Navier-Stokes-Fourier system: dissipative solutions, relative entropies, inviscid and incompressible limits

BIRS, Banff, Canada 17.09-21.09. 2012

Antonín Novotný

IMATH, Université du Sud Toulon-Var, BP 132, 839 57 La Garde, France
http://imath.univ-tln.fr

Joint work with *Eduard Feireisl*

Plan of the talk

1. Dissipative solutions
2. Relative entropy inequality
3. Inviscid limits - weak stratification
4. Inviscid limits - strong stratification

1 Navier-Stokes-Fourier system, weak stratification

1.1 Classical formulation

• $T > 0$, $t \in [0, T]$ is time variable, $\Omega = R^3$ (for simplicity) $x \in \Omega$ is a space variable. We are searching for unknown functions $\varrho(t,x)$ - density, $\vartheta(t,x)$- absolute temperature, $\mathbf{u}(t,x)$ - velocity vector satisfying

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho, \vartheta) = \varepsilon^a \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}), \tag{1.2}$$

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \varepsilon^b \mathrm{div}_x \left(\frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \right) = \sigma, \tag{1.3}$$

$$\sigma = \varepsilon^{2+a} \frac{\mathbb{S}(\vartheta, \nabla_x \mathbf{u})}{\vartheta} : \nabla_x \mathbf{u} - \varepsilon^b \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta^2} \cdot \nabla_x \vartheta, \tag{1.4}$$

$$\text{Mach number} = \varepsilon, \quad \text{Reynolds number} = \varepsilon^{-a}, \quad \text{Péclet number} = \varepsilon^{-b}$$

$$\mathrm{d}e - \vartheta \mathrm{d}s = \frac{p}{\varrho^2} \mathrm{d}\varrho.$$

$$\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) = \mu(\vartheta) \Big(\nabla_x \mathbf{u} + (\nabla_x \mathbf{u})^T - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \Big) + \eta(\vartheta) \mathrm{div}_x \mathbf{u} \mathbb{I}, \tag{1.5}$$

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) = -\kappa(\vartheta) \nabla_x \vartheta \tag{1.6}$$

$$\varrho \to \overline{\varrho},\ \vartheta \to \overline{\vartheta},\ \mathbf{u} \to 0 \text{ as } |x| \to \infty. \tag{1.7}$$

$$\varrho(0,\cdot) = \varrho_{0,\varepsilon} = \overline{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon},\ \vartheta(0,\cdot) = \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon},\ \mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon}. \tag{1.8}$$

1.2 Expected target system as $\varepsilon \to 0$

$$\mathrm{div}_x \mathbf{v} = 0, \tag{1.9}$$

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = 0, \tag{1.10}$$

$$\partial_t T + \mathbf{v} \cdot \nabla_x T = 0. \tag{1.11}$$

1.3 Constitutive relations

1.3.1 Pressure, Internal energy, entropy

• Gibbs relations

We assume that the thermodynamic functions p, e, and s are interrelated through Gibbs' equation

$$\vartheta \mathrm{d}s(\varrho,\vartheta) = \mathrm{d}e(\varrho,\vartheta) - \frac{p(\varrho,\vartheta)}{\varrho^2}\mathrm{d}\varrho. \tag{1.12}$$

• Pressure

$$p(\varrho,\vartheta) = \vartheta^{\gamma/(\gamma-1)} P\left(\frac{\varrho}{\vartheta^{1/(\gamma-1)}}\right) + \frac{a}{3}\vartheta^4,\ a>0,\ \gamma>3/2, \tag{1.13}$$

where

$$P \in C^1[0,\infty),\ P(0)=0,\ P'(Z)>0 \text{ for all } Z \geq 0. \tag{1.14}$$

• Internal Energy

$$e(\varrho,\vartheta) = \frac{1}{\gamma-1}\frac{\vartheta^{\gamma/(\gamma-1)}}{\varrho} P\left(\frac{\varrho}{\vartheta^{1/(\gamma-1)}}\right) + a\frac{\vartheta^4}{\varrho}. \tag{1.15}$$

$$0 < \frac{\gamma P(Z) - P'(Z)Z}{Z} < c \text{ for all } Z > 0. \tag{1.16}$$

Relation (3.5) implies that the function $Z \mapsto P(Z)/Z^\gamma$ is decreasing, and we suppose that

$$\lim_{Z\to\infty} \frac{P(Z)}{Z^\gamma} = P_\infty > 0. \tag{1.17}$$

• Specific entropy

$$s(\varrho,\vartheta) = S\left(\frac{\varrho}{\vartheta^{1/(\gamma-1)}}\right) + \frac{4a}{3}\frac{\vartheta^3}{\varrho}, \tag{1.18}$$

where, in accordance with Third law of thermodynamics,

$$S'(Z) = -\frac{1}{\gamma-1}\frac{\gamma P(Z) - P'(Z)Z}{Z^2} < 0,\ \lim_{Z\to\infty} S(Z) = 0. \tag{1.19}$$

From the point of view of statististical mechanics, the above hypotheses are physically reasonable at least in two cases: if $\gamma = 5/3$ they modelize the monoatomic gas, if $\gamma = 4/3$ they modelize the so called relativistic gas.

1.3.2 Transport coefficients

$$\mu, \eta \in C^1[0,\infty) \text{ are globally Lipschitz and } \mu_0(1+\vartheta) \le \mu(\vartheta),\ 0 \le \eta(\vartheta), \quad \mu_0 > 0, \tag{1.20}$$

$$\kappa \in C^1[0,\infty), \quad \kappa_0(1+\vartheta^3) \le \kappa(\vartheta) \le \kappa_1(1+\vartheta^3),\ 0 < \kappa_0 \le \kappa_1. \tag{1.21}$$

1.4 Weak and dissipative solutions

1.4.1 Ballistic free energy alias Helmholtz function

We introduce ballistic free energy

$$H_\Theta(\vartheta,\varrho) = \varrho e(\varrho,\vartheta) - \Theta\varrho s(\varrho,\vartheta)$$

We suppose that the fluid verifies the *thermodynamic stability conditions,*

$$\frac{\partial p(\varrho,\vartheta)}{\partial\varrho} > 0, \ \frac{\partial e(\varrho,\vartheta)}{\partial\vartheta} > 0 \text{ for all } \varrho,\vartheta > 0. \tag{1.22}$$

We easily verify by using (1.12), that

$$\frac{\partial H_{\overline{\vartheta}}}{\partial\vartheta}(\varrho,\vartheta) = \varrho\frac{\vartheta-\overline{\vartheta}}{\vartheta}\frac{\partial e}{\partial\vartheta}(\varrho,\vartheta) \text{ and } \frac{\partial^2 H_{\overline{\vartheta}}}{\partial\varrho^2}(\varrho,\overline{\vartheta}) = \frac{1}{\varrho}\frac{\partial p}{\partial\varrho}(\varrho,\overline{\vartheta}). \tag{1.23}$$

Thus, the thermodynamic stability in terms of the function $H_{\overline{\vartheta}}$, can be reformulated as follows:

$$\varrho \mapsto H_{\overline{\vartheta}}(\varrho,\overline{\vartheta}) \text{ is strictly convex,} \tag{1.24}$$

while

$$\vartheta \mapsto H_{\overline{\vartheta}}(\varrho,\vartheta) \text{ attains its global minimum at } \vartheta = \overline{\vartheta}. \tag{1.25}$$

We set

$$E(\varrho,\vartheta \mid r,\Theta) = H_\Theta(\varrho,\vartheta) + \partial_\varrho H_\Theta(r,\Theta)(\varrho - r) + H_\Theta(r,\Theta)$$

and notice that

$$E(\varrho,\vartheta \mid r,\Theta) \geq 0 \ \text{ and } E(\varrho,\vartheta \mid r,\Theta) = 0 \Leftrightarrow (\varrho,\vartheta) = (r,\Theta).$$

1.4.2 Very weak solutions

Let $\Omega = R^3$ be a bounded Lipschitz domain. We say that a trio $\{\varrho, \vartheta, \mathbf{u}\}$ is a *very weak solution* to the Navier-Stokes-Fourier system (1.1 - 1.8) if:

(i) the density and the absolute temperature satisfy $\varrho(t,x) \geq 0$, $\vartheta(t,x) > 0$ *for a.a.* $(t,x) \in (0,T) \times \Omega$, $\varrho - \overline{\varrho} \in L^\gamma + L^2(\Omega)$, $\varrho\mathbf{u} \in L^\infty(0,T; L^{2\gamma/(\gamma+1)}_{\rm loc}(\Omega;R^3))$, $\varrho\mathbf{u}^2 \in L^\infty(0,T;L^1(\Omega))$, $\vartheta - \overline{\vartheta} \in L^\infty(0,T;L^4+L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega))$, $\nabla_x \log\vartheta \in L^2(0,T;L^2(\Omega;R^3))$ *and* $\mathbf{u} \in L^2(0,T;W^{1,2}_0(\Omega;R^3))$;

(ii) $\varrho \in C_{\rm weak}([0,T];L^\gamma_{\rm loc}(\Omega))$ *and equation (1.1) is replaced by a family of integral identities*

$$\int_\Omega \varrho\varphi \ {\rm d}x\Big|_0^\tau = \int_0^\tau \int_\Omega \Big(\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi\Big) \ {\rm d}x \ {\rm d}t \tag{1.26}$$

for all $\tau \in [0,T]$ *and for any* $\varphi \in C^1([0,T]\times\overline{\Omega})$;

(iii) $\varrho\mathbf{u} \in C_{\rm weak}([0,T];L^{2\gamma/(\gamma+1)}_{\rm loc}(\Omega;R^3))$ *and momentum equation (1.2) is satisfied in the sense of distributions, specifically,*

$$\int_\Omega \varrho\mathbf{u}\cdot\varphi \ {\rm d}x\Big|_0^\tau = \int_0^\tau \int_\Omega \Big(\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\varphi + p(\varrho,\vartheta){\rm div}_x\varphi - \mathbb{S}(\vartheta,\nabla_x\mathbf{u}) : \nabla_x\varphi\Big) \ {\rm d}x \ {\rm d}t \tag{1.27}$$

for all $\tau \in [0,T]$ *and for any* $\varphi \in C^1_c([0,T]\times\Omega;R^3)$;

(iv) the entropy balance (1.3), (1.8) is replaced by a family of integral inequalities

$$
\begin{aligned}
-\int_\Omega \varrho s(\varrho,\vartheta)\varphi \ \mathrm{d}x\Big|_0^\tau + \int_0^\tau \int_\Omega \frac{\varphi}{\vartheta}\left(\mathbb{S}(\vartheta,\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} - \frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\right) \ \mathrm{d}x \ \mathrm{d}t \\
\leq -\int_0^\tau \int_\Omega \left(\varrho s(\varrho,\vartheta)\partial_t\varphi + \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla_x\varphi + \frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)\cdot\nabla_x\varphi}{\vartheta}\right) \ \mathrm{d}x \ \mathrm{d}t
\end{aligned}
\tag{1.28}
$$

for a.a. $\tau \in (0,T)$ *and for any* $\varphi \in C^1([0,T]\times\overline{\Omega})$, $\varphi \geq 0$*;*

(v) the dissipation inequality holds,

$$
\begin{aligned}
\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + E\left(\varrho,\vartheta \,\middle|\, \overline{\varrho},\overline{\vartheta}\right)\right)(\tau,\cdot)\mathrm{d}x \\
+\int_0^\tau \int_\Omega \frac{\overline{\vartheta}}{\vartheta}\left(\mathbb{S}(\vartheta,\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} - \frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\right)\mathrm{d}x\mathrm{d}t \leq \\
\int_\Omega \left(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + E\left(\varrho_0,\vartheta_0 \,\middle|\, \overline{\varrho},\overline{\vartheta}\right)\right)\mathrm{d}x.
\end{aligned}
\tag{1.29}
$$

for a.a. $\tau \in (0,T)$.

Here and hereafter, the symbol $\int_\Omega g\mathrm{d}x\Big|_0^\tau$ means $\int_\Omega g(x,\tau)\mathrm{d}x - \int_\Omega g_0(x)\mathrm{d}x$.

1.4.3 Renormalized very weak solutions

We say that the triplet $(\varrho, \vartheta, \mathbf{u})$ *is a renormalized weak solution to the Navier-Stokes-Fourier system (1.1 - 1.8) if it is a very weak solution, and if the couple* $(\varrho, \mathbf{u})$ *satisfies the continuity equation in the renormalized sense,*

$$\int_\Omega b(\varrho)\varphi \,\mathrm{d}x\Big|_0^\tau = \int_0^\tau \int_\Omega b(\varrho)\Big(\partial_t \varphi + \mathbf{u}\cdot\nabla_x \varphi\Big)\mathrm{d}x\mathrm{d}t + \int_0^\tau \int_\Omega \Big(\varrho b'(\varrho) - b(\varrho)\Big)\mathrm{div}\mathbf{u}\varphi \mathrm{d}x\mathrm{d}t \tag{1.30}$$

for any $\tau \in [0, T]$*, and any*

$$b \in C[0,\infty), \quad b' \in C_c[0,\infty) \quad \textit{and} \quad \varphi \in C^1_c([0,T)\times\overline{\Omega}).$$

Notice that the set of admissible renormalizing functions b can be extended by density and Lebesgue dominated convergence theorem to

$$b \in C[0,\infty)\cap C^1(0,\infty),\ zb' \in L^\infty(0,1),\ b/z^{5\gamma/6},\ zb'/z^{\gamma/2} \in L^\infty(1,\infty).$$

1.4.4 Dissipative solutions

We say that a trio $\{\varrho, \vartheta, \mathbf{u}\}$ is a *dissipative solution* to the Navier-Stokes-Fourier system (1.1 - 1.8) if:

(i) the density and the absolute temperature satisfy $\varrho(t,x) \geq 0$, $\vartheta(t,x) > 0$ *for a.a.* $(t,x) \in (0,T) \times \Omega$, $\varrho - \overline{\varrho} \in L^\gamma + L^2(\Omega)$, $\varrho \mathbf{u} \in L^\infty(0,T; L^{2\gamma/(\gamma+1)}_{\rm loc}(\Omega; R^3))$, $\varrho \mathbf{u}^2 \in L^\infty(0,T; L^1(\Omega))$, $\vartheta - \overline{\vartheta} \in L^\infty(0,T; L^4 + L^2(\Omega)) \cap L^2(0,T; W^{1,2}(\Omega))$, $\nabla_x \log \vartheta \in L^2(0,T; L^2(\Omega; R^3))$ *and* $\mathbf{u} \in L^2(0,T; W^{1,2}_0(\Omega; R^3))$*;*

(ii) the so called relative entropy inequality holds,

$$\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + E(\varrho, \vartheta | r, \Theta) \right) (\tau, \cdot) \ {\rm d}x \tag{1.31}$$

$$+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \ {\rm d}x \ {\rm d}t$$

$$\leq \int_\Omega \left(\frac{1}{2} \varrho_0 |\mathbf{u}_0 - \mathbf{U}(0,\cdot)|^2 + \mathcal{E}(\varrho_0, \vartheta_0 | r(0,\cdot), \Theta(0,\cdot)) \right) \ {\rm d}x$$

$$+ \int_0^\tau \int_\Omega \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{U} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \Theta \right) \ {\rm d}x{\rm d}t$$

$$+ \int_0^\tau \int_\Omega \left(\varrho \Big(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) - p(\varrho, \vartheta) {\rm div}_x \mathbf{U} \right) \ {\rm d}x \ {\rm d}t$$

$$- \int_0^\tau \int_\Omega \varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \Big(\partial_t \Theta + \mathbf{u} \cdot \nabla_x \Theta \Big) \ {\rm d}x \ {\rm d}t$$

$$+ \int_0^\tau \int_\Omega \left(\left(1 - \frac{\varrho}{r} \right) \partial_t p(r, \Theta) - \frac{\varrho}{r} \mathbf{u} \cdot \nabla_x p(r, \Theta) \right) \ {\rm d}x \ {\rm d}t,$$

for all

$$r, \Theta, \in C^1([0,T] \times \mathbb{R}^3), \ r > 0, \ \Theta > 0, \ \mathbf{U} \in C^1([0,T] \times \mathbb{R}^3).$$

1.5 Existence theorem for the original problem

Theorem 1.1 *Suppose that the thermodynamic functions* p, e, s *satisfy hypotheses (8.6 - 8.9), and that the transport coefficients* μ, η, *and* κ *obey (8.10), (8.11). Finally assume that the initial data (1.8) verify*

$$\int_{\Omega} \Big(\frac{1}{2} \varrho_0 \mathbf{u}_0^2 + \varrho_0 e(\varrho_0, \vartheta_0) + \varrho_0 |s(\varrho_0, \vartheta_0)| \Big) \, \mathrm{d}x < \infty. \tag{1.32}$$

Then the Navier-Stokes-Fourier system (1.1–1.8) admits at least one very weak renormalized dissipative solution.

2 Euler system

$$\mathrm{div}_x \mathbf{v} = 0, \tag{2.1}$$

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = 0, \tag{2.2}$$

Theorem 2.1 *Suppose that $\mathbf{v}_0$ is a given vector field such that*

$$\mathbf{v}_0 \in W^{k,2}(R^3; R^3),\ k > \frac{5}{2},\ \|\mathbf{v}_0\|_{W^{k,2}(\Omega;R^3)} \leq D,\ \mathrm{div}_x \mathbf{v}_0 = 0.$$

Then the Euler system (2.1), (2.2), supplemented with the initial condition

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0.$$

possesses a regular solution $\mathbf{v}$, unique in the class

$$\mathbf{v} \in C([0, T_{\max}); W^{k,2}(R^3; R^3)),\ \partial_t \mathbf{v} \in C([0, T_{\max}); W^{k-1,2}(R^3; R^3)), \tag{2.3}$$

defined on a maximal time interval $[0, T_{\max})$, $T_{\max} = T_{\max}(D)$.

3 Main result

Let the thermodynamic functions p, e, and s comply with hypotheses (1.12 - 8.9), and let the transport coefficients μ and κ satisfy (8.10), (8.11). Let $b > 0$, $0 < a < \frac{10}{3}$. Furthermore, suppose that the initial data (1.8) are chosen in such a way that

$$\{\varrho^{(1)}_{0,\varepsilon}\}_{\varepsilon>0},\ \{\vartheta^{(1)}_{0,\varepsilon}\}_{\varepsilon>0} \text{ are bounded in } L^2\cap L^\infty(R^3),\ \varrho^{(1)}_{0,\varepsilon}\to\varrho^{(1)}_0,\ \vartheta^{(1)}_{0,\varepsilon}\to\vartheta^{(1)}_0 \text{ in } L^2(R^3), \tag{3.1}$$

and

$$\{\mathbf{u}_{0,\varepsilon}\}_{\varepsilon>0} \text{ is bounded in } L^2(R^3;R^3),\ \mathbf{u}_{0,\varepsilon}\to\mathbf{u}_0 \text{ in } L^2(R^3;R^3), \tag{3.2}$$

where

$$\varrho^{(1)}_0,\ \vartheta^{(1)}_0 \in W^{1,2}\cap W^{1,\infty}(R^3),\ \mathbf{H}[\mathbf{u}_0]=\mathbf{v}_0\in W^{k,2}(R^3;R^3) \text{ for a certain } k>\frac{5}{2}. \tag{3.3}$$

Let $T_{\max}\in(0,\infty]$ denote the maximal life-span of the regular solution $\mathbf{v}$ to the Euler system (1.9), (1.10) satisfying $\mathbf{v}(0,\cdot)=\mathbf{v}_0$. Finally, let $\{\varrho_\varepsilon,\vartheta_\varepsilon,\mathbf{u}_\varepsilon\}$ be a dissipative solution of the Navier-Stokes-Fourier system (1.1 - 1.8) in $(0,T)\times R^3$, $T<T_{\max}$.

Then

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|\ \varrho_\varepsilon(t,\cdot)-\overline{\varrho}\ \|_{L^2+L^{5/3}(R^3)} \le \varepsilon c, \tag{3.4}$$

$$\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon \to \sqrt{\overline{\varrho}}\ \mathbf{v} \text{ in } L^\infty_{\rm loc}((0,T];L^2_{\rm loc}(R^3;R^3)) \text{ and weakly-(*) in } L^\infty(0,T;L^2(R^3;R^3)), \tag{3.5}$$

and

$$\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}\to T \text{ in } L^\infty_{\rm loc}((0,T];L^q_{\rm loc}(R^3;R^3)),\ 1\le q<2, \text{ and weakly-(*) in } L^\infty(0,T;L^2(R^3)), \tag{3.6}$$

where $\mathbf{v}$, T is the unique solution of the Euler-Boussinesq system (1.9 - 1.11), with the initial data

$$\mathbf{v}_0=\mathbf{H}[\mathbf{u}_0],\ T_0=\overline{\varrho}\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\vartheta^{(1)}_0-\frac{1}{\overline{\varrho}}\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\varrho^{(1)}_0. \tag{3.7}$$

4 Limit $\varepsilon \to 0$

4.1 Relative entropy inequality

$$
\begin{aligned}
&\left[\mathcal{E}_\varepsilon\left(\varrho,\vartheta,\mathbf{u}\middle| r,\Theta,\mathbf{U}\right)\right]_{t=0}^{\tau} \\
&+\int_0^\tau\int_\Omega \frac{\Theta}{\vartheta}\left(\varepsilon^a\mathbb{S}(\vartheta,\nabla_x\mathbf{u}):\nabla_x\mathbf{u}-\varepsilon^{b-2}\frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\right)\ \mathrm{d}x\ \mathrm{d}t \\
&\le \int_0^\tau \mathcal{R}_\varepsilon\left(\varrho,\vartheta,\mathbf{u}\middle| r,\Theta,\mathbf{U}\right)\mathrm{d}t
\end{aligned}
\tag{4.1}
$$

where

$$
\mathcal{E}_\varepsilon\left(\varrho,\vartheta,\mathbf{u}\middle| r,\Theta,\mathbf{U}\right)=\int_\Omega\left[\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2+\frac{1}{\varepsilon^2}\left(H_\Theta(\varrho,\vartheta)-\frac{\partial H_\Theta(r,\Theta)}{\partial\varrho}(\varrho-r)-H_\Theta(r,\Theta)\right)\right]\ \mathrm{d}x, \tag{4.2}
$$

and

$$
\begin{aligned}
\mathcal{R}_\varepsilon\left(\varrho,\vartheta,\mathbf{u}\middle| r,\Theta,\mathbf{U}\right)&=\int_\Omega\left(\varepsilon^a\mathbb{S}(\vartheta,\nabla_x\mathbf{u}):\nabla_x\mathbf{U}-\varepsilon^b\frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)}{\vartheta}\cdot\nabla_x\Theta\right)\ \mathrm{d}x \\
&+\int_\Omega\left(\varrho\Big(\partial_t\mathbf{U}+\mathbf{u}\cdot\nabla_x\mathbf{U}\Big)\cdot(\mathbf{U}-\mathbf{u})-\frac{1}{\varepsilon^2}\frac{\varrho}{r}\mathbf{u}\cdot\nabla_x p(r,\Theta)\Big)\right)\ \mathrm{d}x \\
&-\frac{1}{\varepsilon^2}\int_\Omega\left[\Big(p(\varrho,\vartheta)-p(\overline{\varrho},\overline{\vartheta})\Big)\mathrm{div}\mathbf{U}\right]\ \mathrm{d}x \\
&-\frac{1}{\varepsilon^2}\int_\Omega\left(\varrho\Big(s(\varrho,\vartheta)-s(r,\Theta)\Big)\partial_t\Theta+\varrho\Big(s(\varrho,\vartheta)-s(r,\Theta)\Big)\mathbf{u}\cdot\nabla_x\Theta\right)\ \mathrm{d}x \\
&+\frac{1}{\varepsilon^2}\int_\Omega\frac{r-\varrho}{r}\partial_t p(r,\Theta)\ \mathrm{d}x
\end{aligned}
$$

4.2 Ill-prepared initial data versus well-prepared initial data

$$\varrho(0,\cdot) = \varrho_{0,\varepsilon} = \overline{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon},\ \vartheta(0,\cdot) = \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon},\ \mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon} \tag{4.3}$$

$$\{\varrho^{(1)}_{0,\varepsilon}\}_{\varepsilon>0},\ \{\vartheta^{(1)}_{0,\varepsilon}\}_{\varepsilon>0} \text{ are bounded in } L^2 \cap L^\infty(R^3),\ \varrho^{(1)}_{0,\varepsilon} \to \varrho^{(1)}_0,\ \vartheta^{(1)}_{0,\varepsilon} \to \vartheta^{(1)}_0 \text{ in } L^2(R^3), \tag{4.4}$$

and

$$\{\mathbf{u}_{0,\varepsilon}\}_{\varepsilon>0} \text{ is bounded in } L^2(R^3;R^3),\ \mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \text{ in } L^2(R^3;R^3), \tag{4.5}$$

where

$$\varrho^{(1)}_0,\ \vartheta^{(1)}_0 \in W^{1,2} \cap W^{1,\infty}(R^3),\ \mathbf{H}[\mathbf{u}_0] = \mathbf{v}_0 \in W^{k,2}(R^3;R^3) \text{ for a certain } k > \frac{5}{2}. \tag{4.6}$$

4.3 Acoustic equation

We set

$$\alpha = \frac{1}{\overline{\varrho}} \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho}, \ \beta = \frac{1}{\overline{\varrho}} \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta}, \delta = \overline{\varrho} \frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta}, \ \omega = \overline{\varrho} \left(\alpha + \frac{\beta^2}{\delta} \right).$$

4.3.1 "Linearization" of the NSF system

The triplet

$$\Big(\frac{\varrho - \overline{\varrho}}{\varepsilon}, \frac{\vartheta - \overline{\vartheta}}{\varepsilon}, \nabla_x \Phi \Big), \quad \text{where } \nabla_x \Phi = \mathbf{H}^{\perp}(\mathbf{u})$$

satisfies

$$\varepsilon \partial_t \Big(\frac{\varrho - \overline{\varrho}}{\varepsilon} \Big) + \Delta \Phi = \varepsilon O(1),$$

$$\varepsilon \partial_t \nabla_x \Phi + \nabla_x \Big(\alpha \frac{\varrho - \overline{\varrho}}{\varepsilon} + \beta \frac{\vartheta - \overline{\vartheta}}{\varepsilon} \Big) = \varepsilon O(1)$$

$$\partial_t \Big(\delta \frac{\vartheta - \overline{\vartheta}}{\varepsilon} - \beta \frac{\varrho - \overline{\varrho}}{\varepsilon} \Big) + \mathbf{u} \cdot \nabla_x \Big(\delta \frac{\vartheta - \overline{\vartheta}}{\varepsilon} - \beta \frac{\varrho - \overline{\varrho}}{\varepsilon} \Big) + \Big(\delta \frac{\vartheta - \overline{\vartheta}}{\varepsilon} - \beta \frac{\varrho - \overline{\varrho}}{\varepsilon} \Big) \Delta \Phi = \varepsilon O(1)$$

with initial data

$$\Big(\frac{\varrho - \overline{\varrho}}{\varepsilon}, \frac{\vartheta - \overline{\vartheta}}{\varepsilon}, \nabla_x \Phi \Big)(0, \cdot) = \Big(\varrho_0^1, \vartheta_0^1, \mathbf{H}^{\perp}(\mathbf{u}_0) \Big)$$

4.4 "Linearization of NSF" an equivalent form

- **Acoustic equation for** $\alpha \frac{\varrho - \overline{\varrho}}{\varepsilon} + \beta \frac{\vartheta - \overline{\vartheta}}{\varepsilon}$

$$\varepsilon \partial_t \Big(\alpha \frac{\varrho - \overline{\varrho}}{\varepsilon} + \beta \frac{\vartheta - \overline{\vartheta}}{\varepsilon} \Big) + \omega \Delta \Phi = \varepsilon O(1),$$

$$\varepsilon \partial_t \nabla_x \Phi + \nabla_x \Big(\alpha \frac{\varrho - \overline{\varrho}}{\varepsilon} + \beta \frac{\vartheta - \overline{\vartheta}}{\varepsilon} \Big) = \varepsilon O(1)$$

- **Transport equation for** $\delta \frac{\vartheta - \overline{\vartheta}}{\varepsilon} - \beta \frac{\varrho - \overline{\varrho}}{\varepsilon}$

$$\partial_t \Big(\delta \frac{\vartheta - \overline{\vartheta}}{\varepsilon} - \beta \frac{\varrho - \overline{\varrho}}{\varepsilon} \Big) + \mathbf{u} \cdot \nabla_x \Big(\delta \frac{\vartheta - \overline{\vartheta}}{\varepsilon} - \beta \frac{\varrho - \overline{\varrho}}{\varepsilon} \Big) + \Big(\delta \frac{\vartheta - \overline{\vartheta}}{\varepsilon} - \beta \frac{\varrho - \overline{\varrho}}{\varepsilon} \Big) \Delta \Phi = \varepsilon O(1)$$

4.5 Choice of test functions $(r, \Theta, \mathbf{U})$ in the relative entropy inequality

$$r = r_\varepsilon = \overline{\varrho} + \varepsilon R_\varepsilon,\ \Theta = \Theta_\varepsilon = \overline{\vartheta} + \varepsilon T_\varepsilon,\ \mathbf{U} = \mathbf{U}_\varepsilon = \mathbf{v} + \nabla_x \Phi_\varepsilon; \tag{4.7}$$

where $\mathbf{v}$ is the solution to the incompressible Euler system (1.9), (1.10), with the initial condition (3.7), and R_ε, T_ε, and Φ_ε solve the *acoustic equation*:

$$\varepsilon \partial_t (\alpha R_\varepsilon + \beta T_\varepsilon) + \omega \Delta \Phi_\varepsilon = 0, \tag{4.8}$$

$$\varepsilon \partial_t \nabla_x \Phi_\varepsilon + \nabla_x (\alpha R_\varepsilon + \beta T_\varepsilon) = 0, \tag{4.9}$$

with the initial data determined by

$$R_\varepsilon(0, \cdot) = R_{0,\varepsilon},\ T_\varepsilon(0, \cdot) = T_{0,\varepsilon},\ \Phi_\varepsilon(0, \cdot) = \Phi_{0,\varepsilon}, \tag{4.10}$$

and the *transport equation*

$$\partial_t (\delta T_\varepsilon - \beta R_\varepsilon) + \mathbf{U}_\varepsilon \cdot \nabla_x (\delta T_\varepsilon - \beta R_\varepsilon) + (\delta T_\varepsilon - \beta R_\varepsilon) \mathrm{div}_x \mathbf{U}_\varepsilon = 0, \tag{4.11}$$

4.6 Goal

To find estimate

$$\int_0^\tau \mathcal{R}_\varepsilon \left(\varrho, \vartheta, \mathbf{u} \Big| r_\varepsilon, \Theta_\varepsilon, \mathbf{U}_\varepsilon \right) \mathrm{d}t \leq \chi^1_\varepsilon(\tau) + \int_0^\tau \chi^2 \mathcal{E}_\varepsilon \left(\varrho, \vartheta, \mathbf{u} \Big| r_\varepsilon, \Theta_\varepsilon, \mathbf{U}_\varepsilon \right)(t) \mathrm{d}(t),$$

where

$$\chi^i_\varepsilon \to 0 \text{ in } L^1(0, T)$$

and finally to get from the relative entropy inequality the inequality

$$\int_0^\tau \mathcal{E}_\varepsilon \left(\varrho, \vartheta, \mathbf{u} \Big| r_\varepsilon, \Theta_\varepsilon, \mathbf{U}_\varepsilon \right) \leq \chi^1_\varepsilon(\tau) + \int_0^\tau \chi^2_\varepsilon(t) \mathcal{E}_\varepsilon \left(\varrho, \vartheta, \mathbf{u} \Big| r_\varepsilon, \Theta_\varepsilon, \mathbf{U}_\varepsilon \right)(t) \mathrm{d}(t)$$

that implies

$$\mathrm{esssup}_{t \in (0,T)} \mathcal{E}_\varepsilon \left(\varrho, \vartheta, \mathbf{u} \Big| r_\varepsilon, \Theta_\varepsilon, \mathbf{U}_\varepsilon \right) \to 0.$$

5 Some elements of the proof

5.1 Dispersive estimates and energy identity for the acoustic equation

The acoustic equation (4.8 - 4.10) possesses a (unique) smooth solution Φ_ε, $Z_\varepsilon = \alpha R_\varepsilon + \beta T_\varepsilon$ satisfying the energy equality

$$\left[\|\nabla_x \Phi_\varepsilon(t,\cdot)\|^2_{W^{k,2}(R^3;R^3)} + \frac{\delta}{\beta^2 + \alpha\delta} \|\alpha R_\varepsilon(t,\cdot) + \beta T_\varepsilon(t,\cdot)\|^2_{W^{k,2}(R^3)} \right]_{t=0}^{t=\tau} = 0 \qquad (5.1)$$

$$\text{for all } \tau \geq 0, \ k = 0, 1, 2, \dots .$$

In addition, we have the dispersive estimates

$$\|\nabla_x \Phi_\varepsilon(t,\cdot)\|_{W^{k,q}(R^3;R^3)} + \|\alpha R_\varepsilon(t,\cdot) + \beta T_\varepsilon(t,\cdot)\|_{W^{k,q}(R^3)} \qquad (5.2)$$

$$\leq c \left(1 + \frac{t}{\varepsilon}\right)^{-\left(\frac{1}{p} - \frac{1}{q}\right)} \left(\|\nabla_x \Phi_{0,\varepsilon}\|_{W^{d+k,p}(R^3;R^3)} + \|\alpha R_{0,\varepsilon} + \beta T_{0,\varepsilon}\|_{W^{d+k,p}(R^3)} \right),$$

for all $t \geq 0$, where

$$2 \leq q \leq \infty, \ \frac{1}{p} + \frac{1}{q} = 1, \ d > 3\left(\frac{1}{p} - \frac{1}{q}\right), \ k = 0, 1, \dots,$$

called Strichartz estimates.

5.2 Estimates for the transport equation

For the solutions of the transport equation (4.11), we have

$$\Big[\int_\Omega |\delta T_\varepsilon - \beta R_\varepsilon|^2 \,\mathrm{d}x\Big]_0^\tau = -\int_0^\tau \int_\Omega \Delta\Phi_\varepsilon |\delta T_\varepsilon - \beta R_\varepsilon|^2 \,\mathrm{d}x \,\mathrm{d}t, \tag{5.3}$$

and

$$\sup_{t\in[0,T]} \|\delta T_\varepsilon - \beta R_\varepsilon\|_{W^{1,q}(\Omega)} \leq c(\eta, T)\, \|\delta T_{0,\varepsilon} - \beta R_{0,\varepsilon}\|_{W^{1,q}(\Omega)}\,,\ 1 \leq q \leq \infty. \tag{5.4}$$

5.3 Uniform bounds for the weak solution

It is convenient to introduce a decomposition

$$h = [h]_{\rm ess} + [h]_{\rm res} \text{ for a measurable function } h,$$

where

$$[h]_{\rm ess} = h \ 1_{\{\overline{\varrho}/2 < \varrho_\varepsilon < 2\overline{\varrho};\ \overline{\vartheta}/2 < \vartheta_\varepsilon < 2\overline{\vartheta}\}},\ [h]_{\rm res} = h - h_{\rm ess}.$$

Consequently, realizing that

$$E(\varrho, \vartheta \,|\, \overline{\varrho}, \overline{\vartheta}) \geq c\Big([1 + \varrho^\gamma + \vartheta^4]_{\rm res} + [\varrho - \overline{\varrho}]^2_{\rm ess} + [\vartheta - \overline{\vartheta}]^2_{\rm ess} \Big)$$

we derive from the relative entropy inequality

$$\operatorname{ess}\sup_{t \in (0,T)} \| \sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon (t, \cdot) \|_{L^2(R^3;R^3)} \leq c, \tag{5.5}$$

$$\operatorname{ess}\sup_{t \in (0,T)} \left\| \left[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} (t, \cdot) \right]_{\rm ess} \right\|_{L^2(R^3;R^3)} + \operatorname{ess}\sup_{t \in (0,T)} \left\| \left[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} (t, \cdot) \right]_{\rm ess} \right\|_{L^2(R^3;R^3)} \leq c, \tag{5.6}$$

$$\operatorname{ess}\sup_{t \in (0,T)} \int_{R^3} \left(\left[\varrho_\varepsilon^{5/3} (t, \cdot) \right]^{5/3}_{\rm res} + [\vartheta_\varepsilon (t, \cdot)]^4_{\rm res} + 1_{\rm res}(t, \cdot) \right) \ {\rm d}x \leq \varepsilon^2 c, \tag{5.7}$$

and

$$\left\| \varepsilon^{a/2} \mathbf{u}_\varepsilon \right\|_{L^2(0,T;W^{1,2}(R^3;R^3))} \leq c, \tag{5.8}$$

$$\left\| \varepsilon^{(b-2)/2} \left(\vartheta_\varepsilon - \overline{\vartheta} \right) \right\|_{L^2(0,T;W^{1,2}(R^3;R^3))} + \left\| \varepsilon^{(b-2)/2} \left(\log(\vartheta_\varepsilon) - \log(\overline{\vartheta}) \right) \right\|_{L^2(0,T;W^{1,2}(R^3;R^3))} \leq c \tag{5.9}$$

.

5.4 Once more the relative entropy

$$\int_0^\tau \mathcal{R}_\varepsilon \left(\varrho, \vartheta, \mathbf{u} \Big| r, \Theta, \mathbf{U}\right) = \int_0^\tau \int_\Omega \left(\varepsilon^a \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{U} - \varepsilon^b \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \Theta \right) \, \mathrm{d}x$$

$$+ \frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) (\mathbf{U} - \mathbf{u}) \cdot \nabla_x \Theta \, \mathrm{d}x$$

$$+ \int_0^\tau \int_\Omega \varrho \Big(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) - \frac{1}{\varepsilon^2} \frac{\varrho}{r} \mathbf{u} \cdot \nabla_x p(r, \Theta) \, \mathrm{d}x$$

$$- \frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \left[\Big(p(\varrho, \vartheta) - p(\overline{\varrho}, \overline{\vartheta}) \Big) \mathrm{div} \mathbf{U} \right] \, \mathrm{d}x$$

$$- \frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \left(\varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \partial_t \Theta + \varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \mathbf{U} \cdot \nabla_x \Theta \right) \, \mathrm{d}x$$

$$+ \frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \frac{r - \varrho}{r} \partial_t p(r, \Theta) \, \mathrm{d}x$$

$$\int_0^\tau \int_\Omega \left(\varepsilon^a \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{U} - \varepsilon^b \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \Theta \right) \, \mathrm{d}x$$

$$+ \frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) (\mathbf{U} - \mathbf{u}) \cdot \nabla_x \Theta \, \mathrm{d}x \le \chi_\varepsilon^1(\tau)$$

$$+ \int_0^\tau \int_\Omega \left[\varrho \Big(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) - \frac{1}{\varepsilon^2} \frac{\varrho}{r} \mathbf{u} \cdot \nabla_x p(r, \Theta) \right] \, \mathrm{d}x$$

$$\le \left[\int_\Omega |\nabla_x \Phi|^2 \, \mathrm{d}x \right]_0^\tau + \chi_\varepsilon^1(\tau) + \int_0^\tau \chi_\varepsilon^2(t) \mathcal{E}(t) \mathrm{d}t$$

$$
\begin{aligned}
&-\frac{1}{\varepsilon^2}\int_0^\tau \int_\Omega \Big[\Big(p(\varrho,\vartheta)-p(\overline{\varrho},\overline{\vartheta})\Big)\mathrm{div}\mathbf{U}\Big]\ \mathrm{d}x \\
&-\frac{1}{\varepsilon^2}\int_0^\tau \int_\Omega \Big(\varrho\Big(s(\varrho,\vartheta)-s(r,\Theta)\Big)\partial_t\Theta+\varrho\Big(s(\varrho,\vartheta)-s(r,\Theta)\Big)\mathbf{U}\cdot\nabla_x\Theta\Big)\ \mathrm{d}x \\
&+\frac{1}{\varepsilon^2}\int_0^\tau \int_\Omega \frac{r-\varrho}{r}\partial_t p(r,\Theta)\ \mathrm{d}x \\
&\leq \Big[\frac{1}{2}\frac{1}{\omega}\int_\Omega \Big|\alpha R+\beta T\Big|^2\ \mathrm{d}x+\frac{1}{2}\frac{1}{\omega}\int_\Omega \Big|\delta R-\beta T\Big|^2\ \mathrm{d}x\Big]_0^\tau+\chi_\varepsilon^1(\tau)
\end{aligned}
$$

where

$$
\chi_\varepsilon^i \rightarrow 0 \text{ in } L^1(0,T).
$$

5.5 Example of the treatment of the "red" term

- **Residual part** goes to zero.

- **Essential part**

$$-\frac{1}{\varepsilon}\int_0^\tau\int_\Omega\Big[\varrho_\varepsilon\Big(s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(r_\varepsilon,\Theta_\varepsilon)\Big)\partial_t T_\varepsilon+\varrho_\varepsilon\Big(s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(r_\varepsilon,\Theta_\varepsilon)\Big)\mathbf{U}_\varepsilon\cdot\nabla_x T_\varepsilon\Big]\ \mathrm{d}x\ \mathrm{d}t \tag{5.10}$$

$$-\frac{1}{\varepsilon^2}\int_0^\tau\int_\Omega\frac{\varrho_\varepsilon-r_\varepsilon}{r_\varepsilon}\partial_t p(r_\varepsilon,\Theta_\varepsilon)\ \mathrm{d}x\ \mathrm{d}t-\frac{1}{\varepsilon^2}\int_0^\tau\int_\Omega\Big(p(\varrho_\varepsilon,\vartheta_\varepsilon)-p(\overline{\varrho},\overline{\vartheta})\Big)\Delta\Phi_\varepsilon\ \mathrm{d}x\ \mathrm{d}t$$

$$=-\int_0^\tau\int_\Omega\Big(\delta\frac{\vartheta_\varepsilon-\Theta_\varepsilon}{\varepsilon}-\beta\frac{\varrho_\varepsilon-r_\varepsilon}{\varepsilon}\Big)\Big(\partial_t T_\varepsilon+\mathbf{U}_\varepsilon\cdot\nabla_x T_\varepsilon\Big)\ \mathrm{d}x\ \mathrm{d}t$$

$$-\int_0^\tau\int_\Omega\frac{\varrho_\varepsilon-r_\varepsilon}{\varepsilon}\partial_t\Big(\alpha R_\varepsilon+\beta T_\varepsilon\Big)\ \mathrm{d}x\mathrm{d}t+\int_0^\tau\int_\Omega\frac{\delta}{\beta^2+\alpha\delta}\Big(\alpha\frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}+\beta\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}\Big)\partial_t\Big(\alpha R_\varepsilon+\beta T_\varepsilon\Big)\ \mathrm{d}x\ \mathrm{d}t+\chi_\varepsilon(\tau)$$

$$=\int_0^\tau\int_\Omega\Big(\delta T_\varepsilon-\beta R_\varepsilon\Big)\partial_t T_\varepsilon\ \mathrm{d}x\mathrm{d}t+\int_0^\tau\int_\Omega R_\varepsilon\partial_t\Big(\alpha R_\varepsilon+\beta T_\varepsilon\Big)\ \mathrm{d}x\ \mathrm{d}t$$

$$-\Big[\int_0^\tau\int_\Omega\Big(\delta\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}-\beta\frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}\Big)\partial_t T_\varepsilon\ \mathrm{d}x\ \mathrm{d}t$$

$$+\int_0^\tau\int_\Omega\Big(\frac{\beta^2}{\beta^2+\alpha\delta}\frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}-\frac{\beta\delta}{\beta^2+\alpha\delta}\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}\Big)\partial_t\Big(\alpha R_\varepsilon+\beta T_\varepsilon\Big)\ \mathrm{d}x\ \mathrm{d}t\Big]$$

$$-\int_0^\tau\int_\Omega\Big(\delta\frac{\vartheta_\varepsilon-\Theta_\varepsilon}{\varepsilon}-\beta\frac{\varrho_\varepsilon-r_\varepsilon}{\varepsilon}\Big)\mathbf{U}_\varepsilon\cdot\nabla_x T_\varepsilon\ \mathrm{d}x\ \mathrm{d}t+\chi_\varepsilon(\tau),$$

$$\int_0^\tau \int_\Omega \Big(\delta T_\varepsilon - \beta R_\varepsilon\Big)\partial_t T_\varepsilon \ \mathrm{d}x \ \mathrm{d}t + \int_0^\tau \int_\Omega R_\varepsilon \partial_t \Big(\alpha R_\varepsilon + \beta T_\varepsilon\Big) \ \mathrm{d}x \ \mathrm{d}t \tag{5.11}$$

$$= \frac{1}{2}\frac{\delta}{\beta^2+\alpha\delta}\left[\int_\Omega |\alpha R_\varepsilon + \beta T_\varepsilon|^2 \ \mathrm{d}x\right]_0^\tau + \frac{1}{2}\frac{\alpha}{\beta^2+\alpha\delta}\left[\int_\Omega |\delta T_\varepsilon - \beta R_\varepsilon|^2 \ \mathrm{d}x\right]_0^\tau .$$

$$-\int_0^\tau \int_\Omega \Big(\delta \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} - \beta \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\Big)\partial_t T_\varepsilon \ \mathrm{d}x \ \mathrm{d}t \tag{5.12}$$

$$-\int_0^\tau \int_\Omega \Big(\frac{\beta^2}{\beta^2+\alpha\delta}\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} - \frac{\beta\delta}{\beta^2+\alpha\delta}\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\Big)\partial_t\Big(\alpha R_\varepsilon + \beta T_\varepsilon\Big) \ \mathrm{d}x \ \mathrm{d}t$$

$$= -\frac{\alpha}{\beta^2+\alpha\delta}\int_0^\tau \int_\Omega \Big(\delta \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} - \beta \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\Big)\partial_t\Big(\delta T_\varepsilon - \beta R_\varepsilon\Big) \ \mathrm{d}x \ \mathrm{d}t$$

$$-\int_0^\tau \int_\Omega \Big(\delta \frac{\vartheta_\varepsilon - \Theta_\varepsilon}{\varepsilon} - \beta \frac{\varrho_\varepsilon - r_\varepsilon}{\varepsilon}\Big)\mathbf{U}_\varepsilon \cdot \nabla_x T_\varepsilon \ \mathrm{d}x \ \mathrm{d}t \tag{5.13}$$

$$= -\frac{\beta}{\beta^2+\alpha\delta}\int_0^\tau \int_\Omega \Big(\delta \frac{\vartheta_\varepsilon - \Theta_\varepsilon}{\varepsilon} - \beta \frac{\varrho_\varepsilon - r_\varepsilon}{\varepsilon}\Big)\mathbf{U}_\varepsilon \cdot \nabla_x\Big(\alpha R_\varepsilon + \beta T_\varepsilon\Big) \ \mathrm{d}x \ \mathrm{d}t$$

$$-\frac{\alpha}{\beta^2+\alpha\delta}\int_0^\tau \int_\Omega \Big(\delta \frac{\vartheta_\varepsilon - \Theta_\varepsilon}{\varepsilon} - \beta \frac{\varrho_\varepsilon - r_\varepsilon}{\varepsilon}\Big)\mathbf{U}_\varepsilon \cdot \nabla_x\Big(\delta T_\varepsilon - \beta R_\varepsilon\Big) \ \mathrm{d}x \ \mathrm{d}t$$

$$\Big[\text{blue}\Big] + \Big[\text{black}\Big] = \frac{\alpha}{\beta^2+\alpha\delta}\int_0^\tau \int_\Omega \Big(\delta \frac{\vartheta_\varepsilon - \Theta_\varepsilon}{\varepsilon} - \beta \frac{\varrho_\varepsilon - r_\varepsilon}{\varepsilon}\Big)\Delta\Phi_\varepsilon \ \mathrm{d}x \ \mathrm{d}t + \chi^1_\varepsilon(\tau) = \chi_\varepsilon(\tau).$$

6 Navier-Stokes-Fourier system, strong stratification - limit to strongly stratified flows remains an open problem

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{6.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho, \vartheta) = \varepsilon^a \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x F, \tag{6.2}$$

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \varepsilon^b \mathrm{div}_x \left(\frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \right) = \sigma, \tag{6.3}$$

$$\sigma = \varepsilon^{2+a} \frac{\mathbb{S}(\vartheta, \nabla_x \mathbf{u})}{\vartheta} : \nabla_x \mathbf{u} - \varepsilon^b \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta^2} \cdot \nabla_x \vartheta, \tag{6.4}$$

$$\varrho \to \overline{\varrho},\ \vartheta \to \overline{\vartheta},\ \mathbf{u} \to 0 \text{ as } |x| \to \infty. \tag{6.5}$$

$$\varrho(0,\cdot) = \varrho_{0,\varepsilon} = \tilde{\varrho} + \varepsilon \varrho_{0,\varepsilon}^{(1)},\ \vartheta(0,\cdot) = \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon \vartheta_{0,\varepsilon}^{(1)},\ \mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon}. \tag{6.6}$$

where

$$\nabla_x p(\tilde{\varrho}, \overline{\vartheta}) = \tilde{\varrho} \nabla_x F,$$
$$\tilde{\varrho} - \overline{\varrho} \in C_c^1(R^3).$$

6.1 Expected target system as $\varepsilon \to 0$

$$\mathrm{div}_x (\tilde{\varrho} \mathbf{v}) = 0, \tag{6.7}$$

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = 0, \tag{6.8}$$

$$\partial_t T + \mathbf{v} \cdot \nabla_x T = 0. \tag{6.9}$$

6.2 Acoustic equation remains still unresolved in this situation

We set

$$\alpha(\tilde{\varrho}) = \frac{1}{\tilde{\varrho}}\frac{\partial p(\tilde{\varrho},\overline{\vartheta})}{\partial \varrho},\ \beta(\tilde{\varrho}) = \frac{1}{\tilde{\varrho}}\frac{\partial p(\tilde{\varrho},\overline{\vartheta})}{\partial \vartheta}, \delta(\tilde{\varrho}) = \tilde{\varrho}\frac{\partial s(\tilde{\varrho},\overline{\vartheta})}{\partial \vartheta}.$$

6.2.1 "Linearization" of the NSF system

The triplet

$$\Big(\frac{\varrho-\tilde{\varrho}}{\varepsilon}, \frac{\vartheta-\overline{\vartheta}}{\varepsilon}, \nabla_x\Phi\Big), \quad \text{where } \nabla_x\Phi = \mathbf{H}^{\perp}(\mathbf{u})$$

satisfies

$$\varepsilon\partial_t\Big(\frac{\varrho-\tilde{\varrho}}{\varepsilon}\Big) + \mathrm{div}\Big(\tilde{\varrho}\nabla_x\Phi\Big) = \varepsilon O(1),$$

$$\varepsilon\partial_t\Big(\tilde{\varrho}\nabla_x\Phi\Big) + \tilde{\varrho}\nabla_x\Big(\alpha(\tilde{\varrho})\frac{\varrho-\tilde{\varrho}}{\varepsilon}\Big) + \nabla_x\Big(\tilde{\varrho}\beta(\tilde{\varrho})\frac{\vartheta-\overline{\vartheta}}{\varepsilon}\Big) = \varepsilon O(1)$$

$$\partial_t\Big(\delta\frac{\vartheta-\overline{\vartheta}}{\varepsilon} - \beta\frac{\varrho-\overline{\varrho}}{\varepsilon}\Big) + \mathbf{u}\cdot\nabla_x\Big(\delta\frac{\vartheta-\overline{\vartheta}}{\varepsilon} - \beta\frac{\varrho-\overline{\varrho}}{\varepsilon}\Big) + \Big(\delta\frac{\vartheta-\overline{\vartheta}}{\varepsilon} - \beta\frac{\varrho-\overline{\varrho}}{\varepsilon}\Big)\Delta\Phi = \varepsilon O(1)$$

with initial data

$$\Big(\frac{\varrho-\tilde{\varrho}}{\varepsilon}, \frac{\vartheta-\overline{\vartheta}}{\varepsilon}, \nabla_x\Phi\Big)(0,\cdot) = \Big(\varrho_0^1, \vartheta_0^1, \mathbf{H}^{\perp}(\mathbf{u}_0)\Big)$$

7 Navier-Stokes-Poisson system and strongly stratified flows

7.1 Navier-Stokes-Poisson system

7.1.1 Equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{7.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2}\nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}_\varepsilon(\nabla_x \mathbf{u}) + \frac{1}{\varepsilon^2}\varrho \nabla_x F, \tag{7.2}$$

$$\mathbb{S}_\varepsilon(\nabla_x \mathbf{u}) = \mu_\varepsilon \left(\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} - \frac{2}{3}\mathrm{div}\mathbf{u}\mathbb{I} \right), \ \mu_\varepsilon > 0, \, \mu_\varepsilon \to 0, \tag{7.3}$$

$$\mathbf{u} \to 0, \ \varrho \to \overline{\varrho} \text{ as } |x| \to \infty, \ \overline{\varrho} > 0. \tag{7.4}$$

7.1.2 Equilibrium state

$$\nabla_x p(\tilde{\varrho}) = \tilde{\varrho} \nabla_x F, \ \tilde{\varrho} \to \overline{\varrho} \text{ as } |x| \to \infty, \tag{7.5}$$

7.1.3 Initial data

$$\varrho_0 = \tilde{\varrho} + \varepsilon \varrho^1_{0,\varepsilon}, \ \mathbf{u}_0 = \mathbf{u}_{0,\varepsilon}$$

7.1.4 Expected target system as $\varepsilon \to 0$ - Lake equations

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = 0, \tag{7.6}$$

$$\mathrm{div}_x(\tilde{\varrho} \mathbf{v}) = 0. \tag{7.7}$$

7.2 Acoustic waves

$$\varepsilon \partial_t s_\varepsilon + \mathrm{div}_x \left[\tilde{\varrho} \nabla_x \Phi_\varepsilon \right] = 0, \tag{7.8}$$

$$\varepsilon \tilde{\varrho} \partial_t \nabla_x \Phi_\varepsilon + \tilde{\varrho} \nabla_x \left[\frac{p'(\tilde{\varrho})}{\tilde{\varrho}} s_\varepsilon \right] = 0, \tag{7.9}$$

7.3 Relative entropy inequality

We set

$$\mathcal{E}\left(\varrho, \mathbf{u} \Big| r, \mathbf{U} \right) = \int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + \frac{1}{\varepsilon^2} \Big(P(\varrho) - P'(r)(\varrho - r) - P(r) \Big) \right] \, \mathrm{d}x. \tag{7.10}$$

We know that any finite energy weak solution to the NSP system satisfies the *relative entropy inequality* in the form:

$$\mathcal{E}\left(\varrho, \mathbf{u} \Big| r, \mathbf{U} \right)(\tau) + \int_0^\tau \int_\Omega \Big(\mathbb{S}_\varepsilon(\nabla_x \mathbf{u}) - \mathbb{S}_\varepsilon(\nabla_x \mathbf{U}) \Big) : \Big(\nabla_x \mathbf{u} - \nabla_x \mathbf{U} \Big) \, \mathrm{d}x \, \mathrm{d}t \tag{7.11}$$
$$\leq \mathcal{E}\left(\varrho, \mathbf{u} \Big| r, \mathbf{U} \right)(0) + \int_0^\tau \mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U}) \, \mathrm{d}t$$

for a.a. $\tau \in [0, T]$, where

$$\mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U}) = \int_\Omega \varrho \Big(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) \, \mathrm{d}x \tag{7.12}$$

$$+ \int_\Omega \mathbb{S}_\varepsilon(\nabla_x \mathbf{U}) : (\nabla_x \mathbf{U} - \nabla_x \mathbf{u}) \, \mathrm{d}x + \frac{1}{\varepsilon^2} \int_\Omega \Big((r - \varrho) \partial_t P'(r) + \nabla_x P'(r) \cdot (r \mathbf{U} - \varrho \mathbf{u}) \Big) \, \mathrm{d}x$$
$$- \frac{1}{\varepsilon^2} \int_\Omega \Big(p(\varrho) - p(r) \Big) \mathrm{div}_x \mathbf{U} \, \mathrm{d}x + \frac{1}{\varepsilon^2} \int_\Omega \varrho \nabla_x F \cdot \Big(\mathbf{u} - \mathbf{U} \Big) \, \mathrm{d}x,$$

where $r > 0$, $\mathbf{U}$ are smooth functions, $r - \overline{\varrho}$, $\mathbf{U}$ compactly supported in R^2.

7.4 Choice of test functions in the relative entropy inequality

$$r = \tilde{\varrho} + \varepsilon s_\varepsilon, \quad \mathbf{U} = \mathbf{v} + \nabla_x \Phi_\varepsilon.$$

7.5 Abstract formulation of the acoustic equation

Consider an abstract linear operator

$$\mathcal{A}_{\tilde{\varrho}}[v] = v \mapsto -\frac{p'(\tilde{\varrho})}{\tilde{\varrho}} \mathrm{div}_x(\tilde{\varrho} \nabla_x v),$$

with the domain of definition

$$\mathcal{D}(\mathcal{A}_{\tilde{\varrho}}) = \Big\{ v \in L^2(R^2) \ \Big| \ \nabla_x v \in L^2(R^2; R^2), \ \int_{R^2} \tilde{\varrho} \nabla_x v \cdot \nabla_x \left(\frac{p'(\tilde{\varrho})}{\tilde{\varrho}} \varphi \right) \ \mathrm{d}x = \int_\Omega g \varphi \ \mathrm{d}x$$

$$\text{for all } \varphi \in C_c^\infty(R^2) \text{ and some } g \in L^2(R^2) \Big\}.$$

Duhamel's formula:

$$\Phi_\varepsilon(t) = \frac{1}{2} \exp\left(\mathrm{i} \sqrt{\mathcal{A}_{\tilde{\varrho}}} \frac{t}{\varepsilon} \right) \left[\Phi_{0,\varepsilon} - \frac{\mathrm{i}}{\sqrt{\mathcal{A}_{\tilde{\varrho}}}} \left[\frac{p'(\tilde{\varrho})}{\tilde{\varrho}} s_{0,\varepsilon} \right] \right] \tag{7.13}$$

$$+ \frac{1}{2} \exp\left(-\mathrm{i} \sqrt{\mathcal{A}_{\tilde{\varrho}}} \frac{t}{\varepsilon} \right) \left[\Phi_{0,\varepsilon} + \frac{\mathrm{i}}{\sqrt{\mathcal{A}_{\tilde{\varrho}}}} \left[\frac{p'(\tilde{\varrho})}{\tilde{\varrho}} s_{0,\varepsilon} \right] \right],$$

$$s_\varepsilon(t) = \frac{\tilde{\varrho}}{p'(\tilde{\varrho})} \left\{ \frac{1}{2} \exp\left(\mathrm{i} \sqrt{\mathcal{A}_{\tilde{\varrho}}} \frac{t}{\varepsilon} \right) \left[\mathrm{i} \sqrt{\mathcal{A}_{\tilde{\varrho}}} [\Phi_{0,\varepsilon}] + \frac{p'(\tilde{\varrho})}{\tilde{\varrho}} s_{0,\varepsilon} \right] \right. \tag{7.14}$$

$$\left. + \frac{1}{2} \exp\left(-\mathrm{i} \sqrt{\mathcal{A}_{\tilde{\varrho}}} \frac{t}{\varepsilon} \right) \left[-\mathrm{i} \sqrt{\mathcal{A}_{\tilde{\varrho}}} [\Phi_{0,\varepsilon}] + \frac{p'(\tilde{\varrho})}{\tilde{\varrho}} s_{0,\varepsilon} \right] \right\}.$$

8 Dispersive estimates (case R^2)

8.1 Global Strichartz estimates in the integral form for $-\Delta$

$$\int_{-\infty}^{\infty} \left\| \exp\left(\mathrm{i}\sqrt{-\Delta} t \right)[h] \right\|_{L^6(R^2)}^6 \ \mathrm{d}t \le c \|h\|_{H^{1/2,2}(R^2)}^6. \tag{8.1}$$

8.2 Local energy decay for $-\Delta$

$$\int_{-\infty}^{\infty} \left\| \varphi \exp\left(\mathrm{i}\sqrt{\mathcal{A}} t \right)[h] \right\|_{H^{\alpha,2}(R^2)}^2 \le c(\varphi) \|h\|_{H^{\alpha,2}(R^2)}^2, \quad \alpha \le 1/2, \quad \varphi \in C_c^1(R^2) \tag{8.2}$$

8.3 Spectrally localized local energy decay for $\mathcal{A}_{\tilde{\varrho}}$

$$\int_{-\infty}^{\infty} \left\| \varphi G(\mathcal{A}_{\tilde{\varrho}}) \exp\left(\mathrm{i}\sqrt{\mathcal{A}_{\tilde{\varrho}}} t \right)[h] \right\|_{H^{\alpha}(R^2)}^2 \le c(\varphi, G) \|h\|_{L^2(R^2)}^2, \tag{8.3}$$

for any $\varphi \in C_c^\infty(R^2)$, $G \in C_c^\infty(0,\infty)$, $\alpha > -1$.

8.4 Spectrally localized energy estimates

$$\int_{-\infty}^{\infty} \left\| H(\mathcal{A}_{\tilde{\varrho}}) \exp\left(\mathrm{i}\sqrt{\mathcal{A}_{\tilde{\varrho}}} t \right)[h] \right\|_{L^6(R^2)}^6 \ \mathrm{d}t \le c(H) \|h\|_{L^2(R^2)}^6 \tag{8.4}$$

for any $H \in C_c^\infty(0,\infty)$.

8.5 Dispersive estimates

$$\int_0^T \Big[\|\Phi_\varepsilon(t,\cdot)\|_{W^{m,\infty}(R^2)} \tag{8.5}$$

$$+\|s_\varepsilon(t,\cdot)\|_{W^{m,\infty}(R^2)}\Big] \,\mathrm{d}t \le \omega(\varepsilon,m,\delta)\Big[\|\nabla_x \Phi_{0,\varepsilon,\delta}\|_{L^2(R^2;R^2)} + \|r_{0,\varepsilon,\delta}\|_{L^2(R^2)}\Big],$$

where

$$\omega(\varepsilon,m,\delta) \to 0 \text{ as } \varepsilon \to 0 \text{ for all fixed values } m \ge 0,\ \delta > 0.$$

8.6 Main result

8.6.1 Hypotheses on pressure and potential force

We suppose the pressure p is a continuously differentiable function of the density such that

$$\left\{ \begin{array}{c} p \in C^1[0,\infty) \cap C^\infty(0,\infty),\ p(0)=0,\ p'(\varrho) > 0 \text{ for all } \varrho > 0, \\ \\ \lim_{\varrho\to\infty} \frac{p'(\varrho)}{\varrho^{\gamma-1}} = p_\infty > 0 \text{ for some } \gamma > 1 \end{array} \right\} \tag{8.6}$$

and

$$F \in C_c^\infty(R^2),\ F \ge 0. \tag{8.7}$$

8.6.2 Main result

Let the pressure p and the potential F comply with the hypotheses (8.6), (8.7) for some $\gamma > 1$.

Let, moreover, the viscosity coefficients satisfy

$$\mu_\varepsilon = \varepsilon^\nu \mu_0,\ \mu_0 > 0,\ 0 < \nu \leq 1,\ \mu_\varepsilon + \lambda_\varepsilon \geq 0,\ \lambda_\varepsilon \to 0 \text{ as } \varepsilon \to 0. \tag{8.8}$$

Suppose that the initial data $\varrho_0 = \varrho_{0,\varepsilon}$, $(\varrho \mathbf{u})_0 = (\varrho \mathbf{u})_{0,\varepsilon}$ take the form

$$\begin{gathered} \varrho_{0,\varepsilon} = \tilde{\varrho} + \varepsilon r_{0,\varepsilon}, \\ \{r_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^2 \cap L^\infty(R^2),\ r_{0,\varepsilon} \to r_0 \text{ in } L^2(R^2), \end{gathered} \tag{8.9}$$

$$\begin{gathered} (\varrho \mathbf{u})_{0,\varepsilon} = \varrho_{0,\varepsilon} \mathbf{u}_{0,\varepsilon},\ \text{with } \mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \text{ in } L^2(R^2), \\ \mathbf{H}_{\tilde{\varrho}}[\mathbf{u}_0] \equiv \mathbf{v}_0 \in W^{m,2}(R^2; R^2) \text{ for some } m \geq 3. \end{gathered} \tag{8.10}$$

Let ϱ_ε, $\mathbf{u}_\varepsilon$ be a family of finite energy weak solutions to the Navier-Stokes system (7.1 - 7.4) in $(0,T) \times R^2$ emanating from the initial data $\varrho_{0,\varepsilon}$, $\mathbf{u}_{0,\varepsilon}$.

Then

$$\operatorname*{ess\,sup}_{t \in (0,T)} \|\varrho_\varepsilon(t,\cdot) - \tilde{\varrho}\|_{L^2 + L^\gamma(R^2)} \leq c\varepsilon^q,\ q = \min\{1, \frac{2}{\gamma}\}, \tag{8.11}$$

$$\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon \to \sqrt{\tilde{\varrho}}\mathbf{v} \left\{ \begin{array}{c} \text{weakly-(*) in } L^\infty(0,T; L^2(R^2;R^2)), \\ \text{and (strongly) in } L^2(0,T; L^2_{\rm loc}(R^2;R^2)), \end{array} \right\} \tag{8.12}$$

where $\mathbf{v}$ is the unique solution of the lake system (3.6) satisfying the initial condition $\mathbf{v}(0,\cdot) = \mathbf{v}_0$.