The additive structure of elliptic homogenization[1]

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Joint work with S. Armstrong & J.-C. Mourrat

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Telč - Czech Republic

Regularity theory for elliptic and parabolic systems and problems in continuum mechanics

[1] arxiv.org/abs/1602.00512
[2] sites.google.com/site/tuomokuusimath/

Scope

The goal is to find optimal quantitative estimates in stochastic homogenization for linear, uniformly elliptic equations of the form

$$-\nabla \cdot \left(\mathbf{a}\left(\tfrac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) = 0 \quad \text{in } U \subseteq \mathbb{R}^d\,,$$

where $\varepsilon > 0$ and coefficients $\mathbf{a}(\cdot)$ are assumed to be $\mathbb{Z}^d$-stationary random field[3] of measurable and elliptic[4] symmetric matrices.

[3] meaning that $\mathbf{a}(x)$ and $\mathbf{a}(x+z)$, $z \in \mathbb{Z}^d$, have the same law; more about this soon

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A convention:

- Here $\mathbf{a}(\cdot)$ denotes in short $\mathbf{a}(\cdot, \omega)$, where $\omega \in \Omega$ denotes a particular random realization sampled from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The variable ω will be suppressed from the notation.

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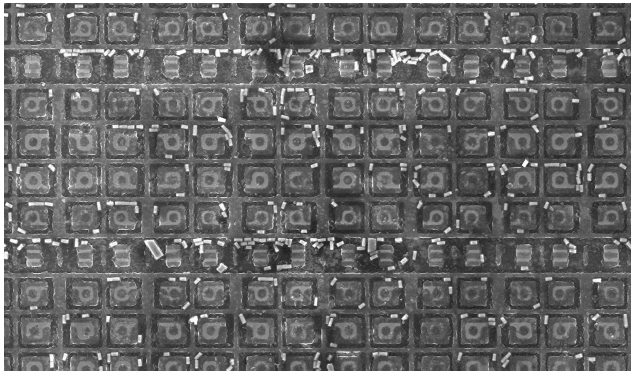
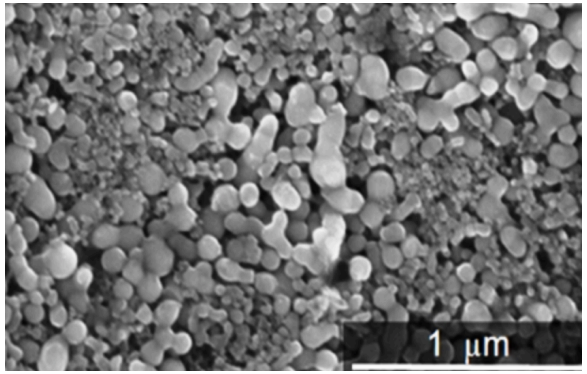
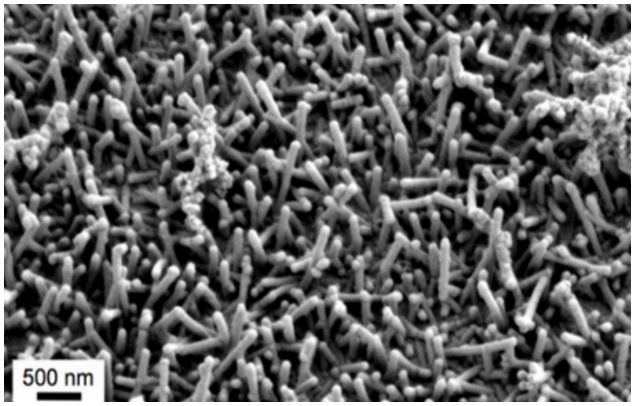


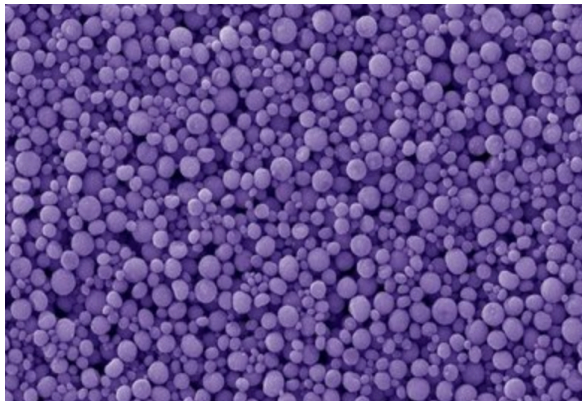
Microprocessor (scale 180nm)



Battery structure



Virus surface



Solar cell structure

Scope

It turns out that so-called correctors play a central role in the theory. First-order corrector $\phi_e \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$, $e \in \mathbb{R}^d$, is the unique weak solution, modulo an additive constant, to the equation

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Correctors yield important properties:

- The effective (homogenized) elliptic, symmetric and deterministic matrix $\bar{\mathbf{a}}$:

$$\bar{\mathbf{a}}e = \mathbb{E}\left[\int_{[0,1]^d} \mathbf{a}(x)\,(e + \phi_e(x))\;dx\right] \qquad \forall\, e \in \mathbb{R}^d\,;$$

  Solutions homogenize $u_\varepsilon \to \overline{u}$ as $\varepsilon \to 0$ with $\nabla \cdot (\bar{\mathbf{a}}\nabla\overline{u}) = 0$.

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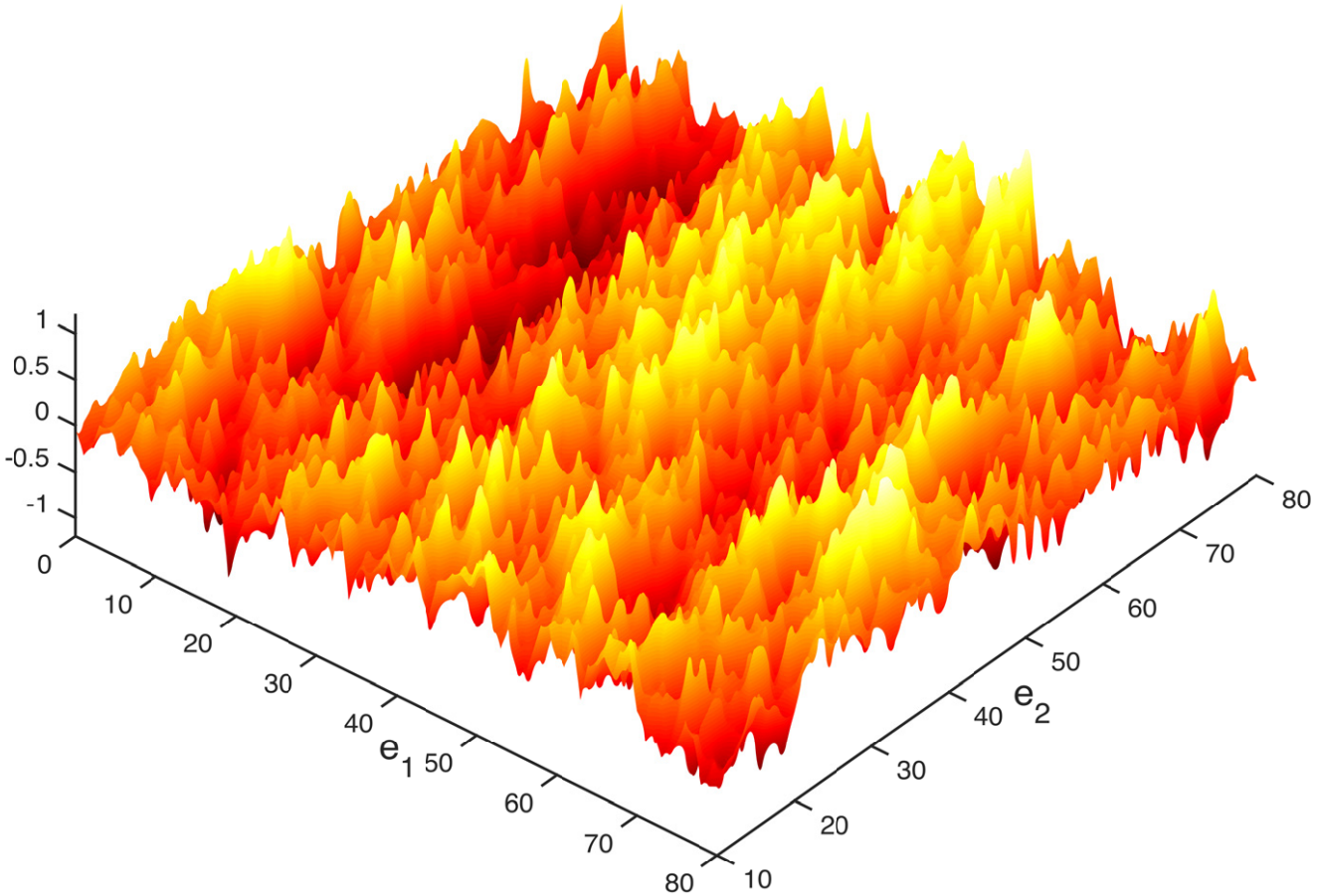
- Correctors replace the planes. Let u solve $\mathrm{div}(\mathbf{a}(x)\nabla u) = 0$ in B_R, $R \gg 1$. Then

$$\inf_{a\in\mathbb{R},\, e\in\mathbb{R}^d} \left(\fint_{B_r} |u - a - e\cdot x - \phi_e|^2\right)^{\frac{1}{2}} \le C\left(\frac{\mathcal{X}_s \wedge r}{r}\right)^{\frac{d}{2}} \left(\frac{r}{R}\right)^2 \left(\fint_{B_R} |u|^2\right)^{\frac{1}{2}}$$

  with $\mathcal{X}_s^s$ being exponentially integrable for all $s \in (0, d)$.

How do the correctors look like?

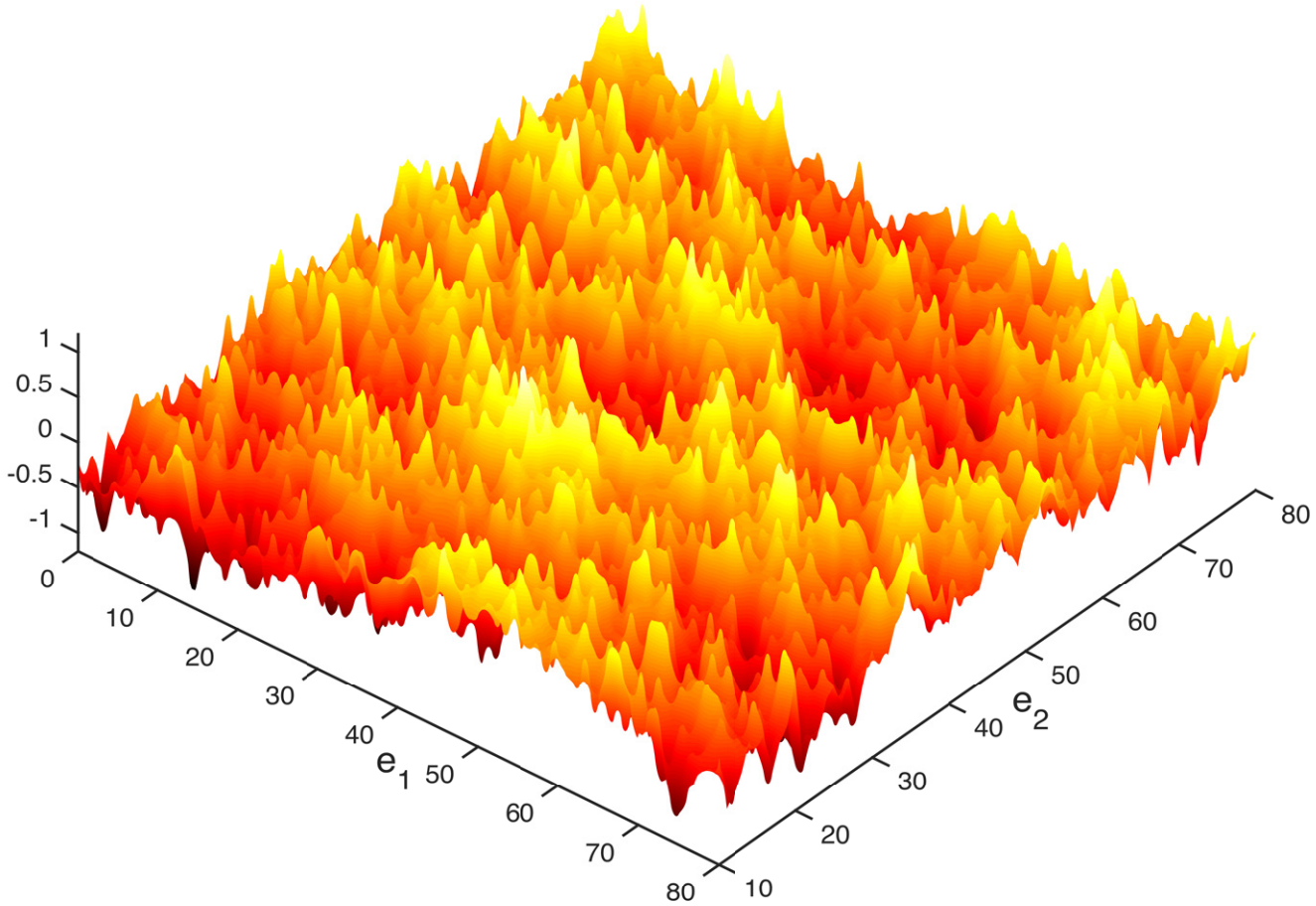
Corrector



Approximation of the graph[5] of ϕ_{e_1} solving $-\mathrm{div}(\mathbf{a}(x)(e_1 + \nabla\phi_{e_1}(x))) = 0$ in $\mathbb{R}^2$

[5] The image is courtesy of Antti Hannukainen (Aalto University)

Corrector



Approximation of the graph[6] of ϕ_{e_2} solving $-\mathrm{div}(\mathbf{a}(x)(e_2 + \nabla\phi_{e_2}(x))) = 0$ in $\mathbb{R}^2$

[6] The image is courtesy of Antti Hannukainen (Aalto University)

More precise description of random coefficients

Denote by Ω the collection of coefficient fields:

$$\Omega := \left\{ \mathbf{a}(\cdot) \,:\, \mathbf{a} : \mathbb{R}^d \to \mathbb{R}^{d\times d} \text{ is symmetric, measurable and elliptic} \right\}.$$

For $U \subseteq \mathbb{R}^d$ Borel, let $\mathcal{F}(U)$ be the σ-algebra on Ω generated by maps

$$\mathbf{a} \longmapsto \int_U p \cdot \mathbf{a}(x) q \, \varphi(x) \, dx, \quad p, q \in \mathbb{R}^d, \; \varphi \in C_c^\infty(\mathbb{R}^d),$$

and set $\mathcal{F} := \mathcal{F}(\mathbb{R}^d)$.

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and set $\mathcal{F} := \mathcal{F}(\mathbb{R}^d)$. The probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is assumed to satisfy the following two conditions:

(P1) $\mathbb{P}$ is invariant under $\mathbb{Z}^d$-translations: for every $z \in \mathbb{Z}^d$ and $A \in \mathcal{F}$,

$$\mathbb{P}[A] = \mathbb{P}[\tau_z A],$$

where τ_z acts on Ω via $(\tau_z \mathbf{a})(x) := \mathbf{a}(x+z)$.

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(P2) $\mathbb{P}$ has a unit range of dependence: for every pair of Borel subsets $U, V \subseteq \mathbb{R}^d$ with $\operatorname{dist}(U, V) \geq 1$,

$$\mathcal{F}(U) \ \text{ and } \ \mathcal{F}(V) \ \text{ are } \ \mathbb{P}\text{-independent.}$$

Some notation

Before the statement of our main theorems, let us introduce some notation.

- We use the heat kernel for the operator $\partial_t - \nabla \cdot \bar{\mathbf{a}} \nabla$ as a smooth mask:

$$\Phi(x,t) := (4\pi t)^{-\frac{d}{2}} (\det \bar{\mathbf{a}})^{-\frac{1}{2}} \exp\left(- \frac{x \cdot \bar{\mathbf{a}}^{-1} x}{4t} \right).$$

  Denote, in short,

$$\Phi_{z,r}(x) := \Phi(x - z, r^2), \qquad \int_{\Phi_{z,r}} f := \int_{\mathbb{R}^d} f(x) \Phi(x - z, r^2)\, dx.$$

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- If X is a random variable and $s, \theta \in (0, \infty)$, then we define the statement

$$X \leq \mathcal{O}_s(\theta)$$

  to mean that

$$\mathbb{E}\left[\exp\left(\left(\theta^{-1} X_+ \right)^s \right) \right] \leq 2,$$

  where $X_+ := X \vee 0 = \max\{X, 0\}$.

Theorem 1 (Armstrong-K-Mourrat): Optimal quantitative estimates for the first-order correctors

Let $s < 2$, $r \geq 1$ and $e \in \partial B_1$. There is $C(s, d, \Lambda) < \infty$ such that

$$\left| \int_{\Phi_r} \nabla \phi_e \right| \leq \mathcal{O}_s \left(C r^{-\frac{d}{2}} \right), \qquad \left| \int_{\Phi_r} \mathbf{a}(e + \nabla \phi_e) - \bar{\mathbf{a}} e \right| \leq \mathcal{O}_s \left(C r^{-\frac{d}{2}} \right),$$

and

$$\left| \int_{\Phi_r} \frac{1}{2} (e + \nabla \phi_e) \cdot \mathbf{a} (e + \nabla \phi_e) - \frac{1}{2} e \cdot \bar{\mathbf{a}} e \right| \leq \mathcal{O}_s \left(C r^{-\frac{d}{2}} \right).$$

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Moreover, there is $\varepsilon(d, \Lambda) > 0$ such that

$$\left(\fint_{B_r} \left| \phi_e - (\phi_e)_{B_r} \right|^2 \right)^{\frac{1}{2}} \leq \begin{cases} \mathcal{O}_s\left(C \log^{\frac{1}{2}} r\right) & \text{if } d = 2, \\ \mathcal{O}_{2+\varepsilon}(C) & \text{if } d > 2. \end{cases}$$

Compare to Central Limit Theorem

Suppose $\{X_1, X_2, \ldots\}$ is a sequence of real-valued i.i.d. (**i**ndependent and **i**dentically **d**istributed) random variables with $\mathbb{E}[X_i] = 0$ and $\mathrm{var}[X_i] = \sigma^2 < \infty$. Then

$$S_n := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \overset{distribution}{\to} N(0, \sigma^2) \qquad \text{as } n \to \infty\,.$$

The above convergence is given by

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \left|\mathbb{P}\left[S_n > t\right] - \mathcal{C}(t/\sigma)\right| = 0\,,$$

where $\mathcal{C}$ is the cumulative distribution function of the normal distribution.

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- Breakthroughs by Gloria and Otto (et. al.) in last 5-10 years
  1. Optimal quantitative bounds on the scaling of the first-order correctors (including their sublinear growth and spatial averages of their energy density).
  2. First to obtain estimates for the correctors at the critical scalings, albeit with suboptimal stochastic integrability (typically finite moment bounds) and with somewhat restrictive ergodic assumptions.
  3. Their central tools are concentration inequalities.

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- The finite range of dependence condition enters in the simplest possible way: the energy quantities are essentially sums of i.i.d. random variables; CLT-type arguments apply!
- The bootstrap argument allows optimal scaling after a finite number of iterations.

Master quantity J

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The energy quantity central to our study is defined, in the simplest case, for each $z \in \mathbb{R}^d$, $r \geq 1$ and $p, q \in \mathbb{R}^d$, by

$$J(z, r, p, q) := \sup_{u \in \mathcal{A}_1} \int_{\Phi_{z,r}} \left(-\frac{1}{2} \nabla u \cdot \mathbf{a} \nabla u - p \cdot \mathbf{a} \nabla u + \nabla u \cdot \bar{\mathbf{a}} q \right).$$

Here $\mathcal{A}_1$ denotes the set of solutions of the equation which have sub-quadratic growth at infinity:

$$\mathcal{A}_1 := \left\{ u \in H^1_{\mathrm{loc}}(\mathbb{R}^d) \, : \, -\operatorname{div}(\mathbf{a} \nabla u) = 0 \text{ in } \mathbb{R}^d, \ \lim_{r \to \infty} \frac{1}{r^4} \fint_{B_r} |u(x)|^2 \, dx = 0 \right\}.$$

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In fact, it turns out that $\mathcal{A}_1$ is a $(d+1)$-dimensional (random) vector space spanned by the constant functions and those of the form $x \mapsto x \cdot e + \phi_e(x)$

Some magical properties of J

Write

$$\mathcal{J}(u,z,r,p,q) := \int_{\Phi_{z,r}} \left(-\frac{1}{2}\nabla u \cdot \mathbf{a}\nabla u - \nabla p \cdot \mathbf{a}\nabla u + \nabla u \cdot \bar{\mathbf{a}}\nabla q \right).$$

so that

$$J(z,r,p,q) := \max_{u\in\mathcal{A}_1} \mathcal{J}(u,z,r,p,q)\,.$$

The function achieving the maximum in the definition of J is denoted by

$$v(\cdot,z,r,p,q) := \text{element of } \mathcal{A}_1 \text{ achieving the maximum for } J(z,r,p,q).$$

Note that $v(\cdot,z,r,p,q)$ is specified uniquely up to an additive constant by the uniform concavity of the maximization problem in the definition of J.

Some magical properties of J

The first variation of the maximization problem is, for all $w \in \mathcal{A}_1$,

$$\int_{\Phi_{z,r}} \nabla w \cdot \mathbf{a} \nabla v(\cdot, z, r, p, q) = \int_{\Phi_{z,r}} (-p \cdot \mathbf{a} \nabla w + q \cdot \bar{\mathbf{a}} \nabla w)$$

and second variation gives the quadratic response to perturbations:

$$J(z, r, p, q) - \mathcal{J}(v(\cdot, z, r, p, q) + w, z, r, p, q) = \frac{1}{2} \int_{\Phi_{z,r}} \nabla w \cdot \mathbf{a} \nabla w.$$

These give us lots of information. For instance,

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- The gradient of J w.r.t (p, q):

$$\nabla_{p,q} J(z, r, p, q)(p', q') = \int_{\Phi_{z,r}} (-\mathbf{a} p' + \bar{\mathbf{a}} q') \cdot \nabla v(\cdot, z, r, p, q)$$

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In fact, almost all of our paper is devoted to the analysis of the quantity J. It's properties are summarized in the following Theorem.

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For every $s < 1$, there exists a constant $C(s, d, \Lambda) < \infty$ such that the following statements hold, for every $R > r \geq 1$ and $p, q \in B_1$, $z \in \mathbb{R}^d$,

(i) Additivity:

$$\left| J(z, R, p, q) - \int_{\Phi_{z, \sqrt{R^2 - r^2}}} J(\cdot, r, p, q) \right| \leq \mathcal{O}_s\left(Cr^{-d}\right).$$

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(ii) Control of the expectation:

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(iii) CLT scaling of the fluctuations:

$$\left| J(z, r, p, q) - \mathbb{E}\left[J(z, r, p, q) \right] \right| \leq \mathcal{O}_{2s} \left(C r^{-\frac{d}{2}} \right).$$

Theorem 2 (Armstrong-K-Mourrat): Additive structure

For every $s < 1$, there exists a constant $C(s, d, \Lambda) < \infty$ such that the following statements hold, for every $R > r \geq 1$ and $p, q \in B_1$, $z \in \mathbb{R}^d$,

(i) *Additivity:*

$$\left| J(z, R, p, q) - \int_{\Phi_{z, \sqrt{R^2 - r^2}}} J(\cdot, r, p, q) \right| \leq \mathcal{O}_s \left(C r^{-d} \right).$$

(ii) *Control of the expectation:*

$$\left| \mathbb{E}\left[J(z, r, p, q) \right] - \frac{1}{2}(q - p) \cdot \bar{\mathbf{a}}(q - p) \right| \leq C r^{-d}.$$

(iii) *CLT scaling of the fluctuations:*

$$\left| J(z, r, p, q) - \mathbb{E}\left[J(z, r, p, q) \right] \right| \leq \mathcal{O}_{2s} \left(C r^{-\frac{d}{2}} \right).$$

(iv) *Localization:* For every $\delta, \varepsilon > 0$, there exist $C(\delta, \varepsilon, s, d, \Lambda) < \infty$ and an $\mathcal{F}(B_{r^{1+\delta}}(z))$-measurable random variable $J^{(\delta)}(z, r, p, q)$ such that, for every $\gamma \in \left(0, \frac{d}{2s} \wedge \left(\frac{d}{2}(1 + \delta) + \delta \right) - \varepsilon \right]$,

$$\left| J(z, r, p, q) - J^{(\delta)}(z, r, p, q) \right| \leq \mathcal{O}_{2s} \left(C r^{-\gamma} \right).$$

The proof of Theorem 1 on first-order correctors is a rather straightforward consequence of Theorem 2 (involving some tools developed for the proof of the latter)

Some ideas behind the proof of Theorem 2

Let $s, \alpha, \delta \in (0, \infty)$, $z \in \mathbb{R}^d$, $r_0(d, \Lambda) \leq r < R$, and $p, q \in B_1$.

$\mathrm{Add}(s, \alpha)$: There exists a constant $C(s, \alpha, d, \Lambda) < \infty$ such that

$$J(z, R, p, q) = \int_{\Phi_{z, \sqrt{R^2 - r^2}}} J(\cdot, r, p, q) + \mathcal{O}_s\left(Cr^{-\alpha}\right).$$

$\mathrm{Fluc}(s, \alpha)$: There is a constant $C(s, \alpha, d, \Lambda) < \infty$ such that

$$J(z, r, p, q) = \mathbb{E}\left[J(z, r, p, q)\right] + \mathcal{O}_s\left(Cr^{-\alpha}\right).$$

$\mathrm{Dual}(\alpha)$: There exists $C(\alpha, d, \Lambda) < \infty$ such that

$$\mathbb{E}\left[\sup_{w \in \mathcal{A}_1(\mathbb{R}^d), \|\nabla w\|_{L^2(\Phi_{z,r})} \leq 1} \left| \int_{\Phi_{z,r}} (\mathbf{a}(x) - \bar{\mathbf{a}}) \nabla w(x)\, dx \right| \right] \leq Cr^{-\alpha}.$$

$\mathrm{Loc}(s, \delta, \alpha)$: There exists $C(s, \delta, \alpha, d, \Lambda) < \infty$ and an $\mathcal{F}(B_{r^{1+\delta}}(z))$-measurable random variable $J^{(\delta)}(z, r, p, q)$ such that

$$J(z, r, p, q) = J^{(\delta)}(z, r, p, q) + \mathcal{O}_s(Cr^{-\alpha}).$$

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- Localization: $\mathrm{Fluc}(s, \alpha)$ and $\mathrm{Dual}(\alpha)$ imply, for all $\delta > 0$, $\mathrm{Loc}\left(s, \delta, (\alpha(1+\delta)+\delta) \wedge \frac{d}{s} -\right)$.

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- Control on fluctuations:
  - *(i)* For every $\beta \in \left(0, \alpha \wedge \frac{d}{2}\right)$ and $\delta \in \left(0, \frac{(\alpha-\beta)(d-2\beta)}{\beta d}\right)$,

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  - *(ii)* For every $\delta > 0$ and $\alpha > \frac{d}{2}(1+\delta)$,

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- $\mathrm{Fluc}(s, \alpha)$ and $\mathrm{Dual}(\beta)$ imply $\mathrm{Dual}\left(2\beta \wedge \alpha\right)$ for $\beta \in \left(0, \frac{d}{2}\right]$.
- Additivity improvement: For every $s \in (0, 2]$ and $\alpha \in \left(0, \frac{d}{s}\right) \cap \left(0, \frac{d}{2}\right]$,
$$\mathrm{Fluc}(s, \alpha) \quad \text{and} \quad \mathrm{Dual}(\alpha) \implies \mathrm{Add}\left(\frac{s}{2}, 2\alpha\right).$$
  Moreover, for each $\varepsilon > 0$, there exists an exponent $\eta(\varepsilon, s, d, \Lambda) > 0$ such that, if $\alpha \in \left(0, \left(\frac{d}{s} - \varepsilon\right) \wedge \frac{d}{2}\right]$, then
$$\mathrm{Fluc}(s, \alpha) \text{ and } \mathrm{Dual}(\alpha) \implies \mathrm{Add}\left(s, \alpha + \eta\right).$$

Thank you for your attention!