Part I: Bellman equation and Vlasov-McKean-functionals

1. Background: ODE - optimal control

Consider

$$\Phi(v; t_0, y) = \int_{t_0}^{T} \underbrace{f(t, x, v)}_{\text{"pay off"}} dt + h(x(T)) = \min!$$

subject to

$$\dot{x} = g(t, x, v), \quad x(t_0) = y, \quad v(t) \in \mathcal{U} = \bar{\mathcal{U}} \subset \mathbb{R}^m$$

existence holds via Cesari-Filippov-theory

2. Bellman equation

Define value function

$$u(\tau, y) = \min_{v} \Phi(v; \tau y)$$

u "satisfies" Bellman equation

$$u_t + H(\cdot, \nabla u) = 0 \quad , \quad u|_T = h$$

where $\quad H(t, x, p) = \min_{\nu \in U} L(t, x, \nu; p)$

with $\quad L(\cdot, \nu; p) = f(\cdot, \nu) + p \cdot g(\cdot, \nu) \quad ,$

H = "Hamiltonian" , L = Lagrangian

Philosophy: solve Bellman $\Rightarrow$ the minimizer $\nu = \nu(t, x; \nabla u(t, x))$ defines the optimal feed back control.

3. Stochastic version:

Replace $\dot{x} = g(\cdot, x, v)$ by

$$dx = g\,dt + d\omega \quad , \quad \omega = \text{white Noise}$$

and $\phi(v, t_0, y)$ by an expectation value

$\Rightarrow$ Bellman equation for the stochastic case

$$-u_t - \Delta u = H(\cdot, \nabla u) \quad , \quad u|_T = h \text{ in } \Omega$$

in $[0,T] \times \Omega$

& Dirichlet or Neumann conditions.

4. Stochastic games can be formulated via Nash-equilibrium

Example of a Bellman system with two players

$$f^i(t,x,v) = \frac{1}{2}|v^i|^2 + f_0^i(t,x) \quad , i = 1,2$$

$$g = A^1 v^1 + A^2 v^2 + b(.)$$

A Nash-point of the Lagrangians satisfies

$$0 = \frac{\partial}{\partial v^i} L^i = \frac{\partial}{\partial v^i}\left(\frac{1}{2}|v^i|^2 + \nabla u^i A^i v^i\right) \Rightarrow v^i = (A^i)^T \nabla u^i \cdot (-1)$$

$$\Rightarrow H^i = \frac{1}{2}|(A^i)^T \nabla u^i|^2 - \nabla u^i\left(\sum_{j=1}^{2} A^j (A^j)^T \nabla u^j + b\right) + f_0^i$$

Problem with quadratic growth!

5. Stochastic Bellman equations via Vlasov-Mc Kean-functionals

Define the mean-field-variable $m = m(v)$ via the forward parabolic equation

$$m_t - \Delta m + \operatorname{div}(m\, g(\cdot, v)) = 0 \qquad m\big|_{t=0} = m_0 \geq 0$$

Neumann cond.

For bounded v $\Rightarrow$ $\exists$ sol. $m \geq 0$, $m \in C^\infty \cap L^2(H^1)$ and better

Vlasov-Mc Kean-functional

$$J(v) = \int_0^T \int_\Omega m(v)\, f(\cdot, v)\, dx\, dt + \int_\Omega h \cdot m(T, \cdot)\, dx$$

Minimizing J leads to the "stochastic Bellman equ." !!

5b

THEOREM. Let v^* be a minimizer of the Vlasov-Mc-Kean-functional (in the compact case).

$$\Rightarrow \quad H(\cdot, \nabla u) \underset{\substack{\uparrow \\ \text{pointwise} \\ \text{for } t,x}}{=} \min L(\cdot, v; \nabla u) = L(\cdot, v^*; \nabla u)$$

where u solves the "stochastic" Bellman eqn.

A similar statement holds for games.

6 "Miracles" concerning the Vlasov-McKean-approach

(i) The solution u does not depend on m_0

$\Rightarrow$ v^* does not depend on m_0

(ii) There holds

$$\int_\Omega u(0,x)\, m_0\, dx = J(v^*)$$

one can choose a sequence of $m_0 \to$ Dirac function, so one sees that u is the pointwise value function

(iii) A parabolic equation (globally) is written as minimization problem of a variational problem

(iv) A "flash"-proof for global existence is possible via lower semicontinuity method.

6b further "miracle"

In the game-theoretic case with N-Vlasov-McKean-functionals and N controls v^i one can prove the existence of a Nash point v^* although the i-th Vlasov-McKean-functional <u>is not convex</u> (!!)

This is based on the uniqueness of the solution of each of the i-th Bellman equations

7 Variants of Vlasov-McKean-functionals

$$\dot{m} - \Delta m + m v = 0 \qquad m\big|_{t=0} = m_0 \geq 0$$

Neumann cond., positive m is possible : $mv \Leftrightarrow m_+ v$

pay off : $f(v) = \frac{1}{p}|v|^p$. $\Rightarrow$ "Bellman"-equation

from $\int_0^T \int_\Omega m(v) f(v)\, dx\, dt = \min$

$$-\dot{u} - \Delta u + \left(1 - \frac{1}{p}\right)|u|^{\frac{p}{p-1}} = 0 \quad , \quad u\big|_T = u_T \text{ given}$$

Neumann-Problem

A "starting" estimate $\iint |u|^{\frac{p}{p-1}} m\, dx\, dt$

is possible