambda}{c(d,\Omega)}}$ to conclude $(8)_1$. Moreover, since we have for any $f \in L^\alpha(\mathbf{R}^d)$ and $\sigma > 0$

$$|\{|f| > \sigma\}| \leq \int_{\mathbf{R}^d} \left(\frac{|f|}{\sigma}\right)^\alpha = \frac{\|f\|^\alpha_{L^\alpha(\mathbf{R}^d)}}{\sigma^\alpha},$$

we obtain for $\alpha > 1$ using the strong type estimate for the maximal function, see [17, Theorem 1]

$$|\{\mathbf{v}_{\theta,\sigma} \neq \mathbf{v}\}| \leq \frac{\|M(\mathbf{v})\|^\alpha_{L^\alpha(\mathbf{R}^d)}}{\lambda^\alpha} + c\frac{\|M(\nabla\mathbf{v})\|^\alpha_{L^\alpha(\mathbf{R}^d)}}{\lambda^\alpha} \leq c\frac{\|\mathbf{v}\|^\alpha_{W^{1,\alpha}(\Omega)}}{\lambda}.$$

For $\alpha = 1$ the estimate $(8)_2$ is a direct consequence of the weak type estimate of the maximal function, see again [17, Theorem 1]. $\square$

2.4. Two-scale convergence

The following concept of convergence was introduced by Nguetseng in his seminal paper [13]: a sequence $\{u^\varepsilon\}$ bounded in $L^2(\Omega)$ is said weakly two-scale convergent to $u^0 \in L^2(\Omega \times Y)$ if for any smooth function $\psi : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}$, which is $Y-$periodic in the second argument,

$$\lim_{\varepsilon \to 0} \int_\Omega u^\varepsilon(x)\psi\left(x, \frac{x}{\varepsilon}\right)\, \mathrm{d}x = \int_{\Omega \times Y} u^0(x,y)\psi(x,y)\, \mathrm{d}x\, \mathrm{d}y. \tag{9}$$

Properties of this notion of convergence were investigated and applied to a number of problems, see [2], and the concept was also extended to $L^p, p \geq 1$. It was shown later that there is an alternative approach, so called periodic unfolding, for the introduction of the weak two-scale convergence, which allows to represent the two-scale convergence by means of the standard weak convergence in a Lebesgue space on the product $\Omega \times Y$. In the same manner the strong two-scale convergence is introduced. Since it is known that both presented notions of the weak two-scale convergence are equivalent, see [18], all properties known for the weak two-scale convergence introduced via (9) hold also for the second approach. We introduce the weak two-scale convergence via periodic unfolding.

Definition 2.4. *We define functions* $n : \mathbf{R} \to \mathbf{Z}$, $r : \mathbf{R} \to [0,1)$, $N : \mathbf{R}^d \to \mathbf{Z}^d$ *and* $R : \mathbf{R}^d \to Y$ *as*

$$n(x) = \max\{n \in \mathbf{Z} : n \leq x\},\ r(x) = x - n(x),$$

$$N(x) = (n(x_1), \ldots, n(x_d)),\ R(x) = x - N(x).$$

Then we have for any $x \in \mathbf{R}^d$ *and* $\varepsilon > 0$, *a two-scale decomposition* $x = \varepsilon\left(N\left(\frac{x}{\varepsilon}\right) + R\left(\frac{x}{\varepsilon}\right)\right)$. *We also define for any* $\varepsilon > 0$ *a two-scale composition function* $T_\varepsilon : \mathbf{R}^d \times Y \to \mathbf{R}^d$ *as* $T_\varepsilon(x,y) = \varepsilon\left(N\left(\frac{x}{\varepsilon}\right) + y\right)$.

Remark 2.2. *It follows that* $T_\varepsilon(x,y) \to x$ *uniformly in* $\mathbf{R}^d \times Y$ *as* $\varepsilon \to 0$ *since* $T_\varepsilon(x,y) = x + \varepsilon\left(y - R\left(\frac{x}{\varepsilon}\right)\right)$.

Definition 2.5. *We say that a sequence of functions* $\{v^\varepsilon\} \subset L^r(\mathbf{R}^d)$

1. *converges to* v^0 *weakly two-scale in* $L^r(\mathbf{R}^d \times Y)$, $v^\varepsilon \xrightharpoonup{2-s} v^0$, *if* $v^\varepsilon \circ T_\varepsilon$ *converges to* v^0 *weakly in* $L^r(\mathbf{R}^d \times Y)$,
2. *converges to* v^0 *strongly two-scale in* $L^r(\mathbf{R}^d \times Y)$, $v^\varepsilon \xrightarrow{2-s} v^0$, *if* $v^\varepsilon \circ T_\varepsilon$ *converges to* v^0 *strongly in* $L^r(\mathbf{R}^d \times Y)$.

Remark 2.3. *We define two-scale convergence in* $L^r(\Omega \times Y)$ *as two-scale convergence in* $L^r(\mathbf{R}^d \times Y)$ *for functions extended by zero to* $\mathbf{R}^d \setminus \Omega$.

Lemma 2.4. *Let* $g \in L^1(\mathbf{R}^d; C_{per}(Y))$. *Then, for any* $\varepsilon > 0$, *the function* $(x,y) \mapsto g(T_\varepsilon(x,y), y)$ *is integrable and*

$$\int_{\mathbf{R}^d} g\left(x, \frac{x}{\varepsilon}\right)\, \mathrm{d}x = \int_{\mathbf{R}^d} \int_Y g(T_\varepsilon(x,y), y)\, \mathrm{d}y\, \mathrm{d}x.$$

Proof. See [18, Lemma 1.1] □

Lemma 2.5.

(i) *Let* $v \in L^r(\Omega; C_{per}(Y)), r \in [1,\infty)$*,* v *be* $Y-$*periodic, define* $v^\epsilon(x) = v(\frac{x}{\epsilon}, x)$ *for* $x \in \Omega$*. Then* $v^\epsilon \xrightarrow{2-s} v$ *in* $L^r(\Omega \times Y)$ *as* $\varepsilon \to 0$*.*

(ii) *Let* $v^\varepsilon \xrightarrow{2-s} v^0$ *in* $L^r(\Omega \times Y)$ *then* $v^\varepsilon \rightharpoonup \int_Y v^0(\cdot, y)\,\mathrm{d}y$ *in* $L^r(\Omega)$*.*

(iii) *Let* $\{v^\varepsilon\}$ *be a bounded sequence in* $L^r(\Omega), r \in (1,\infty)$*. Then there is* $v_0 \in L^r(\Omega \times Y)$ *and a sequence* $\epsilon_k \to 0$ *as* $k \to +\infty$ *such that* $v^{\varepsilon_k} \xrightarrow{2-s} v_0$ *in* $L^r(\Omega \times Y)$ *as* $k \to +\infty$*.*

(iv) *Let* Ω *be bounded and* $\{v^\varepsilon\}$ *converge weakly to* v *in* $W^{1,r}(\Omega), r \in (1,\infty)$ *as* $\varepsilon \to 0$*. Then there is* $v_0 \in L^r(\Omega; W^{1,r}_{per}(Y))$ *and a sequence* $\epsilon_k \to 0$ *as* $k \to +\infty$ *such that* v^{ε_k} *converges strongly to* v *in* $L^r(\Omega)$ *and* ∇v^{ε_k} *converges weakly two-scale to* $\nabla_x v + \nabla_y v_0$ *in* $L^r(\Omega \times Y)^d$ *as* $k \to +\infty$*.*

(v) *Let* $r \in [1,\infty)$ *and* $\Phi : \mathbf{R}^d \times \mathbf{R}^{d\times d}_{sym} \to \mathbf{R}$ *satisfy:*

(a) Φ *is Carathéodory,*

(b) $\Phi(\cdot, \boldsymbol{\xi})$ *is* $Y-$*periodic for any* $\boldsymbol{\xi} \in \mathbf{R}^{d\times d}_{sym}$*,* $\Phi(y, \cdot)$ *is convex for almost all* $y \in Y$*,*

(c) $\Phi \geq 0,\ \Phi(\cdot, 0) = 0$*.*

If $\mathbf{U}^\varepsilon \xrightarrow{2-s} \mathbf{U}^0$ *in* $L^r(\Omega \times Y; \mathbf{R}^{d\times d}_{sym})$ *then*

$$\liminf_{\varepsilon\to 0} \int_\Omega \Phi\left(\frac{x}{\varepsilon}, \mathbf{U}^\varepsilon(x)\right)\,\mathrm{d}x \geq \int_{\Omega\times Y} \Phi(y, \mathbf{U}^0(x,y))\,\mathrm{d}y\,\mathrm{d}x.$$

(vi) *Let* $v^\varepsilon \xrightarrow{2-s} v^0$ *in* $L^r(\Omega\times Y)$ *and* $w^\epsilon \xrightarrow{2-s} w^0$ *in* $L^{r'}(\Omega\times Y)$ *then* $\int_\Omega v^\epsilon w^\epsilon \to \int_\Omega \int_Y v^0 w^0$*.*

Proof. The equalities

$$v^\epsilon \circ T_\epsilon(x,y) = v\left(T_\varepsilon(x,y), \frac{T_\varepsilon(x,y)}{\varepsilon}\right) = v(T_\varepsilon(x,y), y)$$

hold by definition of T_ε and $Y-$periodicity of v. If $v \in C(\overline{\Omega\times Y})$, Remark 2.2 immediately implies

$$\int_{\Omega\times Y} |v(T_\varepsilon(x,y), y) - v(x,y)|^r\,\mathrm{d}x\mathrm{d}y \to 0 \text{ as } \varepsilon \to 0.$$

For general $v \in L^r(\Omega; C_{per}(Y))$ we need to approximate v by a continuous function and then proceed as in the proof of mean continuity of Lebesgue integrable functions.

We obtain (ii) if functions independent of y-variable are considered in the definition (9) of the weak convergence in $L^r(\Omega \times Y)$.

The assertion (iii) is a direct consequence of Lemma 2.4, the weak compactness of bounded sets in $L^r(\Omega \times Y)$ and Definition 2.5_1.

For the proof of (iv) with $r = 2$ see [2, Proposition 1.14. (i)], the proof for general $r \neq 2$ is analogous.

Let us show (v). It follows from Lemma 2.4 and Theorem 2.2, that for $\mathbf{U}^\varepsilon, \mathbf{U}^0$ extended by zero in $\mathbf{R}^d \setminus \Omega$

$$\liminf_{\varepsilon \to 0} \int_\Omega \Phi\left(\frac{x}{\varepsilon}, \mathbf{U}^\varepsilon(x)\right) \,\mathrm{d}x = \liminf_{\varepsilon \to 0} \int_{\Omega \times Y} \Phi\left(y, \mathbf{U}^\varepsilon(T_\varepsilon(x,y))\right) \,\mathrm{d}x \,\mathrm{d}y$$

$$\geq \int_{\Omega \times Y} \Phi(y, \mathbf{U}^0(x,y)) \,\mathrm{d}x \,\mathrm{d}y.$$

Hence we conclude (v).

Statement (vi) follows immediately from definition of the weak and strong two-scale convergence and Lemma 2.5 applied to function $g = v^\epsilon w^\epsilon$ independent of y, see [18, Proposition 1.4]. □

3. Existence of weak solutions of the problems (2) and (4)

Definition 3.1. *Let* $\mathbf{F} \in L^{\beta'}(\Omega; \mathbf{R}^{d\times d}_{sym})$, $\mathbf{S}$ *be defined by* (3) *and* $\varepsilon > 0$ *be fixed and*

$$\mathbf{S}^\varepsilon = \mathbf{S}\left(\frac{x}{\varepsilon}, \mathbf{D}\mathbf{u}^\varepsilon\right). \tag{10}$$

Let s *be determined by*

$$s = \begin{cases} \min\left\{\frac{d\beta}{2(d-\beta)}, \beta'\right\} & \beta < d, \\ \beta' & \beta \geq d. \end{cases} \tag{11}$$

We say that a pair $(\mathbf{u}^\varepsilon, \pi^\varepsilon) \in W^{1,\beta}_{0,\mathrm{div}}(\Omega; \mathbf{R}^d) \times L^s(\Omega)$ *is a weak solution of the problem* (2) *if for any* $\mathbf{w} \in C_0^\infty(\Omega; \mathbf{R}^d)$

$$\int_\Omega (\mathbf{S}^\varepsilon - \mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon - \pi^\varepsilon \mathbf{I}) : \mathbf{D}\mathbf{w} = \int_\Omega \mathbf{F} : \mathbf{D}\mathbf{w}. \tag{12}$$

Theorem 3.1. *Let* $\Omega \subset \mathbf{R}^d$ *be a bounded Lipschitz domain,* $\varepsilon > 0$ *be fixed,* $\mathbf{F} \in L^{\beta'}(\Omega; \mathbf{R}^{d\times d}_{sym})$, $\beta > 2d/(d+2)$, *the measurable function* $p : \mathbf{R}^d \to [1, \beta]$ *be* Y *periodic, Assumption 1.1 be fulfilled. Then there exists a weak solution* $(\mathbf{u}^\varepsilon, \pi^\varepsilon)$ *of* (2). *Moreover, there is* $c > 0$ *independent of* ε *such that*

$$\begin{aligned} \|\mathbf{D}\mathbf{u}^\varepsilon\|_{L^\beta(\Omega)} &\leq c, \\ \|\pi^\varepsilon\|_{L^s(\Omega)} &\leq c. \end{aligned} \tag{13}$$

Proof. Due to assumptions on the function p, namely its boundedness from above by β, we can adopt the technique used for the proof in [6, Theorem 3.1]. □

In accordance with [19] the homogenized tensor $\hat{\mathbf{S}} : \mathbf{R}^{d\times d}_{sym} \to \mathbf{R}^{d\times d}_{sym}$ is determined by

$$\hat{\mathbf{S}}(\boldsymbol{\xi}) = \int_Y \mathbf{S}(y, \boldsymbol{\xi} + \mathbf{V}(y))\, \mathrm{d}y, \tag{14}$$

where the function $\mathbf{V}$ is a solution of the cell problem: Let $\boldsymbol{\xi} \in \mathbf{R}^{d\times d}_{sym}$ be fixed. We seek $\mathbf{V} \in G(Y)$ such that for any $\mathbf{W} \in G(Y)$

$$\int_Y \mathbf{S}(y, \boldsymbol{\xi} + \mathbf{V}(y))\mathbf{W}(y)\, \mathrm{d}y = 0. \tag{15}$$

Since $G(Y)$ is reflexive and the tensor $\mathbf{S}$ is strictly monotone, the existence and uniqueness of $\mathbf{V}$ follows using the theory of monotone operators.

Before we show the existence result for the homogenized problem, we investigate a functional $f : \mathbf{R}^{d\times d}_{sym} \to \mathbf{R}$ defined as

$$f(\boldsymbol{\xi}) = \min_{\mathbf{V}\in G(Y)} \int_Y \frac{|\boldsymbol{\xi} + \mathbf{V}(y)|^\beta}{\beta}\, \mathrm{d}y. \tag{16}$$

We notice that minimum on the right hand side is always attained since a power of the modulus is convex and continuous and $G(Y)$ is reflexive.

Lemma 3.1.

(i) f is positive for $\boldsymbol{\xi} \neq 0$, even and convex,

(ii) there are constants $c_1, c_2 > 0$ such that $c_1|\boldsymbol{\xi}|^\beta \le f(\boldsymbol{\xi}) \le c_2|\boldsymbol{\xi}|^\beta$,

(iii) f is a generalized $N-$function,

(iv) f satisfies the Δ_2-condition.

Proof. First, we recall that by the definition of $G(Y)$

$$G(Y) \cap \mathbf{R}^{d\times d}_{sym} = \{\mathbf{0}\}. \tag{17}$$

Let us assume, contrary to (i), that $f(\boldsymbol{\xi}) = 0$ for some $\boldsymbol{\xi} \neq \mathbf{0}$. Then we obtain from the definition of f that $\mathbf{V} = -\boldsymbol{\xi}$ for almost all $y \in Y$, which is impossible due to (17). The fact that f is even immediately follows since $G(Y)$ is a subspace of $L^\beta(Y; \mathbf{R}^{d\times d}_{sym})$. When showing convexity of f, we choose $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbf{R}^{d\times d}_{sym}$ and find $\mathbf{V}_1, \mathbf{V}_2 \in G(Y)$ such that $f(\boldsymbol{\xi}_i) = \int_Y \frac{|\boldsymbol{\xi}_i + \mathbf{V}_i|^\beta}{\beta}\, \mathrm{d}y, i = 1, 2$. Then for $\lambda \in (0, 1)$ we have by the definition of f and due to the convexity of power of the modulus

$$\begin{aligned} f(\lambda\boldsymbol{\xi}_1 + (1-\lambda)\boldsymbol{\xi}_2) &\le \int_Y \frac{|\lambda\boldsymbol{\xi}_1 + (1-\lambda)\boldsymbol{\xi}_2 + \lambda\mathbf{V}_1 + (1-\lambda)\mathbf{V}_2|^\beta}{\beta} \\ &\le \lambda f(\boldsymbol{\xi}_1) + (1-\lambda) f(\boldsymbol{\xi}_2), \end{aligned}$$

i.e. f is convex.
Since f is convex, it is locally Lipschitz continuous. Especially it is Lipschitz

continuous and therefore continuous on the unit sphere $S_{\mathbf{R}^{d\times d}_{sym}}$, which is compact in $\mathbf{R}^{d\times d}_{sym}$. Hence f attains its minimum value f_{min} and maximum value f_{max} on $S_{\mathbf{R}^{d\times d}_{sym}}$. We have for $\boldsymbol{\xi} \neq 0$ that

$$f(\boldsymbol{\xi}) = \min_{\mathbf{V}\in G(Y)} \int_Y \frac{|\boldsymbol{\xi} + \mathbf{V}(y)|^\beta}{\beta} \,\mathrm{d}y = |\boldsymbol{\xi}|^\beta \min_{\mathbf{V}\in G(Y)} \int_Y \frac{\left|\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} + \mathbf{V}(y)\right|^\beta}{\beta} \,\mathrm{d}y$$

since $G(Y)$ is a subspace of $L^\beta(Y;\mathbf{R}^{d\times d}_{sym})$. Hence we realize $|\boldsymbol{\xi}|^\beta f_{min} \leq f(\boldsymbol{\xi}) \leq |\boldsymbol{\xi}|^\beta f_{max}$, i.e. we have shown (ii).
To verify that f is an $N-$function it suffices to show its superlinear growth which immediately follows from inequalities in (ii).
To show that f satisfies Δ_2-condition, we choose $\boldsymbol{\xi} \in \mathbf{R}^{d\times d}_{sym}$ and find $\mathbf{V} \in G(Y)$ that minimizes the integral in (16). Then we obtain due to the upper and lower bound for p

$$f(2\boldsymbol{\xi}) \leq \int_Y \frac{|2\boldsymbol{\xi} + 2\mathbf{V}(y)|^\beta}{\beta} \,\mathrm{d}y \leq 2^\beta f(\boldsymbol{\xi}).$$

□

Lemma 3.2. *A conjugate functional to f is given by*

$$f^*(\boldsymbol{\xi}) = \min_{\substack{\mathbf{V}^*\in G^\perp(Y)\\ \int_Y \mathbf{V}^*(y)\,\mathrm{d}y = \boldsymbol{\xi}}} \int_Y \frac{|\mathbf{V}^*(y)|^{\beta'}}{\beta'} \,\mathrm{d}y. \tag{18}$$

Proof. Since

$$\begin{aligned}
f^*(\boldsymbol{\xi}) &= \sup_{\boldsymbol{\eta}\in\mathbf{R}^{d\times d}_{sym}} \left\{ \boldsymbol{\eta}\cdot\boldsymbol{\xi} - \inf_{\mathbf{V}\in G(Y)} \int_Y \frac{|\mathbf{V}+\boldsymbol{\eta}|^\beta}{\beta} \right\} \\
&= \sup_{\boldsymbol{\eta}\in\mathbf{R}^{d\times d}_{sym}} \left\{ - \inf_{\mathbf{V}\in G(Y)} \int_Y \frac{|\mathbf{V}+\boldsymbol{\eta}|^\beta}{\beta} - (\mathbf{V}+\boldsymbol{\eta})\cdot\xi \right\} \\
&= - \inf_{\boldsymbol{\eta}\in\mathbf{R}^{d\times d}_{sym}} \inf_{\mathbf{V}\in G(Y)} \left\{ \int_Y \frac{|\mathbf{V}+\boldsymbol{\eta}|^\beta}{\beta} - (\mathbf{V}+\boldsymbol{\eta})\cdot\xi \right\} \\
&= - \inf_{\mathbf{V}\in\mathbf{R}^{d\times d}_{sym}\oplus G(Y)} \left\{ \int_Y \frac{|\mathbf{V}|^\beta}{\beta} - \mathbf{V}\cdot\boldsymbol{\xi} \right\},
\end{aligned}$$

where we also used that $\int_Y \mathbf{V} = 0$ since $\mathbf{V}$ is a symmetric gradient of some $Y-$periodic function, and $F(\mathbf{V}) = \int_Y |\mathbf{V}|^\beta$ is closed because it is continuous with respect to the strong topology of $L^\beta(Y;\mathbf{R}^{d\times d}_{sym})$. The identity (7) and Proposition 2.1 yield

$$f^*(\xi) = \inf_{\mathbf{V}^*\in(G^\perp(Y))_0} \int_Y \frac{|\mathbf{V}^* + \xi|^{\beta'}}{\beta'}.$$

Hence we conclude (18). □

Proposition 3.1. *f^* is convex and satisfies Δ_2-condition.*

Proof. To show convexity of f^*, we choose $\boldsymbol{\xi}_i \in \mathbf{R}^{d\times d}_{sym}$ and denote by $\mathbf{V}^*_i \in G^{\perp}(Y)$ the corresponding minimizers of (18), i.e. $f^*(\boldsymbol{\xi}_i) = \int_Y \frac{|\mathbf{V}^*_i(y)|^{\beta'}}{\beta'}\,\mathrm{d}y, i = 1,2$. Then we have for $\lambda \in (0,1)$

$$f^*(\lambda\boldsymbol{\xi}_1 + (1-\lambda)\boldsymbol{\xi}_2) \le \int_Y \frac{|\lambda\mathbf{V}^*_1 + (1-\lambda)\mathbf{V}^*_2|^{\beta'}}{\beta'} \le \lambda f^*(\boldsymbol{\xi}_1) + (1-\lambda)f^*(\boldsymbol{\xi}_2).$$

Let us show that f^* satisfies Δ_2-condition. We choose $\boldsymbol{\xi} \in \mathbf{R}^{d\times d}_{sym}$ and the corresponding minimizer of (18) denoted by $\mathbf{V}^* \in G^{\perp}(Y)$. Then we infer

$$f^*(2\boldsymbol{\xi}) \le \int_Y \frac{|2\mathbf{V}^*|^{\beta'}}{\beta'}\,\mathrm{d}y \le 2^{\beta'} f^*(\boldsymbol{\xi}).$$

□

The reason why we deal with the generalized $N-$functions f and f^* is that they indicate properties of the homogenized tensor, namely its coerciveness and growth.

Proposition 3.2. *There are constants $\hat{c}_1, \hat{c}_2 > 0$ such that for any $\boldsymbol{\xi} \in \mathbf{R}^{d\times d}_{sym}$*

$$\hat{\mathbf{S}}(\boldsymbol{\xi}) \cdot \boldsymbol{\xi} \le \hat{c}_1 f(\boldsymbol{\xi}) - 1,$$

$$f^*(\hat{\mathbf{S}}(\boldsymbol{\xi})) \le \hat{c}_2 f(\boldsymbol{\xi}) + 1.$$

Moreover, $\hat{\mathbf{S}}$ is monotone.

Proof. See [19, Proposition 2]. □

Regarding to the properties of the functionals f, f^*, it is meaningful to introduce spaces $L^f(\Omega)$ and $L^{f^*}(\Omega)$ in the sense of Definition 2.3. Moreover, we define

$$W^f_{0,\mathrm{div}}(\Omega) = \{\mathbf{u} \in W^{1,1}_{0,\mathrm{div}}(\Omega; \mathbf{R}^d) : \mathbf{D}\mathbf{u} \in L^f(\Omega)\}.$$

Definition 3.2. *Let s be determined by* (11). *We say that a pair $(\mathbf{u}, \pi) \in W^f_{0,\mathrm{div}}(\Omega) \times L^s(\Omega)$ is a weak solution of the problem* (4) *if for any $\mathbf{w} \in C^\infty_0(\Omega; \mathbf{R}^d)$*

$$\int_\Omega (\hat{\mathbf{S}}(\mathbf{D}\mathbf{u}) - \mathbf{u}\otimes\mathbf{u} - \pi\mathbf{I}) : \mathbf{D}\mathbf{w} = \int_\Omega \mathbf{F} : \mathbf{D}\mathbf{w}.$$

Theorem 3.2. *Let $\mathbf{F} \in L^{\beta'}(\Omega; \mathbf{R}^{d\times d}_{sym})$ and $\beta > \frac{2d}{d+2}$. Then there exists a weak solution $(\mathbf{u}, \pi)$ of the problem* (4).

Proof. One proceeds in the same manner as in the proof of [6, Theorem 3.1] since the tensor $\hat{\mathbf{S}}$ enjoys properties listed in Proposition 3.2, $W^f_{0,\mathrm{div}}(\Omega)$ is compactly embedded into $L^2(\Omega; \mathbf{R}^d)$ due to the assumed lower bound for β and Lemma 3.1 (*ii*), which also implies that $C^\infty_{0,\mathrm{div}}(\Omega; \mathbf{R}^d)$ is dense in $W^f_{0,\mathrm{div}}(\Omega)$. □

4. Proof of the main theorem

Lemma 4.1. *Let s be given by (11) and the functions $\mathbf{u}^\varepsilon, \pi^\varepsilon, \mathbf{S}^\varepsilon, \mathbf{F}$ be extended by zero on $\mathbf{R}^d \setminus \Omega$ for $\varepsilon > 0$. Let functions $\pi^{\varepsilon,1} \in L^{\beta'}(\mathbf{R}^d), \pi^{\varepsilon,2} \in L^{\frac{\beta^*}{2}}(\mathbf{R}^d), \pi^{\varepsilon,3} \in L^{\beta'}(\Omega)$ be defined as*

$$\pi^{\varepsilon,1} = \operatorname{div}\operatorname{div}\mathcal{N}\left(\mathbf{S}^\varepsilon\right),$$

$$\pi^{\varepsilon,2} = -\operatorname{div}\operatorname{div}\mathcal{N}\left(\mathbf{F} + \mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon\right),$$

$$\pi^{\varepsilon,3} = \pi^\varepsilon - \pi^{\varepsilon,1} - \pi^{\varepsilon,2}.$$

Here $\mathcal{N}\left(\mathbf{S}^\varepsilon\right)$ denotes the componentwise Newton potential of $\mathbf{S}^\varepsilon$. Then

$$\begin{aligned} &\{\pi^{\varepsilon,1}\} \textit{ is bounded in } L^{\beta'}(\mathbf{R}^d), \\ &\{\pi^{\varepsilon,2}\} \textit{ is precompact in } L^q(\mathbf{R}^d) \textit{ for any } q \in [1,s), \\ &\{\pi^{\varepsilon,3}\} \textit{ is precompact in } L^{\beta'}(O) \textit{ for any } O \Subset \Omega. \end{aligned} \tag{19}$$

Proof. Applying the theory of Calderon-Zygmund operators, see [5, Section 6.3], yields the estimate

$$\|\pi^{\varepsilon,1}\|_{L^{\beta'}(\mathbf{R}^d)} \le c\|\mathbf{S}^\varepsilon\|_{L^{\beta'}(\mathbf{R}^d)} \tag{20}$$

and the precompactness of $\{\pi^{\varepsilon,2}\}$ in $L^q(\mathbf{R}^d), q \in [1,s)$ since $\{\mathbf{F} + \mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon\}$ is precompact in $L^q(\mathbf{R}^d; \mathbf{R}^{d\times d}_{sym})$. It follows from (12) and (13) that $\{\pi^{\varepsilon,3}\}$ are harmonic functions in Ω and bounded in $L^1(\Omega)$. Hence $\{\pi^{\varepsilon,3}\}$ is precompact in $L^{\beta'}(O)$ for any $O \Subset \Omega$. □

Lemma 4.2. *Let $\mathbf{S}^\varepsilon$ be defined by (10) and $\pi^{\varepsilon,1}$ be defined in Lemma 4.1. Then there is a sequence $\epsilon_k \to 0$ as $k \to +\infty$ and a sequence of measurable sets $\Omega_1 \subset \Omega_2 \subset \dots \subset \Omega_n \subset \dots \subset \Omega$ with $|\Omega \setminus \Omega_n| \to 0$ as $n \to +\infty$ such that for any $n \in \mathbf{N}$ and $\theta > 0$ there is $\delta > 0$ such that for any $k \in \mathbf{N}$ and $K \subset \Omega_n$ with $|K| < \delta$*

$$\|\mathbf{S}^k\|_{L^{\beta'}(K)} + \|\pi^{k,1}\|_{L^\beta(K)} < \theta^{\frac{1}{\beta}}, \tag{21}$$

where we denoted $\mathbf{S}^k = \mathbf{S}^{\epsilon_k}$ and $\pi^{k,1} = \pi^{\epsilon_k,1}$ for $k \in \mathbf{N}$.

Proof. Let us denote $g^\varepsilon = |\mathbf{S}^\varepsilon|^{\beta'} + |\pi^{\varepsilon,1}|^{\beta'}$. The apriori estimate $(13)_1$, the growth condition on $\mathbf{S}$ and $(19)_1$ imply the boundedness of $\{g^\varepsilon\}$ in $L^1(\Omega)$. The application of Chacon's biting lemma 2.1 on $\{g^\varepsilon\}$ yields the existence of sets $\Omega_n \subset \Omega$ with $|\Omega \setminus \Omega_n| \to 0$ as $n \to +\infty$ and the existence of a subsequence $\{g^{\epsilon_k}\}$ and a function $g \in L^1(\Omega)$ such that $g^{\epsilon_k} \rightharpoonup g$ in $L^1(\Omega_n)$ as $k \to +\infty$. The equiintegrability of $\{g^{\epsilon_k}\}$ on Ω_n follows according to Dunford theorem 2.1. □

¿From now on we deal only with sequences $\{\mathbf{S}^k\}$, $\{\mathbf{u}^k\}$, $\{\pi^k\}$, $\{\pi^{k,1}\}$, $\{\pi^{k,2}\}$, $\{\pi^{k,3}\}$ extracted in the previous Lemma.

Lemma 4.3. *Let s be determined by (11). There exists a subsequence of $\{(\mathbf{u}^k,\pi^k)\}$ (that will not be relabeled), functions $\mathbf{u}\in W^{1,\beta}_{0,\mathrm{div}}(\Omega;\mathbf{R}^d),\pi\in L^s(\Omega)$, $\overline{\mathbf{S}}\in L^{\beta'}(\Omega;\mathbf{R}^{d\times d}_{sym})$ and $\pi^1\in L^{\beta'}(\Omega)$ such that as $k\to+\infty$*

$$
\begin{aligned}
\mathbf{u}^k &\rightharpoonup \mathbf{u} \quad \text{in } W^{1,\beta}_0(\Omega;\mathbf{R}^d),\\
\mathbf{u}^k &\to \mathbf{u} \quad \text{in } L^{\beta*}(\Omega;\mathbf{R}^d),\\
\pi^k &\rightharpoonup \pi \quad \text{in } L^s(\Omega),\\
\pi^{k,1} &\rightharpoonup \pi^1 \quad \text{in } L^{\beta'}(\Omega),\\
\mathbf{S}^k &\rightharpoonup \overline{\mathbf{S}} \quad \text{in } L^{\beta'}(\Omega;\mathbf{R}^{d\times d}_{sym}).
\end{aligned}
\tag{22}
$$

The limit functions $\mathbf{u},\pi,\overline{\mathbf{S}}$ satisfy for any $\mathbf{w}\in C^\infty_0(\Omega)$

$$
\int_\Omega(\overline{\mathbf{S}}-\mathbf{u}\otimes\mathbf{u}-\pi\mathbf{I}):\mathbf{Dw}=\int_\Omega\mathbf{F}:\mathbf{Dw}.
$$

Proof. The statement follows in a standard way from (13), Sobolev embedding theorem, $(19)_1$, (3) and (12). □

The rest of the paper is devoted to finding the relation between $\overline{\mathbf{S}}$ and $\mathbf{Du}$.

Lemma 4.4. *There exist subsequences of $\{\mathbf{u}^k\},\{\mathbf{S}^k\},\{\pi^{k,1}\}$ (that will not be relabeled) and functions $\mathbf{u}^0\in L^\beta\left(\Omega;W^{1,\beta}_{per}(Y)^d\right)$, $\overline{\mathbf{S}^0}\in L^{\beta'}(\Omega\times Y;\mathbf{R}^{d\times d}_{sym})$ and $\overline{\pi^1}\in L^{\beta'}(\Omega\times Y)$ such that as $k\to+\infty$*

$$
\mathbf{Du}^k \xrightarrow{2-s} \mathbf{Du}+\mathbf{D}_y\mathbf{u}^0 \quad \text{in } L^\beta(\Omega\times Y;\mathbf{R}^{d\times d}_{sym}), \tag{23}
$$

$$
\mathbf{S}^k \xrightarrow{2-s} \overline{\mathbf{S}^0} \quad \text{in } L^{\beta'}(\Omega\times Y;\mathbf{R}^{d\times d}_{sym}), \tag{24}
$$

$$
\pi^{k,1} \xrightarrow{2-s} \overline{\pi^1} \quad \text{in } L^{\beta'}(\Omega\times Y). \tag{25}
$$

Moreover, the limit functions satisfy

$$
\text{for almost all } x\in\Omega:\ \mathbf{D}_y\mathbf{u}^0(x,\cdot)\in G(Y), \tag{26}
$$

$$
\text{for almost all } x\in\Omega:\ \overline{\mathbf{S}^0}(x,\cdot)\in G^\perp(Y), \tag{27}
$$

$$
\text{for almost all } x\in\Omega:\ \overline{\pi^1}(x,\cdot)\mathbf{I}\in G^\perp(Y), \tag{28}
$$

$$
\mathbf{u}\in W^f_{0,\mathrm{div}}(\Omega), \tag{29}
$$

$$
\int_Y\overline{\mathbf{S}^0}=\overline{\mathbf{S}},\ \overline{\mathbf{S}}\in L^{f^*}(\Omega), \tag{30}
$$

$$
\int_Y\overline{\pi^1}=\pi^1,\ \pi^1 I\in L^{f^*}(\Omega), \tag{31}
$$

where the functions $\overline{\mathbf{S}}$ *and* π^1 *come from Lemma 4.3.*

Proof. Lemma 4.3 and Lemma 2.5 (*iv*) imply (23). Statements (24) and (25) follow from $(13)_1$, $(19)_1$, the assumption on growth of $\mathbf{S}$ and Lemma 2.5 (*iii*). Let us show (26). The convergence (23) means that for any $\boldsymbol{\psi} \in \mathcal{D}\left(\Omega; C^\infty_{per}(Y)^{d\times d}\right)$

$$\lim_{k\to+\infty} \int_\Omega \mathbf{Du}^k(x)\boldsymbol{\psi}\left(x, \frac{x}{\varepsilon_k}\right) \,\mathrm{d}x = \int_\Omega \int_Y \left(\mathbf{Du}(x) + \mathbf{D}_y\mathbf{u}^0(x,y)\right)\boldsymbol{\psi}(x,y) \,\mathrm{d}x \,\mathrm{d}y. \tag{32}$$

We pick $a \in \mathcal{D}(\Omega), b \in C^\infty_{per}(Y)$ and put $\boldsymbol{\psi}(x,y) = a(x)b(y)\mathbf{I}$ in (32). Obviously, we get using the weak convergence of $\{\mathbf{u}^k\}$ in $W^{1,\beta}_0(\Omega;\mathbf{R}^d)$

$$0 = \lim_{k\to+\infty}\int_\Omega \operatorname{div}\mathbf{u}^k(x)a(x)b\left(\frac{x}{\varepsilon_k}\right)\,\mathrm{d}x = \lim_{k\to+\infty}\int_\Omega \mathbf{Du}^k(x)a(x)b\left(\frac{x}{\varepsilon_k}\right) : \mathbf{I}\,\mathrm{d}x$$

$$= \int_\Omega\int_Y \left(\mathbf{Du}(x) + \mathbf{D}_y\mathbf{u}^0(x,y)\right)a(x)b(y) : \mathbf{I}\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_\Omega \operatorname{div}\mathbf{u}(x)a(x)\int_Y b(y)\,\mathrm{d}y\,\mathrm{d}x + \int_\Omega\int_Y \operatorname{div}_y \mathbf{u}^0(x,y)b(y)\,\mathrm{d}y\ a(x)\,\mathrm{d}x$$

$$= \int_\Omega\int_Y \operatorname{div}_y \mathbf{u}^0(x,y)b(y)\,\mathrm{d}y\ a(x)\,\mathrm{d}x.$$

Hence for a.a. $x \in \Omega$ $\operatorname{div}_y \mathbf{u}(x,\cdot)^0 = 0$ a.e. in Y, i.e. we conclude (26).
We show that for any $\sigma \in C^\infty_0(\Omega)$ and $\mathbf{h} \in C^\infty_{per,\mathrm{div}}(Y;\mathbf{R}^d)$

$$\int_\Omega\int_Y \overline{\mathbf{S}^0}(x,y)\mathbf{Dh}(y)\,\mathrm{d}y\sigma(x)\,\mathrm{d}x = 0. \tag{33}$$

Since $\varepsilon_k\sigma(x)\mathbf{h}\left(\frac{x}{\varepsilon_k}\right)$ is not solenoidal, the correction $\mathbf{B}^k(x) = \mathcal{B}\left(\varepsilon_k\mathbf{h}\left(\frac{x}{\varepsilon_k}\right)\nabla\sigma(x)\right)$ that satisfies

$$\operatorname{div}\mathbf{B}^k(x) = \varepsilon_k\mathbf{h}(\frac{x}{\epsilon_k})\nabla\sigma(x) \text{ in } x\in\Omega, \quad \mathbf{B}^k = 0 \text{ on } \partial\Omega,$$

$$\|\nabla\mathbf{B}^k\|_{L^\gamma(\Omega)} \le c(\gamma, \|\mathbf{h}\|_{L^\infty(Y)}, \|\nabla\sigma\|_{L^\infty(\Omega)})\varepsilon_k$$

with an arbitrary $\gamma \in (1,\infty)$, is introduced to allow using $\varepsilon_k\sigma(x)\mathbf{h}\left(\frac{x}{\varepsilon_k}\right) - \mathbf{B}^k$ as a test function in (12). Then we employ convergences as $k\to+\infty$

$$\mathbf{B}^k \to 0 \qquad \text{in } L^\gamma(\Omega;\mathbf{R}^d),$$

$$\mathbf{DB}^k \to 0 \qquad \text{in } L^\gamma(\Omega;\mathbf{R}^{d\times d}_{sym}),$$

$$\sigma\mathbf{u}^k\otimes\mathbf{u}^k \to \sigma\mathbf{u}\otimes\mathbf{u} \text{ in } L^1(\Omega;\mathbf{R}^{d\times d}_{sym}),$$

$$\mathbf{S}^k \xrightharpoonup{2-s} \overline{\mathbf{S}^0} \quad \text{in } L^{\beta'}(\Omega;\mathbf{R}^{d\times d}_{sym}),$$

to obtain from (12) by (9) and Lemma 2.5_1 that

$$\int_\Omega \int_Y (\overline{\mathbf{S}^0}(x,y) - \mathbf{u}(x)\otimes\mathbf{u}(x) - \mathbf{F}(x))\mathbf{D}_y\mathbf{h}(y)\ \mathrm{d}y\sigma(x)\ \mathrm{d}x = 0.$$

Hence (33) and thus (27) follow due to an obvious fact $\int_Y \mathbf{D}_y\mathbf{h}(y) = 0$.
We use (23) and Lemma 2.5 (v) to infer

$$\int_\Omega \int_Y |\mathbf{D}\mathbf{u}(x) + \mathbf{D}_y\mathbf{u}^0(x,y)|^\beta\ \mathrm{d}y\ \mathrm{d}x \le \liminf_{\varepsilon\to 0} \int_\Omega |\mathbf{D}\mathbf{u}^\varepsilon(x)|^\beta\ \mathrm{d}x.$$

Hence we obtain due to the definition of f, (26) and the apriori estimate $(13)_1$

$$\int_\Omega f(\mathbf{D}\mathbf{u}(x))\ \mathrm{d}x \le \int_\Omega \int_Y |\mathbf{D}\mathbf{u}(x) + \mathbf{D}_y\mathbf{u}^0(x,y)|^\beta\ \mathrm{d}y\ \mathrm{d}x < \infty. \tag{34}$$

Similarly, $(13)_1$, the assumption on growth of $\mathbf{S}$, Lemma 2.5 (v) and (ii) together with the definition of f^* imply (30).
Finally, we infer that for any $\mathbf{W} = \mathbf{D}\mathbf{w} \in G(Y)$ and almost all $x \in \Omega$

$$\int_Y \overline{\pi^1}(x,y)\mathbf{I} : \mathbf{W}(y)\ \mathrm{d}y = \int_Y \overline{\pi^1}(x,y)\,\mathrm{div}_y\,\mathbf{w}(y)\ \mathrm{d}y = 0.$$

Hence $\overline{\pi^1}(x,\cdot)\mathbf{I} \in G^\perp(Y)$, Lemma 2.5 (v) and (ii) together with the definition of f^* imply (31). □

Lemma 4.5. *There is $c > 0$ and a subsequence of $\{\mathbf{u}^k\}_{k=1}^{+\infty}$ (that will not be relabelled) such that*

$$\forall k,\lambda \in \mathbf{N} : \|\mathbf{D}\mathbf{u}^{k,\lambda}\|_{L^\beta(\Omega)} \le c, \tag{35}$$

$$\forall \lambda \in \mathbf{N} : \mathbf{D}\mathbf{u}^{k,\lambda} \rightharpoonup^* \mathbf{D}\mathbf{u}^\lambda \text{ as } k \to +\infty \text{ in } L^\infty(\Omega;\mathbf{R}^d), \tag{36}$$

where we denoted by $\mathbf{u}^{k,\lambda}$ functions constructed to $\mathbf{u}^k$ by Theorem 2.4. Moreover, a subsequence $\{\mathbf{u}^{\lambda_k}\}_{k=1}^{+\infty}$ can be selected such that

$$\mathbf{D}\mathbf{u}^{\lambda_k} \rightharpoonup \mathbf{D}\mathbf{u} \text{ as } k \to +\infty \text{ in } L^\beta(\Omega;\mathbf{R}^{d\times d}_{sym}),. \tag{37}$$

Proof. The application of Theorem 2.4 to the sequence $\{\mathbf{u}^k\}$ yields for any $\lambda \in \mathbf{N}$ the existence of a sequence $\{\mathbf{u}^{k,\lambda}\} \subset W^{1,\infty}_0(\Omega;\mathbf{R}^d)$ satisfying

$$\|\mathbf{u}^{k,\lambda}\|_{W^{1,\infty}(\Omega)} \le \lambda, \quad |\{x\in\Omega : \mathbf{u}^k(x) \ne \mathbf{u}^{k,\lambda}(x)\}| \le c\frac{\|\mathbf{u}^k\|^\beta_{W^{1,\beta}(\Omega)}}{\lambda^\beta}. \tag{38}$$

Utilizing (38), Friedrichs and Korn's inequalities, we obtain

$$\begin{aligned}\int_\Omega |\mathbf{D}\mathbf{u}^{k,\lambda}|^\beta &= \int_{\{\mathbf{u}^\varepsilon = \mathbf{u}^{k,\lambda}\}} |\mathbf{D}\mathbf{u}^{k,\lambda}|^\beta + \int_{\{\mathbf{u}^\varepsilon \ne \mathbf{u}^{k,\lambda}\}} |\mathbf{D}\mathbf{u}^{k,\lambda}|^\beta \\ &\le \int_\Omega |\mathbf{D}\mathbf{u}^k|^\beta + \lambda^\beta|\{x\in\Omega : \mathbf{u}^k(x)\ne\mathbf{u}^{k,\lambda}(x)\}| \le c\|\mathbf{D}\mathbf{u}^k\|^\beta_{L^\beta(\Omega)},\end{aligned}$$

which implies (35) due to (13).

The convergence (36) follows from $(38)_1$ by diagonal procedure. Moreover, the estimate (35), (36) and the weak lower semicontinuity of the $L^\beta-$norm imply the existence of a positive constant c such that

$$\forall \lambda \in \mathbf{N} : \|\mathbf{D}\mathbf{u}^\lambda\|_{L^\beta(\Omega)} \leq c.$$

Hence we can pick a function $\tilde{\mathbf{u}} \in W_0^{1,\beta}(\Omega; \mathbf{R}^d)$ and a subsequence $\{\lambda_k\}_{k=1}^{+\infty}$ such that

$$\mathbf{u}^{\lambda_k} \rightharpoonup \tilde{\mathbf{u}} \text{ as } k \to +\infty \text{ in } W^{1,\beta}(\Omega; \mathbf{R}^d).$$

It remains to show $\tilde{\mathbf{u}} = \mathbf{u}$. Using the boundedness of the sequences $\{\mathbf{u}^k\}, \{\mathbf{u}^{k,\lambda}\}$ in $W^{1,\beta}(\Omega; \mathbf{R}^d)$ and the estimate $(38)_2$, we obtain

$$\int_\Omega |\mathbf{u}^{k,\lambda} - \mathbf{u}^k| = \int_{\{\mathbf{u}^{k,\lambda} \neq \mathbf{u}^\varepsilon\}} |\mathbf{u}^{k,\lambda} - \mathbf{u}^k| \leq \|\mathbf{u}^{k,\lambda} - \mathbf{u}^k\|_{L^\beta(\Omega)} |\{\mathbf{u}^{k,\lambda} \neq \mathbf{u}^k\}|^{\frac{1}{\beta'}} \leq \frac{c}{\lambda^{\beta-1}}.$$

Moreover, the compact embedding $W^{1,\beta}(\Omega; \mathbf{R}^d) \hookrightarrow L^1(\Omega; \mathbf{R}^d)$ implies

$$\|\mathbf{u}^\lambda - \mathbf{u}\|_{L^1(\Omega)} = \lim_{k \to +\infty} \|\mathbf{u}^{k,\lambda} - \mathbf{u}^k\|_{L^1(\Omega)}.$$

Therefore $\mathbf{u}^\lambda \to \mathbf{u}$ in $L^1(\Omega)$ and we conclude $\tilde{\mathbf{u}} = \mathbf{u}$ a.e. in Ω. □

Remark 4.1. *The improvement of Lemma 4.5 for the case when* $\delta = 0$ *in* (3) *requires a variant of Theorem 2.4. The proof of Theorem 2.4 is based on the boundedness of the maximal operator in* $L^\beta(\mathbf{R}^d)$*. The above expressed case of the stress tensor leads to spaces* $L^{p\left(\frac{x}{\varepsilon}\right)}$*. For the boundedness of the maximal operator on Lebesgue space with variable exponent* p *it is necessary to assume that* p *is log-Hölder continuous, i.e. there are* $c_{p_{log}} > 0$ *and* $p_\infty \in \mathbf{R}$ *such that for any* $x, y \in \mathbf{R}^d$

$$|p(x) - p(y)| \leq \frac{c}{\log\left(e + \frac{1}{|x-y|}\right)}, |p(x) - p_\infty| \leq \frac{c}{\log(e + |x|)},$$

see [5, Chapter 4 of Part I]. As estimates of the norm of the maximal operator in the proof of Theorem 2.4 are independent of ε*, the same is needed also in the case of Lebesgue spaces* $L^{p\left(\frac{x}{\varepsilon}\right)}$*. But it is unclear whether this uniform boundedness of the maximal operator holds or not. In particular, it does not follow from the log-Hölder continuity of* p*.*

In the rest of the paper we denote for any $k, l \in \mathbf{N}$ the function $\mathbf{u}^{k,l} := \mathbf{u}^{k,\lambda_l}$, where $\{\lambda_l\}$ is sequence constructed in Lemma 4.5.

Lemma 4.6. *Let* $\{\Omega_n\}, \{\mathbf{S}^k\}$ *be from Lemma 4.2,* $O \Subset \Omega$ *be arbitrary open and denote* $\tilde{\Omega}_n = \Omega_n \cap O$*. Then for each* $n \in \mathbf{N}$

$$\lim_{k \to +\infty} \int_{\tilde{\Omega}_n} \mathbf{S}^k : \mathbf{D}\mathbf{u}^k = \int_{\tilde{\Omega}_n} \overline{\mathbf{S}} : \mathbf{D}\mathbf{u}. \tag{39}$$

Proof. We observe that whenever there exist $n_0 \in \mathbf{N}$ and a sequence of measurable sets $\{E_j\}$ such that $\bigcup E_j \subset \Omega_{n_0}$ and $|E_j| \to 0$ as $j \to 0$ then

$$\|\mathbf{S}^k\|_{L^{\beta'}(E_j)} + \|\pi^{k,1}\|_{L^{\beta'}(E_j)} \to 0 \quad \text{as } j \to +\infty \tag{40}$$

uniformly with respect to $k \in \mathbf{N}$, which is a direct consequence of Lemma 4.2.

For fixed $n \in \mathbf{N}$ and any $k, l \in \mathbf{N}$ we decompose using the solenoidality of $\mathbf{u}^k$

$$\int_{\tilde{\Omega}_n} \mathbf{S}^k : \mathbf{D}\mathbf{u}^k = \int_{\tilde{\Omega}_n} (\mathbf{S}^k - \pi^{k,1}\mathbf{I}) : \mathbf{D}\mathbf{u}^k = \int_{\tilde{\Omega}_n} (\mathbf{S}^k - \pi^{k,1}\mathbf{I}) : \mathbf{D}\left(\mathbf{u}^k - \mathbf{u}^{k,l}\right)$$
$$+ \int_{\tilde{\Omega}_n} (\mathbf{S}^k - \pi^{k,1}\mathbf{I}) : \mathbf{D}\mathbf{u}^{k,l} = I^{k,l} + II^{k,l}.$$

We want to perform the limit passage $k \to +\infty$ and then $l \to +\infty$ in both terms on the right hand side of the latter equality. We denote $\tilde{\Omega}_n^{k,l} = \tilde{\Omega}_n \cap \{\mathbf{u}^k \neq \mathbf{u}^{k,l}\}$ and estimate using Hölder's inequality, (20), $(13)_1$ and (35)

$$|I^{k,l}| \le 2\|\mathbf{S}^k - \pi^{k,1}\mathbf{I}\|_{L^{\beta'}(\tilde{\Omega}_n^{k,l})} \|\mathbf{D}(\mathbf{u}^k - \mathbf{u}^{k,l})\|_{L^{\beta}(\tilde{\Omega}_n^{k,l})}$$
$$\le c\left(\|\mathbf{S}^k\|_{L^{\beta'}(\tilde{\Omega}_n^{k,l})} + \|\pi^{k,1}\|_{L^{\beta'}(\tilde{\Omega}_n^{k,l})}\right).$$

As $|\tilde{\Omega}_n^{k,l}| \le c\lambda_l^{-\beta}$ by $(38)_2$, we get by (40) that for any $\theta > 0$ there exists $l_0 \in \mathbf{N}$ such that for any $l > l_0$ and $k \in \mathbf{N}$ we have $|I^{k,l}| < \theta$ and therefore

$$\lim_{l \to +\infty} \lim_{k \to +\infty} I^{k,l} = \lim_{k \to +\infty} \lim_{l \to +\infty} I^{k,l} = 0.$$

For the limit passage $k \to +\infty$ in $II^{k,l}$ we employ Lemma 2.2. Let us pick $q \in (1, s)$, where s is determined by (11). We have for any $\mathbf{w} \in W_0^{1,q'}(O; \mathbf{R}^d)$

$$\left\langle \operatorname{div}(\mathbf{S}^k - \pi^{k,1}\mathbf{I}), \mathbf{w} \right\rangle = -\int_O \left(\mathbf{F} + \mathbf{u}^k \otimes \mathbf{u}^k + (\pi^{k,2} + \pi^{k,3})\mathbf{I}\right) : \mathbf{D}\mathbf{w}.$$

It follows from Lemma 4.1 that $\{\mathbf{F} + \mathbf{u}^k \otimes \mathbf{u}^k + (\pi^{k,2} + \pi^{k,3})\mathbf{I}\}$ is precompact in $L^q(O; \mathbf{R}^{d\times d}_{sym})$. Therefore we obtain that $\{\operatorname{div}(\mathbf{S}^k + \pi^{k,1}\mathbf{I})\}$ is precompact in $W^{-1,q}(O; \mathbf{R}^d)$. We observe that $\operatorname{curl}(\nabla \mathbf{u}^{k,l}) = 0$. Then Lemma 2.2 and the convergences $(22)_{4,5}$ and (36) imply

$$(\mathbf{S}^k - \pi^{k,1}\mathbf{I}) : \mathbf{D}\mathbf{u}^{k,l} = (\mathbf{S}^k - \pi^{k,1}\mathbf{I}) : \nabla \mathbf{u}^{k,l} \rightharpoonup (\overline{\mathbf{S}} - \pi^1\mathbf{I}) : \nabla \mathbf{u}^l = (\overline{\mathbf{S}} - \pi^1\mathbf{I}) : \mathbf{D}\mathbf{u}^l$$

in $L^r(O)$ for any $r > 1$ as $k \to +\infty$. Hence we deduce using (37) and the solenoidality of $\mathbf{u}$

$$\lim_{l \to +\infty} \lim_{k \to +\infty} II^{k,l} = \lim_{l \to +\infty} \lim_{k \to +\infty} \int_O (\mathbf{S}^k - \pi^{k,1}\mathbf{I}) : \mathbf{D}\mathbf{u}^{k,l} \chi_{\tilde{\Omega}_n}$$
$$= \lim_{l \to +\infty} \int_O (\overline{\mathbf{S}} - \pi^1\mathbf{I}) : \mathbf{D}\mathbf{u}^l \chi_{\tilde{\Omega}_n} = \int_{\tilde{\Omega}_n} \overline{\mathbf{S}} : \mathbf{D}\mathbf{u}. \tag{41}$$

□

Proof of Theorem 1.1. It remains to show the relation

$$\overline{\mathbf{S}}(x) = \int_Y \mathbf{S}(x, \mathbf{D}\mathbf{u}(x) + \mathbf{D}_y \mathbf{u}^0(x,y))\, \mathrm{d}y \text{ for almost all } x \in \Omega. \tag{42}$$

We fix $n \in \mathbf{N}$, a corresponding Ω_n from Lemma 4.2 and $O \Subset \Omega$. Keeping the notation of Lemma 4.6, using (30), (26) and (27), it follows from (39) that

$$\lim_{k\to+\infty} \int_{\tilde{\Omega}_n} \mathbf{S}^k : \mathbf{D}\mathbf{u}^k = \int_{\tilde{\Omega}_n} \int_Y \overline{\mathbf{S}^0} : (\mathbf{D}\mathbf{u} + \mathbf{D}_y \mathbf{u}^0). \tag{43}$$

We choose $\mathbf{U} \in L^\beta(\tilde{\Omega}_n; C_{per}(Y; \mathbf{R}^{d\times d}_{sym}))$. The monotonicity of $\mathbf{S}$ implies

$$\begin{aligned}
0 &\le \int_{\tilde{\Omega}_n} \left(\mathbf{S}^k(x) - \mathbf{S}\left(x\varepsilon_k^{-1}, \mathbf{U}\left(x, x\varepsilon_k^{-1}\right)\right)\right) : \left(\mathbf{D}\mathbf{u}^k(x) - \mathbf{U}\left(x, x\varepsilon_k^{-1}\right)\right)\, \mathrm{d}x \\
&= \int_{\tilde{\Omega}_n} \mathbf{S}^k(x) : \mathbf{D}\mathbf{u}^k(x)\, \mathrm{d}x - \int_{\tilde{\Omega}_n} \mathbf{S}\left(x\varepsilon_k^{-1}, \mathbf{U}\left(x, x\varepsilon_k^{-1}\right)\right) : \mathbf{D}\mathbf{u}^k(x)\, \mathrm{d}x \\
&- \int_{\tilde{\Omega}_n} \mathbf{S}^k(x) : \mathbf{U}\left(x, x\varepsilon_k^{-1}\right)\, \mathrm{d}x + \int_{\tilde{\Omega}_n} \mathbf{S}\left(x\varepsilon_k^{-1}, \mathbf{U}\left(x, x\varepsilon_k^{-1}\right)\right) : \mathbf{U}\left(x, x\varepsilon_k^{-1}\right)\, \mathrm{d}x \\
&= I^k - II^k - III^k + IV^k.
\end{aligned}$$

We want to pass to the limit as $k \to +\infty$ in I^k, II^k, III^k, IV^k. We use (43) for the passage in I^k. Applying Lemma 2.5 (i) to $\mathbf{S}(y, \mathbf{U}(x,y))$ and $\mathbf{U}$ yields

$$\mathbf{S}\left(x\varepsilon_k^{-1}, \mathbf{U}\left(x, x\varepsilon_k^{-1}\right)\right) \xrightarrow{2-s} \mathbf{S}(y, \mathbf{U}(x,y)) \text{ in } L^{\beta'}(\Omega \times Y; \mathbf{R}^{d\times d}_{sym}),$$

$$\mathbf{U}\left(x, x\varepsilon_k^{-1}\right) \xrightarrow{2-s} \mathbf{U}(x,y) \text{ in } L^{\beta}(\Omega \times Y; \mathbf{R}^{d\times d}_{sym})$$

as $k \to +\infty$. Employing these convergences and (23) we infer

$$\lim_{k\to+\infty} II^k = \int_{\tilde{\Omega}_n} \int_Y \mathbf{S}(y, \mathbf{U}(x,y)) : (\mathbf{D}\mathbf{u}(x) + \mathbf{D}_y \mathbf{u}^0(x,y)\, \mathrm{d}y\, \mathrm{d}x,$$

$$\lim_{k\to+\infty} III^k = \int_{\tilde{\Omega}_n} \int_Y \overline{\mathbf{S}^0}(x,y) : \mathbf{U}(x,y)\, \mathrm{d}y\, \mathrm{d}x,$$

$$\lim_{k\to+\infty} IV^k = \int_{\tilde{\Omega}_n} \int_Y \mathbf{S}(y, \mathbf{U}(x,y)) : \mathbf{U}(x,y)\, \mathrm{d}y\, \mathrm{d}x.$$

Thus one obtains for any $n \in \mathbf{N}$ and $\mathbf{U} \in L^\beta(\tilde{\Omega}_n; C_{per}(Y; \mathbf{R}^{d\times d}_{sym}))$

$$\int_{\tilde{\Omega}_n} \int_Y \left(\overline{\mathbf{S}^0}(x,y) - \mathbf{S}(y, \mathbf{U}(x,y))\right) : \left(\mathbf{D}\mathbf{u}(x) + \mathbf{D}_y \mathbf{u}^0(x,y) - \mathbf{U}(x,y)\right) \ge 0. \tag{44}$$

To be able to apply Minty's trick, we need (44) to be satisfied for any $\mathbf{U} \in L^\beta(\tilde{\Omega}_n \times Y; \mathbf{R}^{d\times d}_{sym})$. In order to obtain that we consider $\mathbf{U} \in L^\beta(\tilde{\Omega}_n \times Y; \mathbf{R}^{d\times d}_{sym})$ and $\{\mathbf{U}^k\} \subset L^\beta(\tilde{\Omega}_n; C_{per}(Y; \mathbf{R}^{d\times d}_{sym}))$ such that $\mathbf{U}^k \to \mathbf{U}$ in $L^\beta(\tilde{\Omega}_n \times Y; \mathbf{R}^{d\times d}_{sym})$. Then we have due to the growth of $\mathbf{S}$ and theory of Nemytskii operators that $\mathbf{S}(y, \mathbf{U}^k) \to \mathbf{S}(y, \mathbf{U})$ in $L^{\beta'}(\tilde{\Omega}_n \times Y; \mathbf{R}^{d\times d}_{sym})$. Therefore one deduces the accomplishment of (44) for any $\mathbf{U} \in L^\beta(\tilde{\Omega}_n \times Y; \mathbf{R}^{d\times d}_{sym})$. Minty's trick yields that $\overline{\mathbf{S}^0}(x,y) = \mathbf{S}(y, \mathbf{Du}(x) + \mathbf{D}_y\mathbf{u}^0(x,y))$ for almost all $(x,y) \in \tilde{\Omega}_n \times Y$. Since $|\Omega \setminus \Omega_n| \to 0$ and $O \Subset \Omega$ was arbitrary, we have for almost all $(x,y) \in \Omega \times Y$ $\overline{\mathbf{S}^0}(x,y) = \mathbf{S}(y, \mathbf{Du}(x) + \mathbf{D}_y\mathbf{u}^0(x,y))$. Moreover, due to the properties (26) and (27) we obtain that $\mathbf{D}_y\mathbf{u}^0$ is a solution of the cell problem (15) with $\xi = \mathbf{Du}(x)$. Consequently, (30) and the definition of the homogenized tensor imply for $x \in \Omega$

$$\overline{\mathbf{S}}(x) = \int_Y \mathbf{S}(y, \mathbf{Du}(x) + \mathbf{D}_y\mathbf{u}^0(x,y))\,\mathrm{d}y = \hat{\mathbf{S}}(\mathbf{Du}(x)).$$

□

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6. Bibliography

[1] E. Acerbi and N. Fusco, *An approximation lemma for $W^{1,p}$ functions*, Material instabilities in continuum mechanics (Edinburgh, 1985–1986), Oxford Sci. Publ., Oxford Univ. Press, New York, 1988, pp. 1–5. MR 970512 (89m:46060)

[2] G. Allaire, *Homogenization and two-scale convergence*, SIAM J. Math. Anal. **23** (1992), no. 6, 1482–1518. MR 1185639 (93k:35022)

[3] J. M. Ball and F. Murat, *Remarks on Chacon's biting lemma*, Proc. Amer. Math. Soc. **107** (1989), no. 3, 655–663. MR 984807 (90g:46064)

[4] M. Bulíček, P. Gwiazda, J. Málek, A. Świerczewska-Gwiazda, *On unsteady flows of implicitly constituted incompressible fluids*, SIAM J. Math. Anal. 44 (2012), no. 4, 2756-2801.

[5] L. Diening, P. Harjulehto, P. Hästö, and M. Růžička, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, vol. 2017, Springer, Heidelberg, 2011. MR 2790542

[6] L. Diening, J. Málek, and M. Steinhauer, *On Lipschitz truncations of Sobolev functions (with variable exponent) and their selected applications*, ESAIM Control Optim. Calc. Var. **14** (2008), no. 2, 211–232. MR 2394508 (2009e:35054)

[7] L. Diening, M. Růžička and J. Wolf, *Existence of weak solutions for unsteady motions of generalized Newtonian fluids*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **9** (2010), no. 1, 146.

[8] J. Diestel and J.J. Uhl, *Vector measures*, Mathematical surveys and monographs, American Mathematical Society, 1977.

[9] E. Feireisl and A. Novotný, *Singular limits in thermodynamics of viscous fluids*, Advances in Mathematical Fluid Mechanics, Birkhäuser Verlag, Basel, 2009. MR 2499296 (2011b:35001)

[10] J. Frehse, J. Málek and M. Steinhauer, *On analysis of steady flows of fluids with shear-dependent viscosity based on the Lipschitz truncation method*, SIAM J. Math. Anal. **34** (2003), no. 5, 1064-1083.

[11] E. Giusti, *Direct methods in the calculus of variations*, World Scientific Publishing Co., Inc., River Edge, NJ, 2003. MR 1962933 (2004g:49003)

[12] V. V. Zhikov, S. M. Kozlov, and O. A. Oleĭnik, *Homogenization of differential operators and integral functionals*, Springer-Verlag, Berlin, 1994, Translated from the Russian by G. A. Yosifian [G. A. Iosif′yan]. MR 1329546 (96h:35003b)

[13] G. Nguetseng, *A general convergence result for a functional related to the theory of homogenization*, SIAM J. Math. Anal. **20** (1989), no. 3, 608–623. MR 990867 (90j:35030)

[14] M. Růžička, *Electrorheological fluids: modeling and mathematical theory*, Lecture Notes in Mathematics, vol. 1748, Springer-Verlag, Berlin, 2000. MR 1810360 (2002a:76004)

[15] M. S. Skaff, *Vector valued Orlicz spaces generalized N-functions. I*, Pacific J. Math. **28** (1969), 193–206. MR 0415305 (54 #3395a)

[16] ———, *Vector valued Orlicz spaces. II*, Pacific J. Math. **28** (1969), 413–430. MR 0415306 (54 #3395b)

[17] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970. MR 0290095 (44 #7280)

[18] A. Visintin, *Towards a two-scale calculus*, ESAIM Control Optim. Calc. Var. **12** (2006), no. 3, 371–397 (electronic). MR 2224819 (2007b:35034)

[19] V. V. Zhikov, *Homogenization of a Navier-Stokes-type system for electrorheological fluid*, Complex Var. Elliptic Equ. **56** (2011), no. 7-9, 545–558. MR 2832202 (2012i:35022)