Global weak solutions to Cauchy problem for Navier-Stokes equations with non-energy initial data

T. Barker
G. Seregin
V. Sverak

Initial boundary value problem

$$\left.\begin{aligned} \partial_t v + v\cdot\nabla v - \nu\Delta v &= -\nabla q \\ \operatorname{div} v &= 0 \end{aligned}\right\} \text{ in } Q_\infty = \Omega \times]0,\infty[$$

(BC) $\Omega = \mathbb{R}^3$ or $\mathbb{R}^3_+ = \{x = (x_i),\ x_3 > 0\}$

$$v(x,t) \xrightarrow[|x|\to 0]{} 0 \qquad \Longleftrightarrow \qquad v\big|_{x_3=0} = 0$$

(IC) $v\big|_{t=0} = u_0$ in Ω, $\boxed{\operatorname{div} u_0 = 0}$

$\boxed{\nu = 1}$

$$\Omega = \mathbb{R}^3$$

<u>Energy initial data:</u> $u_0 \in L_2(\mathbb{R}^3) = L_2$

<u>weak Leray-Hopf</u> (Leray) solution

- $v \in L_\infty(0,\infty; \mathring{J}) \cap L_2(0,\infty; \mathring{J}^1_2)$

$$\mathring{J} := \left[C^\infty_{0,0}\right]^{L_2}, \quad \mathring{J}^1_2 := \left[C^\infty_{0,0}\right]^{L^1_2}$$

$$C^\infty_{0,0}(\mathbb{R}^3) := \left\{ v \in C^\infty_0(\mathbb{R}^3) : \operatorname{div} v = 0 \right\}$$

- $$\int_0^\infty \int_{\mathbb{R}^3} \left(v \cdot \partial_t w + v \otimes v : \nabla w - \nabla v : \nabla w \right) dx\, dt = 0$$

$$\forall\, w \in C^\infty_{0,0}(\mathbb{R}^3 \times \mathbb{R})$$

- $$t \longmapsto \int_{\mathbb{R}^3} v(x,t) \cdot w(x)\, dx \in C([0,T]), \quad \forall\, w \in L_2, \ \forall\, T > 0$$

- $$\| v(\cdot, t) - u_0(\cdot) \|_{L_2} \longrightarrow 0 \quad \text{as} \quad t \to 0$$

- $$\frac{1}{2} \int_{\mathbb{R}^3} |v(x,t)|^2 dx + \int_0^t \int_{\mathbb{R}^3} |\nabla v|^2 dx\, dt' \leq \frac{1}{2} \int_{\mathbb{R}^3} |u_0(x)|^2 dx$$

$$\forall\, t \geq 0$$

Non-energy initial data

$u_0 \in L_3$ or $L^{3,\infty}$

Lorentz spaces

$$d_f(\alpha) := \left| \{ x : |f(x)| > \alpha \} \right|$$

$$L^{s,\ell} = \left\{ \|u\|_{L^{s,\ell}} = \left(s \int_0^{+\infty} \alpha^{\ell} \left[d_f(\alpha) \right]^{\ell/s} \frac{d\alpha}{\alpha} \right)^{1/\ell} < \infty \right\}$$

$$L^{s,\infty} = \left\{ \|u\|_{L^{s,\infty}} = \sup_{\alpha > 0} \alpha \left[d_f(\alpha) \right]^{1/s} < \infty \right\}$$

$$L_s := L^{s,s}$$

Mild solutions (Kato's scheme)

$$v(x,t) = S(t)u_0(x) + \int_0^t \int_{\mathbb{R}^3} K(x-y, t-\tau)\big(v(y,\tau)\otimes v(y,\tau)\big)\,dy\,d\tau$$

Heat semigroup

$$|K(x,t)| \sim \frac{C}{(|x|^2+t)^2}$$

Strong local in time solutions

(in $L_\sigma^{3,\infty}$-case additional assumptions on initial data)

Motivation (for global solutions)

NSE's scaling: $v^\lambda(x,t) = \lambda\, v(\lambda x, \lambda^2 t)$

$$q^\lambda(x,t) = \lambda^2\, q(\lambda x, \lambda^2 t)$$

Scale invariant norms:

Examples: $X = L_3$, $L^{3,\infty}$ and so on

Question: Let T be a potential blowup time. Show that

$$\lim_{t \uparrow T} \|v(\cdot,t)\|_X = \infty$$

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$$\|v(\cdot,t)\|_{L_6} \geq \frac{c(6)}{(T-t)^{\frac{3}{2}(\frac{1}{3}-\frac{1}{6})}} \quad \text{if } 6>3$$

no universal function $f(t)$ exists s.t.

$$\|v(\cdot,t)\|_{L_3} \geq f(T-t) \qquad 0 \leq t < T$$

and $f(t) \longrightarrow \infty$ as $t \to 0$

$$v^\lambda(y,s) = \lambda v(x,t), \qquad x = \lambda y, \quad t = \lambda^2 s$$

$$0 \leq s < \frac{T}{\lambda^2}$$

$$\|v^\lambda(\cdot,0)\|_{X^-} = \|v(\cdot,0)\|_{X} \geq f\left(\frac{T}{\lambda^2}\right)$$

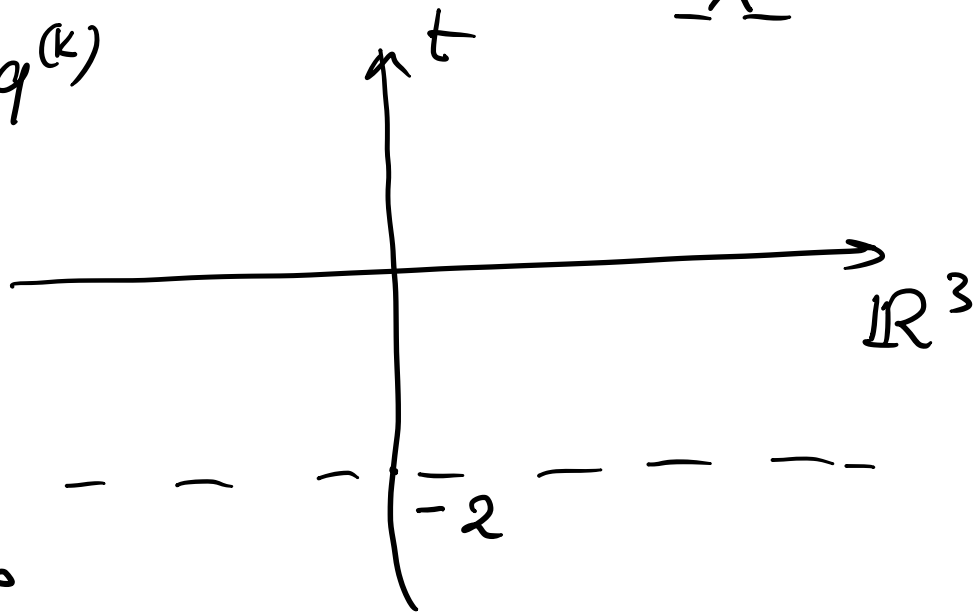
$$\longrightarrow \infty \quad \text{as } \lambda \to \infty$$

Ad absurdum:

$$\exists\, t_k \longrightarrow T : \qquad M = \sup_k \|v(\cdot, t_k)\|_X < \infty$$

$$\begin{cases} \partial_t v^{(k)} + v^{(k)} \cdot \nabla v^{(k)} - \Delta v^{(k)} = -\nabla q^{(k)} \\ \operatorname{div} v^{(k)} = 0 \end{cases}$$

in $\mathbb{R}^3 \times]0, \infty[$



$$M = \sup_k \|v^{(k)}(\cdot, -2)\|_X < +\infty$$

$$v^{(k)}(\cdot, -2) = u_0^{(k)}(\cdot) \longrightarrow \left(\xrightarrow{*} \right) u_0(\cdot)$$

$$v^{(k)} \longrightarrow v \quad ?$$

Wish list

- Global solvability
- "weak continuity" with respect to initial data

or

"weak stability"

Lemarie - Riesset Solutions

Phase space $\quad \mathring{E}_2 = \left[C^\infty_{0,0} \right]^{L_{2,\,unif}}$

$$L_{s,\,unif} = \left\{ \|v\|_{L_{s,\,unif}} = \sup_{x \in \mathbb{R}^3} \|v\|_{L_2(B(x,1))} < \infty \right\}$$

Local Energy Inequality (for certain pressure class)

Initial data: $\quad \|v(\cdot,t) - u_0\|_{L_{2,\,unif}} \longrightarrow 0$

as $t \longrightarrow 0$

Why not Lemarie-Riesset solutions?

- Too wide class for uniqueness
- Does not work if $\Omega \neq \mathbb{R}^3$
- "weak continuity" is not yet known

Our definition

$$u_0 \in X \ (L_3 \text{ or } L^{3,\infty}), \quad \operatorname{div} u_0 = 0,$$

$$V(x,t) = S(t)\, u_0(x)$$

v is weak X-solution in $Q_T = \mathbb{R}^3 \times]0,T[$

if $\exists\, q \in L_{3/2,\,loc}(Q_T)$ s. t. $u = v - V$

and q satisfy the following requirements:

- $u \in L_\infty(0,T;\, \mathring{J}) \cap L_2(0,T;\, \mathring{J}^1_2)$
- $$\left.\begin{aligned} \partial_t u + v\cdot\nabla v - \Delta u &= -\nabla q \\ \operatorname{div} u &= 0 \end{aligned}\right\} \text{ in } Q_T$$

- $$t \longmapsto \int_{\mathbb{R}^3} u(x,t)\cdot w(x)\,dx \in C([0,T])$$

$$\forall\, w \in L_2$$

- $$\frac{1}{2}\int_{\mathbb{R}^3} |u(x,t)|^2 dx + \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 dx\, dt' \leq$$

$$\leq \int_0^t \int_{\mathbb{R}^3} \left(V \otimes u + V \otimes V \right) : \nabla u\, dx\, dt'$$

for any $\quad 0 \leq t \leq T$

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$$\int_{\mathbb{R}^3} |v(x,t)|^2 \varphi^2(x,t)\,dx + 2\int_0^t \int_{\mathbb{R}^3} |\nabla v|^2 \varphi^2\,dx\,dt' \leq$$

$$\leq \int_0^t \int_{\mathbb{R}^3} |v|^2 \left(\Delta\varphi + \partial_t \varphi\right) + v\cdot\nabla\varphi\left(|v|^2 + 2q\right)\,dx\,dt'$$

$$\forall\, \varphi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R})$$

Initial data: $u(\cdot,t) \xrightarrow[t\to 0]{} 0$ in L_2

$X = L_3$

$v = V + u$

$V(\cdot,t) \longrightarrow u_0$ in L_3

$X = L^{3,\infty}$, $V(\cdot,t) \xrightarrow[\text{in } L^{3,\infty}]{*} u_0$

$V(\cdot,t) \to u_0$ in L_s, unif $\forall s < 3$

$$u_0 = \left(\frac{x_2}{|x|^2}, -\frac{x_1}{|x|^2}, 0 \right)$$

$$V(x,t) = \frac{1}{\sqrt{t}} V\left(\frac{x}{\sqrt{t}}, 1\right)$$

$$\|V(\cdot,t)\|_{L^{3,\infty}} = \|V(\cdot,1)\|_{L^{3,\infty}}$$

Global forward self-similar solutions constructed by H. Jia, V. Sverak are global $L^{3,\infty}$-solutions

-15- **Th. 1** Let $u_0^{(k)} \xrightarrow{*} u_0$ in X

$$v^{(k)} = V + u^{(k)} \longrightarrow v = V + u$$

in the sense of distributions at least

Proof A priori estimates

$$\exists \quad M := \sup_k \|u_0^{(k)}\|_X < +\infty$$

$$\Rightarrow \sup_{0 \le t \le T} \int_{\mathbb{R}^3} |u^{(k)}(x,t)|^2 dx + \int_0^T \int_{\mathbb{R}^3} |\nabla u^{(k)}|^2 dx dt \le C(M, T)$$

Corollary: Existence

Given $u_0 \in X$, find $u_0^{(k)} \in C_{0,0}^{\infty} \xrightarrow{*} u_0$ in X

In $L^{3,\infty}$, some work should be done

Given $u_0^{(k)}$, $\exists\ v^{(k)} = V + u_0^{(k)}$

that is weak X-solution.

Existence follows from Theorem 1.

-17- **Lemma** Let v be global weak $L^{3,\infty}$-solution with u and q from definition. Then

$$u = u^1 + u^2 + u^3, \qquad q = p_1 + p_2 + p_3 \quad \forall\, T > 0$$

$$u^i \in W^{2,1}_{s_i,\ell_i}(Q_T), \qquad \nabla p_i \in L_{s_i,\ell_i}(Q_T)$$

$$(s_1,\ell_1) = \left(9/8, \tfrac{3}{2}\right), \quad s_2 = \ell_2 = \tfrac{11}{4}, \quad (s_2,\ell_2) = \left(5/4, 3/2\right)$$

and

$$\left.\begin{cases} \partial_t u^1 - \Delta u^1 + \nabla p_1 = -u\cdot\nabla u \\ \partial_t u^2 - \Delta u^2 + \nabla p_2 = -V\cdot\nabla V \\ \partial_t u^3 - \Delta u^3 + \nabla p_3 = -V\cdot\nabla u - u\cdot\nabla V \end{cases}\right\} \text{ in } Q_\infty$$

$$\operatorname{div} u^i = 0 \quad \text{in } Q_\infty \qquad i = 1,2,3$$

$$u^i(\cdot,0) = 0 \quad \text{in } \mathbb{R}^3$$

Uniqueness and Regularity. $X = L_3$

Th. 2 (Uniqueness) Let v and $\tilde{v}$ be two weak L_3-solutions with the same initial data $u_0 \in L_3(\mathbb{R}^3)$.

Let, in addition,

$$\boxed{v \in L_\infty(0,T;L_3)}$$

Then $v = \tilde{v}$ in $Q_T = \mathbb{R}^3 \times]0,T[$.

Th. 3 (Regularity) Let v be a weak L_3-solution with initial data $u_0 \in L_3$. Then $\exists\, T(u_0) > 0$ s.t.

$$v \in L_\infty(0, T(u_0); L_3)$$

and

$$v \in C^\infty \quad \text{on} \quad \mathbb{R}^3 \times]0, T(u_0)]$$

Outline of Proof of Th. 2 and 3

Pr. 1 Let v and $\tilde{v}$ be two weak L_3-solutions with the same initial data $u_0 \in L_3$. Let $v \in L_\infty(0,T; L_3)$

$\exists\, \varepsilon > 0$ s. t. if, for some $T_1 \in]0, T]$

s. t. $$\sup_{0 < t \le T_1} \|v(\cdot, t) - u_0(\cdot)\|_{L_3} < \varepsilon .$$

Then $v = \tilde{v}$ in Q_T.

Pr. 2 Let v and $\tilde{v}$ be two weak L_3-solutions with the same initial data $u_0 \in L_3$. If

$$v \in L_5(Q_T)$$

Then $v = \tilde{v}$ in Q_T

Pr. 3 Let v be weak L_3-solution with initial data $u_0 \in L_3$. Then

$\exists\, T(u_0) > 0$: $v \in L_5(Q_{T(u_0)})$.

$$\Longrightarrow \quad v \in L_\infty(0, T(u_0); L_3)$$

Uniqueness and Regularity. $X = L^{3,\infty}$

Pr. 3 Let v be a weak $L^{3,\infty}$-solution with initial data $u_0 \in L^{3,\infty}$. There exists universal constant $\varepsilon_0 > 0$ s.t. if

$$\limsup_{R \to 0} \|u_0\|_{L^{3,\infty}(B(x_0,R))} < \varepsilon_0$$

for all $x_0 \in \mathbb{R}^3$ and

$$\|v(\cdot, t) - u_0\|_{L^{3,\infty}} < \varepsilon_0$$

for all $0 < t \leq T$, then $v \in C^\infty(\mathbb{R}^3 \times]0, T])$.

Moreover, if $\tilde{v}$ is another solution with the same initial data, then

$$v = \tilde{v} \quad \text{in } Q_T$$

Corollary Let v and $\tilde{v}$ be two weak $L^{3,\infty}$-solutions with the same initial data $u_0 \in L^{3,\infty}$. Assume that

$$v \in C([0,T]; L^{3,\infty}).$$

Then $v = \tilde{v}$ in Q_T

Th. 4 Let $u_0 \in L^{3,\infty}$. There exists universal constant $\varepsilon_0 > 0$ s.t. if

$$\langle V \rangle_{Q_T} := \sup_{0<t\le T} t^{1/5} \|V(\cdot,t)\|_{L_5} < \varepsilon_0,$$

$V(\cdot,t) := S(t)u_0(\cdot)$, then $\exists$ a global weak $L^{3,\infty}$-solution with initial data u_0 such that

$$\langle v \rangle_{Q_T} \le 2\langle V \rangle_{Q_T}.$$

Moreover, $\|v - V\|_{L_\infty(0,T;L_3)} < \langle V \rangle_{Q_T}$ and $v \in C^\infty$ in $\mathbb{R}^3 \times]0,T]$

$$u_0 = \left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}, 0 \right)$$

Then

$$t^{1/5} \| V(\cdot, t) \|_{L_5} = \| V(\cdot, 1) \|_{L_5}$$

$$= \| S(1) u_0 \|_{L_5}$$

$$\boxed{u_0 \in \left[C_{c\,3}^{\infty} \right]^{L^{3,\infty}} + \text{small perturbations}}$$

Th. 5 Let $u_0 \in L^{3,\infty}$. There exists universal constant $\varepsilon > 0$ such that if

$$(*) \quad \limsup_{\alpha \to \infty} \alpha \, d_{u_0}^{1/3}(\alpha) < \varepsilon,$$

then $\exists\, T = T(u_0) > 0$ such that all global weak $L^{3,\infty}$-solutions, with initial data u_0, coincide in Q_T

$(*)$ - Yamazaki - Kozono condition