On steady compressible Navier–Stokes–Fourier system

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Model reduction in continuum thermodynamics

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1 System of equations in the steady regime

- Balance of mass

$$\operatorname{div}(\rho\mathbf{u}) = 0 \tag{1}$$

$\rho(x)$: $\Omega \mapsto \mathbb{R}$ . . . density of the fluid
$\mathbf{u}(x)$: $\Omega \mapsto \mathbb{R}^3$ . . . velocity field

- Balance of momentum

$$\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) - \operatorname{div}\mathbb{S} + \nabla p = \rho\mathbf{f} \tag{2}$$

$\mathbb{S}$ . . . viscous part of the stress tensor (symmetric tensor)
$\mathbf{f}(x)$: $\Omega \mapsto \mathbb{R}^3$ . . . specific volume force
p. . . pressure (scalar quantity)

- Balance of total energy

$$\operatorname{div}\big(\rho E\mathbf{u}\big) + \operatorname{div}(\mathbf{q} + p\mathbf{u}) = \rho\mathbf{f}\cdot\mathbf{u} + \operatorname{div}\big(\mathbb{S}\mathbf{u}\big) \tag{3}$$

$E = \frac{1}{2}|\mathbf{u}|^2 + e\ldots$ specific total energy
e . . . specific internal energy (scalar quantity)
$\mathbf{q}$ . . . heat flux (vector field)
(no energy sources assumed)

2 Thermodynamics

We will work with basic quantities: density ρ and temperature ϑ

We assume: $e = e(\rho, \vartheta)$, $p = p(\rho, \vartheta)$

- Gibbs' relation

$$\frac{1}{\vartheta}\left(De(\rho, \vartheta) + p(\rho, \vartheta)D\left(\frac{1}{\rho}\right)\right) = Ds(\rho, \vartheta) \tag{4}$$

with $s(\rho, \vartheta)$ the specific entropy.

The entropy fulfills

- Entropy balance

$$\operatorname{div}(\rho s \mathbf{u}) + \operatorname{div}\left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma = \frac{\mathbb{S} : \nabla \mathbf{u}}{\vartheta} - \frac{\mathbf{q} \cdot \nabla \vartheta}{\vartheta^2} \tag{5}$$

- Second law of thermodynamics

$$\sigma = \frac{\mathbb{S} : \nabla \mathbf{u}}{\vartheta} - \frac{\mathbf{q} \cdot \nabla \vartheta}{\vartheta^2} \geq 0 \tag{6}$$

3 Constitutive relations

- Newtonian fluid

$$\mathbb{S} = \mathbb{S}(\vartheta, \nabla \mathbf{u}) = \mu(\vartheta)\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T - \frac{2}{3}\mathrm{div}\,\mathbf{u}\mathbb{I}\right] + \xi(\vartheta)\mathrm{div}\,\mathbf{u}\mathbb{I} \tag{7}$$

μ, ξ: viscosity coefficients

- Fourier's law

$$\mathbf{q} = \mathbf{q}(\vartheta, \nabla\vartheta) = -\kappa(\vartheta)\nabla\vartheta \tag{8}$$

$\kappa(\cdot)$: $\mathbb{R}^+ \mapsto \mathbb{R}^+$... heat conductivity

- Pressure law

$$
\begin{aligned}
p = p(\rho,\vartheta) \quad &= \quad \rho^\gamma + \rho\vartheta \\
\text{or} \quad &= \quad (\gamma - 1)\rho e(\rho,\vartheta)
\end{aligned}
\tag{9}
$$

(in what follows, to avoid technicalities, we consider the former)

- Internal energy

$$
e(\rho,\vartheta) = c_v\vartheta + \frac{\rho^{\gamma-1}}{\gamma - 1} \tag{10}
$$

- Heat conductivity

$$\kappa(\vartheta) \sim (1 + \vartheta^m) \tag{11}$$

$0 < m \in \mathbb{R}$

- Viscosity coefficients

$$\begin{gathered} C_1(1+\vartheta)^\alpha \leq \mu(\vartheta) \leq C_2(1+\vartheta)^\alpha \\ 0 \leq \xi(\vartheta) \leq C_2(1+\vartheta)^\alpha \end{gathered} \tag{12}$$

$0 \leq \alpha \leq 1$

4 Classical formulation of the problem

We consider steady solutions in a bounded domain $\Omega \subset \mathbb{R}^3$:

Steady compressible Navier–Stokes–Fourier system

$$
\begin{gathered}
\operatorname{div}(\rho \mathbf{u}) = 0 \\
\operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \operatorname{div} \mathbb{S}(\vartheta, \nabla \mathbf{u}) + \nabla p(\rho, \vartheta) = \rho \mathbf{f} \\
\operatorname{div} \Big(\rho \big(\frac{1}{2}|\mathbf{u}|^2 + e(\rho, \vartheta)\big) \mathbf{u} \Big) - \operatorname{div}(\kappa(\vartheta) \nabla \vartheta) \\
= \operatorname{div} \big(- p(\rho, \vartheta) \mathbf{u} + \mathbb{S}(\vartheta, \nabla \mathbf{u}) \mathbf{u} \big) + \rho \mathbf{f} \cdot \mathbf{u}
\end{gathered} \tag{13}
$$

Boundary conditions at $\partial\Omega$: velocity

$$\begin{gathered}\mathbf{u} = \mathbf{0} \\ \text{or} \\ \mathbf{u}\cdot\mathbf{n} = 0 \\ (\mathbb{I} - \mathbf{n}\otimes\mathbf{n})(\mathbb{S}(\vartheta, \nabla\mathbf{u})\mathbf{n} + \lambda\mathbf{u}) = \mathbf{0}\end{gathered} \tag{14}$$

Boundary conditions at $\partial\Omega$: temperature

$$\kappa(\vartheta)\frac{\partial\vartheta}{\partial\mathbf{n}} + L(\vartheta)(\vartheta - \Theta_0) = 0 \tag{15}$$

Total mass

$$\int_\Omega \rho \,\mathrm{d}x = M > 0 \tag{16}$$

Instead of total energy balance we can consider the entropy balance

Entropy balance

$$\begin{aligned}\operatorname{div}\left(\rho s(\rho,\vartheta)\mathbf{u}\right) - \operatorname{div}\left(\kappa(\vartheta)\frac{\nabla\vartheta}{\vartheta}\right) &= \sigma \\ = \frac{\mathbb{S}(\vartheta,\nabla\mathbf{u}) : \nabla\mathbf{u}}{\vartheta} &+ \frac{\kappa(\vartheta)|\nabla\vartheta|^2}{\vartheta^2}\end{aligned} \tag{17}$$

5 Weak solution to our problem

- Weak formulation of the continuity equation

$$\int_\Omega \varrho \mathbf{u} \cdot \nabla \psi \, \mathrm{d}x = 0 \qquad \forall \psi \in C^1(\overline{\Omega}) \tag{18}$$

- Renormalized continuity equation

$(\varrho, \mathbf{u})$ extended by zero outside Ω

$$\int_\Omega b(\varrho) \mathbf{u} \cdot \nabla \psi \, \mathrm{d}x + \int_\Omega \big(\mathbf{u} b'(\rho) - b(\rho) \big) \operatorname{div} \mathbf{u} \, \mathrm{d}x = 0 \forall \psi \in C_0^1(\mathbb{R}^3) \tag{19}$$

for all $b \in C^1([0,\infty)) \cap W^{1,\infty}(0,\infty)$ with $zb'(z) \in L^\infty(0,\infty)$

- Weak formulation of the momentum equation

$$\int_\Omega \big(-\rho(\mathbf{u}\otimes\mathbf{u}) : \nabla\boldsymbol{\varphi} - p(\rho,\vartheta)\mathrm{div}\,\boldsymbol{\varphi} + \mathbb{S}(\vartheta,\nabla\mathbf{u}) : \nabla\boldsymbol{\varphi}\big)\,\mathrm{d}x \\ = \int_\Omega \rho\mathbf{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}x \quad \forall\boldsymbol{\varphi}\in C_0^1(\Omega;\mathbb{R}^3) \tag{20}$$

$$\int_\Omega \big(-\rho(\mathbf{u}\otimes\mathbf{u}) : \nabla\boldsymbol{\varphi} - p(\rho,\vartheta)\mathrm{div}\,\boldsymbol{\varphi} + \mathbb{S}(\vartheta,\nabla\mathbf{u}) : \nabla\boldsymbol{\varphi}\big)\,\mathrm{d}x \\ +\lambda\int_{\partial\Omega}\mathbf{u}\cdot\boldsymbol{\varphi}\,\mathrm{d}\sigma = \int_\Omega \rho\mathbf{f}\cdot\boldsymbol{\varphi}\,\mathrm{dx} \\ \forall\boldsymbol{\varphi}\in C_{\mathbf{n}}^1(\Omega;\mathbb{R}^3) \tag{21}$$

- Weak formulation of the total energy balance

$$
\begin{aligned}
&\int_\Omega -\Big(\frac{1}{2}\rho|\mathbf{u}|^2 + \rho e(\rho,\vartheta)\Big)\mathbf{u}\cdot\nabla\psi\,\mathrm{d}x \\
&= \int_\Omega \big(\rho\mathbf{f}\cdot\mathbf{u}\psi + p(\rho,\vartheta)\mathbf{u}\cdot\nabla\psi\big)\,\mathrm{d}x \\
&- \int_\Omega \big(\big(\mathbb{S}(\vartheta,\mathbf{u})\mathbf{u}\big)\cdot\nabla\psi + \kappa(\vartheta)\nabla\vartheta\cdot\nabla\psi\big)\,\mathrm{d}x \\
&- \int_{\partial\Omega} \big(L(\vartheta-\Theta_0)\psi + \lambda|\mathbf{u}|^2\psi\big)\,\mathrm{d}\sigma \\
&\forall\psi\in C^1(\overline{\Omega})
\end{aligned}
\tag{22}
$$

Definition 1. *The triple* $(\rho,\mathbf{u},\vartheta)$, $\rho\geq 0$, $\vartheta>0$ *is called a renormalized weak solution to our system (13)–(16) if* $\int_\Omega \rho\ \mathrm{d}x = M$*, (18), (19), (20) (or (21)) and (22) hold true.*

6 Entropy variational solution to our problem

- Weak formulation of the entropy inequality

$$\int_\Omega \Big(\frac{\mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u}}{\vartheta} + \kappa(\vartheta)\frac{|\nabla\vartheta|^2}{\vartheta^2}\Big)\psi \,\mathrm{d}x + \int_{\partial\Omega} \frac{L}{\vartheta}\Theta_0 \psi \,\mathrm{d}\sigma$$
$$\leq \int_{\partial\Omega} L\psi \,\mathrm{d}\sigma + \int_\Omega \Big(\kappa(\vartheta)\frac{\nabla\vartheta : \nabla\psi}{\vartheta} - \rho s(\rho,\vartheta)\mathbf{u}\cdot\nabla\psi\Big)\,\mathrm{d}x$$
$$\forall \text{ nonnegative } \psi \in C^1(\overline{\Omega}) \tag{23}$$

- Global total energy balance

$$\int_{\partial\Omega} \left(L(\vartheta - \Theta_0) + \lambda|\mathbf{u}|^2\right) \mathrm{d}\sigma = \int_\Omega \rho \mathbf{f} \cdot \mathbf{u} \, \mathrm{d}x \qquad (24)$$

Definition 2. *The triple* $(\rho, \mathbf{u}, \vartheta)$, $\rho \geq 0$, $\vartheta > 0$ *is called a renormalized variational entropy solution to our system (13)–(16), if* $\int_\Omega \rho \ \mathrm{d}x = M$ *(18), (19) and (20) or (21) are satisfied in the same sense as in Definition 1, and we have the entropy inequality (23) together with the global total energy balance (24).*

Both type of solutions are reasonable in the sense that any smooth weak or entropy variational solution is actually a classical solution to (13)–(16).

7 Mathematical results

Until 2009, in the literature there was no existence results except for small data results or one result by P.L. Lions, where, however, the fixed mass was replaced by the finite L^p norm of the density for p sufficiently large.

Mucha, M.P.: Commun. Math. Phys. (2009)

Assumptions: constant viscosity, slip boundary conditions for the velocity, in the boundary conditions for the temperature $L(\vartheta) \sim (1+\vartheta)^l$:

Aim: to find solutions with maximal possible regularity, i.e. bounded density and gradient of temperature and velocity in any $L^q(\Omega)$, $q < \infty$

Approximate scheme: special approximation which gives bounded density solutions with uniform control (goes back to our papers in Nonlinearity (2006) and DCDS: Series S (2007) for the compressible Navier–Stokes equations).

A priori estimates:

a) Global total energy balance

$$\int_{\partial\Omega} L(\vartheta)(\vartheta - \Theta_0)\,\mathrm{d}\sigma \leq C\Big(1 + \int_{\Omega} |\varrho\mathbf{u}\cdot\mathbf{f}|\,\mathrm{d}x\Big). \tag{25}$$

b) Entropy inequality

$$\int_{\Omega} \frac{\mathbb{S}(\nabla\mathbf{u}) : \nabla\mathbf{u}}{\vartheta}\,\mathrm{d}x + \int_{\Omega} \frac{1+\vartheta^m}{\vartheta^2}|\nabla\vartheta|^2\,\mathrm{d}x + \int_{\partial\Omega} \frac{L(\vartheta)\Theta_0}{\vartheta}\,\mathrm{d}\sigma \leq C\int_{\partial\Omega} L(\vartheta)\,\mathrm{d}\sigma. \tag{26}$$

c) Take $m = l + 1$. Then

$$\|\vartheta\|_{3m} \le C\left(1 + \int_\Omega |\rho\mathbf{f}\cdot\mathbf{u}|\,\mathrm{d}x\right)^{1/m}. \tag{27}$$

d) Multiply the momentum equation by the solution to

$$\operatorname{div}\mathbf{H} = \varrho^\gamma - \frac{1}{|\Omega|}\int_\Omega \varrho^\gamma\,\mathrm{d}x$$

with $\mathbf{H} = \mathbf{0}$ at $\partial\Omega$ such that

$$\|\mathbf{H}\|_{1,q} \le C\|\varrho^\gamma\|_q, \quad 1 < q < \infty.$$

This gives control of density by velocity and temperature

$$\int_\Omega \varrho^{2\gamma} \, \mathrm{d}x \leq RHS.$$

e) Finally, test the momentum equation by the velocity. This gives the control of velocity by temperature and density

$$\int_\Omega \mathbb{S}(\nabla \mathbf{u}) : \nabla \mathbf{u} \, \mathrm{d}x \leq RHS.$$

This procedure can be closed, i.e. we get the estimates, if $\gamma > \mathbf{3}$, $m = l + 1 > \frac{3\gamma - 1}{3\gamma - 7}$.

Higher regularity: We cut off the continuity equation in the approximate scheme for large ϱ:

$$\mathrm{div}\,(K(\varrho)\varrho\mathbf{u}) = l.o.t.$$

Thus the density is bounded uniformly throughout the approximation procedure. The slip boundary condition and constant viscosity allow to write a nice elliptic problem for the vorticity, which leads finally to higher regularity for the velocity, consequently also for the temperature.

Limit passage: We use a version of the effective viscous flux identity, but due to high regularity of the density we have no problems with renormalized continuity equation and we even do not use it. The solution even fulfills the internal energy balance.

Mucha, M.P.: M3AS (2010)

Assumptions: constant viscosity, slip or homogeneous Dirichlet boundary conditions for the velocity, in the boundary conditions for the temperature $L(\vartheta) \sim (1+\vartheta)^l$:

Aim: to extend the results from the previous paper to situations with lower γ

Approximate scheme: Since we do not expect anymore solutions with bounded density (reasons: either $\gamma < 3$ or Dirichlet boundary condition), we use standard elliptic regularization of the continuity equations.

A priori estimates: The only difference is the fact that we allow weaker estimates for the density. In d) we test by

$$\operatorname{div}\mathbf{H} = \varrho^{s(\gamma)} - \frac{1}{|\Omega|}\int_\Omega \varrho^{s(\gamma)}\,\mathrm{d}x.$$

We can close the estimates for $\gamma > \frac{7}{3}$, the bound for m and l is the same.

Limit passage: We use the effective viscous flux identity as well as the renormalized continuity equation. Due to high γ our limit fulfills the renormalized continuity equation directly. The solution fulfills only the total energy balance.

Novotný, M.P.: J. Differential Equations (2011)

Assumptions: viscosity dependent on temperature:

$$\mu(\vartheta), \xi(\vartheta) \sim (1+\vartheta)$$

$(\alpha = 1)$, $L \sim const$ $(l = 0)$ homogeneous Dirichlet condition for the velocity. (But slip b.c. can be treated via the same method.)

Aim: to extend the interval for γ to include also some physically interesting cases as e.g. $\gamma = \frac{5}{3}$ or $\gamma = \frac{4}{3}$. Another goal was to present in details construction of approximation if the viscosity is temperature dependent.

Approximate scheme: elliptic regularization of the continuity equation, more steps than in the previous case with constant viscosity.

A priori estimates:

a) Global total energy balance

$$\int_{\partial\Omega} L(\vartheta - \Theta_0)\,\mathrm{d}\sigma \leq C\Big(1 + \int_{\Omega} |\varrho\mathbf{u}\cdot\mathbf{f}|\,\mathrm{d}x\Big). \tag{28}$$

b) Entropy inequality

$$\begin{gathered}\int_{\Omega} \frac{\mathbb{S}(\vartheta, \nabla\mathbf{u}) : \nabla\mathbf{u}}{\vartheta}\,\mathrm{d}x + \int_{\Omega} \frac{1+\vartheta^m}{\vartheta^2}|\nabla\vartheta|^2\,\mathrm{d}x \\ + \int_{\partial\Omega} \frac{L\Theta_0}{\vartheta}\,\mathrm{d}\sigma \leq C\int_{\partial\Omega} L\,\mathrm{d}\sigma.\end{gathered} \tag{29}$$

c) Main difference: due to Korn's inequality we immediately have

$$\|\mathbf{u}\|_{1,2} \leq C$$

while for the temperature we get again

$$\|\vartheta\|_{3m} \leq C\Big(1 + \int_\Omega |\rho \mathbf{f} \cdot \mathbf{u}| \,\mathrm{d}x\Big). \tag{30}$$

d) Multiply the momentum equation by the solution to

$$\operatorname{div} \mathbf{H} = \varrho^{s(\gamma)} - \frac{1}{|\Omega|} \int_\Omega \varrho^{s(\gamma)} \,\mathrm{d}x$$

with $\mathbf{H} = \mathbf{0}$ at $\partial\Omega$ such that

$$\|\mathbf{H}\|_{1,q} \leq C\|\rho^{s(\gamma)}\|_q, \quad 1 < q < \infty.$$

These estimates imply the restriction: $\gamma > \frac{3}{2}$! Under additional assumptions on m we get a solution for any $\gamma > \frac{3}{2}$.

Limit passage: We use a version of the effective viscous flux identity, and the renormalized continuity equation to get the strong convergence of the velocity. But for small γ we do not have for free the renormalized continuity equation for the limit functions. We use the technique of E. Feireisl developed for the evolutionary case: the control of oscillation defect measure implies the renormalized continuity equation. For $\gamma > \frac{5}{3}$ and sufficiently large m we get the total energy balance, in the other case only entropy inequality and global total energy balance — the variational entropy solution.

Novotný, M.P.: SIAM J. Math. Anal. (2011)

Assumptions: the same as in the previous case.

Aim: to extend the interval for γ.

Approximate scheme: the same as before

A priori estimates: The main difference is that, following the idea of Frehse, Steinhauer, Weigant (used for the Navier–Stokes equations), we are able to get additional estimates for the density of the form

$$\sup_{y\in\overline{\Omega}} \int_\Omega \frac{p(\varrho,\vartheta)}{|x-y|^t}\,\mathrm{d}x < +\infty$$

with $t = t(m)$. This gives the a priori estimates for any $\gamma > 1$, with some additional bounds on m.

Limit passage: More or less the same as above. But due to not the best possible choice of carrying out this limit passage in the convective term we got additional restriction $\gamma > \frac{3+\sqrt{41}}{8}$.

We improved the interval for weak solution: for $\gamma > \frac{4}{3}$ and sufficiently large m we get the total energy balance, in the other cases only the entropy inequality and global total energy balance — the variational entropy solution.

Kreml, Nečasová, M.P.: to appear in ZAMP

Assumptions: it involves also a model for radiation, more complex than just adding ϑ^4 to the pressure. We consider also viscosity of the type $\mu(\vartheta)$, $\xi(\vartheta) \sim (1+\vartheta)^\alpha$, $0 < \alpha \leq 1$. More details about the technique see the talk of O. Kreml.

Aim: to include also the physically relevant case $\alpha = \frac{1}{2}$.

Approximate scheme: the same as before.

A priori estimates: We use only estimates based on Bogovskii operator estimates; hence we must restrict ourselves to $\gamma > \frac{3}{2}$. Another result with O. Kreml: we combine the local pressure estimates with $\alpha < 1$. This allows to treat the case $\gamma < \frac{3}{2}$.

Limit passage: More or less the same as above.

Jesslé, Novotný, M.P.: submitted

Assumptions: The case of temperature dependent viscosity with $\alpha = 1$, with slip boundary condition.

Aim: to extend the interval for γ.

Approximate scheme: the same as before

A priori estimates: Based on the ideas from Jiang, Zhou and Jesslé, Novotný for Navier–Stokes system, we are able to get additional estimates for the density of the form

$$\sup_{y\in\overline{\Omega}} \int_\Omega \frac{p(\varrho,\vartheta) + \varrho|\mathbf{u}|^2}{|x-y|^t}\,\mathrm{d}x < +\infty$$

with $t = t(m)$, bigger than in the previous paper. This gives the a priori estimates for any $\gamma > 1$, with some additional bounds on m.

More precisely, for $1 \leq a \leq \gamma$, $0 < b < 1$ we consider the quantity

$$\mathcal{A} = \int_\Omega (\rho^a |\mathbf{u}|^2 + \rho^b |\mathbf{u}|^{2b+2}) \, \mathrm{d}x \tag{31}$$

We have

$$
\begin{aligned}
\|\mathbf{u}\|_{1,2} &\leq C \\
\|\vartheta\|_{3m} &\leq C\Big(1+\mathcal{A}^{\frac{a-b}{2(ab+a-2b)}}\Big) \\
\int_\Omega \Big(\rho^{s\gamma}+\rho^{(s-1)\gamma}p(\rho,\vartheta)+(\rho|\mathbf{u}|^2)^s+\delta\rho^{\beta+(s-1)\gamma}\Big)\ \mathrm{d}x & \\
&\leq C(1+\mathcal{A}^{\frac{sa-b}{ab+a-b}}),
\end{aligned}
\tag{32}
$$

provided $1<s<\frac{1}{2-a}$, $0<(s-1)\frac{a}{a-1}<b<1$, $s\leq\frac{6m}{3m+2}$, $m>\frac{2}{3}$. The last estimates follows from Bogovskii type estimates.

Next we want to use of the test functions of the type

$$\varphi_i(x) = \frac{(x-y)_i}{|x-y|^t}.$$

We have to work separately near the boundary and in the interior.

Lemma 1. *Let* $y \in \Omega$, $R_0 < \frac{1}{3} dist\,(y, \partial\Omega)$. *Then*

$$\int_{B_{R_0}(y)} \Big(\frac{p(\rho,\vartheta)}{|x-y|^t} + \frac{\rho|\mathbf{u}|^2}{|x-y|^t}\Big)\,\mathrm{d}x \\ \leq C\Big(1 + \|p(\rho,\vartheta)\|_1 + \|\mathbf{u}\|_{1,2}(1+\|\vartheta\|_{3m}) + \|\rho|\mathbf{u}|^2\|_1\Big), \tag{33}$$

provided $t < \min\left\{\frac{3m-2}{2m}, 1\right\}$.

Proof. We use as test function in the approximative momentum balance

$$\varphi_i(x) = \frac{(x-y)_i}{|x-y|^t}\tau^2$$

with $\tau \equiv 1$ in $B_{R_0}(y)$, R_0 as above, $\tau \equiv 0$ outside $B_{2R_0}(y)$, $|\nabla\tau| \le \frac{C}{R_0}$. Note that

$$\operatorname{div}\boldsymbol{\varphi} = \frac{3-t}{|x-y|^t}\tau^2 + g_1(x),$$

$$\partial_i\varphi_j = \Big(\frac{\delta_{ij}}{|x-y|^t} - t\frac{(x-y)_i(x-y)_j}{|x-y|^{t+2}}\Big)\tau^2 + g_2(x)$$

with g_1, g_2 in $L^\infty(\Omega)$. Thus we get the estimates from the pressure term and the convective term. We control the elliptic term provided $\frac{1}{q} = 1 - \frac{1}{2} - \frac{1}{3m} > \frac{t}{3}$, implying $t < \frac{3m-2}{2m}$ for $m > \frac{2}{3}$. $\square$

Near the boundary, we use to use a similar test function. The idea of Frehse, Steinhauer and Weigant, which can be used for both slip and no slip boundary conditions, leads to artificial restrictions on m and γ. The recent improvement by Jiang, Zhou allows to treat the Dirichlet b.c. with less restriction, but the method presented below for the slip b.c. is better. Assume for a moment that we deal with a flat part of the boundary which is described by $x_3 = 0$, i.e. $a(x') = 0$, $x' \in \mathcal{O} \subset \mathbb{R}^2$ with the normal vector $\mathbf{n} = (0,0,-1)$ and $\mathbf{t}^1 = (1,0,0)$, $\mathbf{t}^2 = (0,1,0)$ the tangent vectors. Consider the points in the neighborhood of the origin. Then the test function which replaces the test function above can be taken in the form

$\mathbf{w}(x) = \mathbf{v}(x - y)$, where

$$\mathbf{v}(z) = \begin{cases} \dfrac{1}{|z|^t}(z_1, z_2, z_3) = (z \cdot \mathbf{t}^1)\mathbf{t}^1 + (z \cdot \mathbf{t}^2)\mathbf{t}^2 \\ +((0, 0, z_3 - a(z')) \cdot \mathbf{n})\mathbf{n}, z_3 \geq 0, \\ \dfrac{1}{|z|^t}(z_1, z_2, 0) = (z \cdot \mathbf{t}^1)\mathbf{t}^1 + (z \cdot \mathbf{t}^2)\mathbf{t}^2, z_3 < 0. \end{cases}$$

Note that if $y \in \overline{\Omega}$ (i.e. $y_3 \geq 0$), then $(\mathbf{w} \cdot \mathbf{n})(x) = w_3(x) = 0$ for $x_3 = 0$. For a general C^2 domain we use partition of unity and local flattening of the boundary. Therefore we get the same result as in Lemma 1 also in the neighborhood of the boundary, i.e. for any point in $\overline{\Omega}$.

We distinguish two cases. For $m \geq 2$ we have $\frac{3m-2}{2m} \geq 1$, hence $t < 1$ is the only restriction. If $m \in (\frac{2}{3}, 2)$, we have $t < \frac{3m-2}{2m}$.

For $m > 2$, passing $t \to 1^-$

Lemma 2. *Let* $b \in ((s-1)\frac{\gamma}{\gamma-1}, 1)$, $1 < s < \frac{2}{2-\gamma}$, $m \geq 2$, $s \leq \frac{6m}{3m+2}$. *Then there exists* C *independent of* δ *such that for any* $y \in \overline{\Omega}$

$$\int_\Omega \frac{p(\rho,\vartheta) + (\rho|\mathbf{u}|^2)^b}{|x-y|} \, \mathrm{d}x \leq C\big(1 + \|p(\rho,\vartheta)\|_1 + (1 + \|\vartheta\|_{3m})\|\mathbf{u}\|_{1,2} + \|\rho|\mathbf{u}|^2\|_1\big). \tag{34}$$

If $m < 2$, we take $1 \leq a < \gamma$ and relatively easily by Hölder's inequality

Lemma 3. *Let* $b \in ((s-1)\frac{\gamma}{\gamma-1}, 1)$, $1 < s < \frac{2}{2-\gamma}$, $t > \max\{\frac{3a-2\gamma}{a}, \frac{3b-2}{b}\}$, $m \in (\frac{2}{3}, 2)$. *Then there exists* C *independent of* δ *such that for any* $y \in \overline{\Omega}$

$$\begin{aligned} &\int_\Omega \frac{\rho^a + (\rho|\mathbf{u}|^2)^b}{|x-y|} \,\mathrm{d}x \\ &\leq C\Big(1 + \|p(\rho,\vartheta)\|_1 + (1 + \|\vartheta\|_{3m})\|\mathbf{u}\|_{1,2} \\ &\quad + \|\rho|\mathbf{u}|^2\|_1\Big)^{\max\{\frac{a}{\gamma}, b\}}. \end{aligned} \tag{35}$$

Let us consider

$$
\begin{aligned}
-\Delta h = \rho^a + \rho^b |\mathbf{u}|^{2b} - \frac{1}{|\Omega|} \int_\Omega \left(\rho^a + \rho^b |\mathbf{u}|^{2b} \right) \mathrm{d}x, \\
\frac{\partial h}{\partial \mathbf{n}}|_{\partial\Omega} = 0.
\end{aligned}
\tag{36}
$$

The unique strong solution can be written

$$
h(y) = \int_\Omega G(x,y)(\rho_\delta^a + \rho_\delta^b |\mathbf{u}_\delta|^{2b}) \,\mathrm{d}x + l.o.t.; \tag{37}
$$

as $G(x,y) \le C|x-y|^{-1}$, we get

- $m \geq 2$

$$\|h\|_\infty \leq C(1 + \mathcal{A}^{\frac{\gamma - b/s}{b\gamma + \gamma - 2b}}), \tag{38}$$

- $\frac{2}{3} < m < 2$

$$\|h\|_\infty \leq C(1 + \mathcal{A}^{\frac{a-b/s}{ab+a-2b}\frac{a}{\gamma}} + \mathcal{A}^{\frac{a-b/s}{ab+a-2b}b}), \tag{39}$$

plus some additional restrictions. Now

$$\mathcal{A} = \int_\Omega -\Delta h \mathbf{u}^2 \, \mathrm{d}x = \int_\Omega \nabla h \cdot \nabla |\mathbf{u}|^2 \, \mathrm{d}x \leq 2\|\nabla \mathbf{u}\|_2 B^{\frac{1}{2}}, \tag{40}$$

$$B = \int_\Omega |\nabla h \otimes \mathbf{u}|^2 \, \mathrm{d}x. \tag{41}$$

Employing once more integration by parts

$$\begin{aligned} B &= -\int_\Omega h\Delta h|\mathbf{u}|^2 \ \mathrm{d}x - \int_\Omega h\nabla h\cdot\nabla\mathbf{u}\cdot\mathbf{u} \ \mathrm{d}x \\ &\leq \|h\|_\infty(\mathcal{A} + \|\nabla\mathbf{u}\|_2 B^{\frac{1}{2}}), \end{aligned}$$

i.e.

$$B \leq \|h\|_\infty\mathcal{A} + \frac{1}{2}\|\nabla\mathbf{u}\|_2^2\|h\|_\infty^2. \tag{42}$$

Therefore

$$\mathcal{A} \leq C\|\nabla\mathbf{u}\|_2^2\|h\|_\infty. \tag{43}$$

Analyzing all conditions we have

Lemma 4. *Let* $\gamma > 1$ *and* $m > \frac{2}{4\gamma - 3}$. *Then there exists* $s > 1$ *such that*

$$\begin{array}{lll} \sup_{\delta>0} \|\rho\|_{\gamma s} & < & +\infty \\ \sup_{\delta>0} \|\rho \mathbf{u}\|_{s} & < & +\infty \\ \sup_{\delta>0} \|\rho |\mathbf{u}|^2\|_{s} & < & +\infty \\ \sup_{\delta>0} \|\mathbf{u}\|_{1,2} & < & +\infty \\ \sup_{\delta>0} \|\vartheta\|_{3m} & < & +\infty \\ \sup_{\delta>0} \|\vartheta^{m/2}\|_{1,2} & < & +\infty \end{array} \tag{44}$$

Moreover, we can take $s > \frac{6}{5}$ *provided* $\gamma > \frac{5}{4}$, $m > \max\{1, \frac{2\gamma+10}{17\gamma-15}\}$.

Limit passage: More or less the same as above; it is enough to show the strong convergence of the density sequence; this can be achieved by standard technique (effective viscous flux identity, oscillation defect measure estimate implying the renormalized continuity equation for the limit). Some improvements with respect to the previous paper are needed to avoid additional restrictions on γ. However, we get some additional restrictions on $m = m(\gamma)$. We get the existence of variational entropy solutions for $\gamma > 1$. We also improve the interval for weak solution: for $\gamma > \frac{5}{4}$ and sufficiently large m we get the total energy balance, hence existence of weak solutions.

THANK YOU

FOR YOUR

ATTENTION!