On Implicit Constitutive Theories

K.R. Rajagopal

Department of Mechanical Engineering
Texas A&M University
College Station, Texas 77845

krajagopal@tamu.edu

- *We are falsely led to regard slightly related beginnings, vague tracks, hazy indications, which are found, as evidences of a real insight, which disposes us to 'promote one above another'. Hence a mythological process results, comparable to that which, in former times, thrust all conceivable feats of strength on the one Hercules.*

  – Einstein (As quoted in Moszkowski(1972))

- *In the presentation of a novel outlook with wide ramifications a single line of communication from premises to conclusions is not sufficient for intelligibility. Your audience will construe whatever you say in conformity with pre-existing outlook.*

  – A.N. Whitehead

- *People think they are thinking when they are merely rearranging their prejudices.*

  – William James

- *We are falsely led to regard slightly related beginnings, vague tracks, hazy indications, which are found, as evidences of a real insight, which disposes us to 'promote one above another'. Hence a mythological process results, comparable to that which, in former times, thrust all conceivable feats of strength on the one Hercules.*

– Einstein (As quoted in Moszkowski(1972))

- *In the presentation of a novel outlook with wide ramifications a single line of communication from premises to conclusions is not sufficient for intelligibility. Your audience will construe whatever you say in conformity with pre-existing outlook.*

– A.N. Whitehead

- *People think they are thinking when they are merely rearranging their prejudices.*

– William James

- *We are falsely led to regard slightly related beginnings, vague tracks, hazy indications, which are found, as evidences of a real insight, which disposes us to 'promote one above another'. Hence a mythological process results, comparable to that which, in former times, thrust all conceivable feats of strength on the one Hercules.*

  – Einstein (As quoted in Moszkowski(1972))

- *In the presentation of a novel outlook with wide ramifications a single line of communication from premises to conclusions is not sufficient for intelligibility. Your audience will construe whatever you say in conformity with pre-existing outlook.*

  – A.N. Whitehead

- *People think they are thinking when they are merely rearranging their prejudices.*

  – William James

Constitutive Equations

- How a body responds to stimuli, depends on how it is constituted and its constitution is expressed by "constitutive equations".
- The coinage "constitutive equation" unfortunately does not describe how a material is constituted. It is an incorrect usage of the English word "constitutive".
- The difference between how a body is constituted and what one means by "constitutive equations" can be best understood if we think in terms of a black box responding to an input by exhibiting a certain output, the input-output relation does not reveal the contents of the black box.
- There is nothing to prevent two different black boxes having the same input-output relation, similarly there is nothing that prevents two different bodies to respond in the same manner.

Constitutive Equations

- How a body responds to stimuli, depends on how it is constituted and its constitution is expressed by "constitutive equations".
- The coinage "constitutive equation" unfortunately does not describe how a material is constituted. It is an incorrect usage of the English word "constitutive".
- The difference between how a body is constituted and what one means by "constitutive equations" can be best understood if we think in terms of a black box responding to an input by exhibiting a certain output, the input-output relation does not reveal the contents of the black box.
- There is nothing to prevent two different black boxes having the same input-output relation, similarly there is nothing that prevents two different bodies to respond in the same manner.

Constitutive Equations

- How a body responds to stimuli, depends on how it is constituted and its constitution is expressed by "constitutive equations".
- The coinage "constitutive equation" unfortunately does not describe how a material is constituted. It is an incorrect usage of the English word "constitutive".
- The difference between how a body is constituted and what one means by "constitutive equations" can be best understood if we think in terms of a black box responding to an input by exhibiting a certain output, the input-output relation does not reveal the contents of the black box.
- There is nothing to prevent two different black boxes having the same input-output relation, similarly there is nothing that prevents two different bodies to respond in the same manner.

Constitutive Equations

- How a body responds to stimuli, depends on how it is constituted and its constitution is expressed by "constitutive equations".
- The coinage "constitutive equation" unfortunately does not describe how a material is constituted. It is an incorrect usage of the English word "constitutive".
- The difference between how a body is constituted and what one means by "constitutive equations" can be best understood if we think in terms of a black box responding to an input by exhibiting a certain output, the input-output relation does not reveal the contents of the black box.
- There is nothing to prevent two different black boxes having the same input-output relation, similarly there is nothing that prevents two different bodies to respond in the same manner.

Constitutive Equations

- How a body responds to stimuli, depends on how it is constituted and its constitution is expressed by "constitutive equations".
- The coinage "constitutive equation" unfortunately does not describe how a material is constituted. It is an incorrect usage of the English word "constitutive".
- The difference between how a body is constituted and what one means by "constitutive equations" can be best understood if we think in terms of a black box responding to an input by exhibiting a certain output, the input-output relation does not reveal the contents of the black box.
- There is nothing to prevent two different black boxes having the same input-output relation, similarly there is nothing that prevents two different bodies to respond in the same manner.

The terms “constitutive relation”, “constitutive function”, “constitutive equation” and “constitutive expression” are used interchangeably in continuum mechanics. This imprecise, careless and slipshod usage of these terms, as though they have the same signification, masks crucial differences and obscures fundamental and profound implications with regard to describing the response characteristics of bodies, and this point cannot be overemphasized. The term “constitutive function” suggests that the characterization of material is through the specification of explicit expressions for a certain variable, say the stress, in terms of kinematical quantities such as the strain, or the velocity gradient. The term “relation” (binary relation), on the other hand, implies that given two sets A and B, the member of one is related to the members of the other, usually expressed as $x\mathrm{R}y$ wherein $x \in A$ and $y \in B$.

Newton is unequivocal about the fact that force is the cause and motion is the effect as evidenced by the following sentiments:

- *The causes by which true and relative motion are distinguished, one from the other, are the forces impressed upon bodies to generate motion.*

- *The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.*

– Newton
Principia, 1687

- *A constitutive equation is a relation between forces and motions. In popular terms, force is applied to a body to "cause" it to undergo a motion, and the motion "caused" differs according to the nature of the body. In continuum mechanics the forces of interest are contact forces, which are specified by the stress tensor $\boldsymbol{T}$.*

– Clifford Truesdell, 1984

Newton is unequivocal about the fact that force is the cause and motion is the effect as evidenced by the following sentiments:

- *The causes by which true and relative motion are distinguished, one from the other, are the forces impressed upon bodies to generate motion.*

- *The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.*

– Newton
Principia, 1687

- *A constitutive equation is a relation between forces and motions. In popular terms, force is applied to a body to "cause" it to undergo a motion, and the motion "caused" differs according to the nature of the body. In continuum mechanics the forces of interest are contact forces, which are specified by the stress tensor $\boldsymbol{T}$.*

– Clifford Truesdell, 1984

Newton is unequivocal about the fact that force is the cause and motion is the effect as evidenced by the following sentiments:

- *The causes by which true and relative motion are distinguished, one from the other, are the forces impressed upon bodies to generate motion.*

- *The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.*

– Newton
Principia, 1687

- *A constitutive equation is a relation between forces and motions. In popular terms, force is applied to a body to "cause" it to undergo a motion, and the motion "caused" differs according to the nature of the body. In continuum mechanics the forces of interest are contact forces, which are specified by the stress tensor $\boldsymbol{T}$.*

– Clifford Truesdell, 1984

Newton is unequivocal about the fact that force is the cause and motion is the effect as evidenced by the following sentiments:

- *The causes by which true and relative motion are distinguished, one from the other, are the forces impressed upon bodies to generate motion.*

- *The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.*

– Newton
Principia, 1687

- *A constitutive equation is a relation between forces and motions. In popular terms, force is applied to a body to "cause" it to undergo a motion, and the motion "caused" differs according to the nature of the body. In continuum mechanics the forces of interest are contact forces, which are specified by the stress tensor* $\boldsymbol{T}$.

– Clifford Truesdell, 1984

- One should provide expressions for kinematical quantities (effects) in terms of stress (cause).
- This may not be possible, in which case one might have the more complicated situation of relations between causes and effects, which is forces and kinematical quantities.
- In classical theories like linearized elasticity it is done both ways as it is in linearized viscoelasticity.

Example

$\boldsymbol{T\epsilon} = 2\mu\boldsymbol{\epsilon} + \lambda \mathrm{tr}(\boldsymbol{\epsilon})\mathbf{1}$
where λ, μ are the Lame constants.
While the Lame constant μ is the shear modulus and has clear physical underpinning and can be measured directly, the Lame constant λ cannot be measured directly, $(3\lambda + 2\mu)$ has a physical basis, it is the bulk modulus and can be measured directly.

Equivalently, $\boldsymbol{\epsilon} = \frac{1}{1+E}\boldsymbol{T} - \nu \mathrm{tr}(\boldsymbol{T})\mathbf{1}$, E is the Young's modulus and ν is the Poisson's ratio.

- One should provide expressions for kinematical quantities (effects) in terms of stress (cause).
- This may not be possible, in which case one might have the more complicated situation of relations between causes and effects, which is forces and kinematical quantities.
- In classical theories like linearized elasticity it is done both ways as it is in linearized viscoelasticity.

Example

$\boldsymbol{T}\boldsymbol{\epsilon} = 2\mu\boldsymbol{\epsilon} + \lambda \mathrm{tr}(\boldsymbol{\epsilon})\mathbf{1}$
where λ, μ are the Lame constants.
While the Lame constant μ is the shear modulus and has clear physical underpinning and can be measured directly, the Lame constant λ cannot be measured directly, $(3\lambda + 2\mu)$ has a physical basis, it is the bulk modulus and can be measured directly.

Equivalently, $\boldsymbol{\epsilon} = \frac{1}{1+E}\boldsymbol{T} - \nu \mathrm{tr}(\boldsymbol{T})\mathbf{1}$, E is the Young's modulus and ν is the Poisson's ratio.

- One should provide expressions for kinematical quantities (effects) in terms of stress (cause).
- This may not be possible, in which case one might have the more complicated situation of relations between causes and effects, which is forces and kinematical quantities.
- In classical theories like linearized elasticity it is done both ways as it is in linearized viscoelasticity.

Example

$\boldsymbol{T\epsilon} = 2\mu\boldsymbol{\epsilon} + \lambda \mathrm{tr}(\boldsymbol{\epsilon})\mathbf{1}$
where λ, μ are the Lame constants.
While the Lame constant μ is the shear modulus and has clear physical underpinning and can be measured directly, the Lame constant λ cannot be measured directly, $(3\lambda + 2\mu)$ has a physical basis, it is the bulk modulus and can be measured directly.

Equivalently, $\boldsymbol{\epsilon} = \frac{1}{1+E}\boldsymbol{T} - \nu \mathrm{tr}(\boldsymbol{T})\mathbf{1}$, E is the Young's modulus and ν is the Poisson's ratio.

- One should provide expressions for kinematical quantities (effects) in terms of stress (cause).
- This may not be possible, in which case one might have the more complicated situation of relations between causes and effects, which is forces and kinematical quantities.
- In classical theories like linearized elasticity it is done both ways as it is in linearized viscoelasticity.

Example

$\boldsymbol{T\epsilon} = 2\mu\boldsymbol{\epsilon} + \lambda \mathrm{tr}(\boldsymbol{\epsilon})\mathbf{1}$
where λ, μ are the Lame constants.
While the Lame constant μ is the shear modulus and has clear physical underpinning and can be measured directly, the Lame constant λ cannot be measured directly, $(3\lambda + 2\mu)$ has a physical basis, it is the bulk modulus and can be measured directly.

Equivalently, $\boldsymbol{\epsilon} = \frac{1}{1+E}\boldsymbol{T} - \nu \mathrm{tr}(\boldsymbol{T})\mathbf{1}$, E is the Young's modulus and ν is the Poisson's ratio.

One could also do so in the Navier-Stokes theory though it is never done so. In fact, it makes much more sense to do so especially when it comes to enforcing constraints such as incompressibility.

$$\boldsymbol{T} = -p_{\mathrm{th}}(\rho,\theta)\mathbf{1} + \lambda(\rho,\theta)\mathrm{tr}(\boldsymbol{D})\mathbf{1} + 2\mu(\rho,\theta)\boldsymbol{D} \tag{1}$$

Suppose $3\lambda + 2\mu \neq 0$. Then one can rewrite the above as (Rajagopal, 2012)

$$\boldsymbol{D} = \frac{p_{\mathrm{th}}}{3\lambda + 2\mu}\mathbf{1} - \frac{\lambda\mathrm{tr}(\boldsymbol{T})}{2\mu(3\lambda + 2\mu)}\mathbf{1} + \frac{1}{2\mu}\boldsymbol{T} \tag{2}$$

The question is whether $3\lambda + 2\mu$ can be zero. In fact Stokes makes the assumption that it is zero. One can show that this assumption is untenable. It is WRONG (As Scheherazade said "that is another story"). Books like that by Batchelor use incorrect mathematics to be in conformity with the Great Stokes.

One could also do so in the Navier-Stokes theory though it is never done so. In fact, it makes much more sense to do so especially when it comes to enforcing constraints such as incompressibility.

$$\boldsymbol{T} = -p_{\mathrm{th}}(\rho,\theta)\mathbf{1} + \lambda(\rho,\theta)\mathrm{tr}(\boldsymbol{D})\mathbf{1} + 2\mu(\rho,\theta)\boldsymbol{D} \tag{1}$$

Suppose $3\lambda + 2\mu \neq 0$. Then one can rewrite the above as (Rajagopal, 2012)

$$\boldsymbol{D} = \frac{p_{\mathrm{th}}}{3\lambda + 2\mu}\mathbf{1} - \frac{\lambda\mathrm{tr}(\boldsymbol{T})}{2\mu(3\lambda + 2\mu)}\mathbf{1} + \frac{1}{2\mu}\boldsymbol{T} \tag{2}$$

The question is whether $3\lambda + 2\mu$ can be zero. In fact Stokes makes the assumption that it is zero. One can show that this assumption is untenable. It is WRONG (As Scheherazade said "that is another story"). Books like that by Batchelor use incorrect mathematics to be in conformity with the Great Stokes.

One could also do so in the Navier-Stokes theory though it is never done so. In fact, it makes much more sense to do so especially when it comes to enforcing constraints such as incompressibility.

$$\boldsymbol{T} = -p_{\mathrm{th}}(\rho,\theta)\mathbf{1} + \lambda(\rho,\theta)\mathrm{tr}(\boldsymbol{D})\mathbf{1} + 2\mu(\rho,\theta)\boldsymbol{D} \tag{1}$$

Suppose $3\lambda + 2\mu \neq 0$. Then one can rewrite the above as (Rajagopal, 2012)

$$\boldsymbol{D} = \frac{p_{\mathrm{th}}}{3\lambda + 2\mu}\mathbf{1} - \frac{\lambda \mathrm{tr}(\boldsymbol{T})}{2\mu(3\lambda + 2\mu)}\mathbf{1} + \frac{1}{2\mu}\boldsymbol{T} \tag{2}$$

The question is whether $3\lambda + 2\mu$ can be zero. In fact Stokes makes the assumption that it is zero. One can show that this assumption is untenable. It is WRONG (As Scheherazade said "that is another story"). Books like that by Batchelor use incorrect mathematics to be in conformity with the Great Stokes.

One could also do so in the Navier-Stokes theory though it is never done so. In fact, it makes much more sense to do so especially when it comes to enforcing constraints such as incompressibility.

$$\boldsymbol{T} = -p_{\text{th}}(\rho,\theta)\mathbf{1} + \lambda(\rho,\theta)\text{tr}(\boldsymbol{D})\mathbf{1} + 2\mu(\rho,\theta)\boldsymbol{D} \tag{1}$$

Suppose $3\lambda + 2\mu \neq 0$. Then one can rewrite the above as (Rajagopal, 2012)

$$\boldsymbol{D} = \frac{p_{\text{th}}}{3\lambda + 2\mu}\mathbf{1} - \frac{\lambda\text{tr}(\boldsymbol{T})}{2\mu(3\lambda + 2\mu)}\mathbf{1} + \frac{1}{2\mu}\boldsymbol{T} \tag{2}$$

The question is whether $3\lambda + 2\mu$ can be zero. In fact Stokes makes the assumption that it is zero. One can show that this assumption is untenable. It is WRONG (As Scheherazade said "that is another story"). Books like that by Batchelor use incorrect mathematics to be in conformity with the Great Stokes.

One could also do so in the Navier-Stokes theory though it is never done so. In fact, it makes much more sense to do so especially when it comes to enforcing constraints such as incompressibility.

$$\boldsymbol{T} = -p_{\text{th}}(\rho, \theta)\mathbf{1} + \lambda(\rho, \theta)\text{tr}(\boldsymbol{D})\mathbf{1} + 2\mu(\rho, \theta)\boldsymbol{D} \tag{1}$$

Suppose $3\lambda + 2\mu \neq 0$. Then one can rewrite the above as (Rajagopal, 2012)

$$\boldsymbol{D} = \frac{p_{\text{th}}}{3\lambda + 2\mu}\mathbf{1} - \frac{\lambda \text{tr}(\boldsymbol{T})}{2\mu(3\lambda + 2\mu)}\mathbf{1} + \frac{1}{2\mu}\boldsymbol{T} \tag{2}$$

The question is whether $3\lambda + 2\mu$ can be zero. In fact Stokes makes the assumption that it is zero. One can show that this assumption is untenable. It is WRONG (As Scheherazade said "that is another story"). Books like that by Batchelor use incorrect mathematics to be in conformity with the Great Stokes.

Early implicit rate type theories for fluids are due to Maxwell, Burgers and Oldroyd. Implicit theories have been discussed in great generality including implicit theories wherein the material moduli depend on both the invariants of the stress and the velocity gradient has been carried out by Rajagopal (2003), (2006), (2007). A reasonably general implicit model (Prusa and Rajagopal (2012)):

$$\mathfrak{F}_{s=0}^{\infty}\{\rho(t-s), \theta(t-s), \boldsymbol{T}(t-s), \boldsymbol{F}(t-s)\} = \boldsymbol{0} \tag{3}$$

A special sub-class (Rajagopal 2012): $\overset{(n)}{\overset{\nabla}{\boldsymbol{T}}}$

$$\mathfrak{R}\{\rho, \theta, \boldsymbol{T}, \boldsymbol{D}, \overset{\nabla}{\boldsymbol{T}}, \overset{\nabla}{\boldsymbol{D}}, \cdots \overset{(n)}{\overset{\nabla}{\boldsymbol{T}}}, \overset{(n)}{\overset{\nabla}{\boldsymbol{D}}}\} = \boldsymbol{0} \tag{4}$$

where ∇ denotes a frame indifferent material time derivative and $\overset{(n)}{\nabla}$ denotes the frame indifferent n^{th} time derivative. The Navier-Stokes, Maxwell, Oldroyd and Burgers models are special sub-classes of the above model.

Early implicit rate type theories for fluids are due to Maxwell, Burgers and Oldroyd. Implicit theories have been discussed in great generality including implicit theories wherein the material moduli depend on both the invariants of the stress and the velocity gradient has been carried out by Rajagopal (2003), (2006), (2007).

A reasonably general implicit model (Prusa and Rajagopal (2012)):

$$\mathfrak{F}_{s=0}^{\infty}\{\rho(t-s),\theta(t-s),\boldsymbol{T}(t-s),\boldsymbol{F}(t-s)\}=\boldsymbol{0} \tag{3}$$

A special sub-class (Rajagopal 2012): $\overset{(n)}{\overset{\nabla}{\boldsymbol{T}}}$

$$\mathfrak{R}\{\rho,\theta,\boldsymbol{T},\boldsymbol{D},\overset{\nabla}{\boldsymbol{T}},\overset{\nabla}{\boldsymbol{D}},\cdots\overset{(n)}{\overset{\nabla}{\boldsymbol{T}}},\overset{(n)}{\overset{\nabla}{\boldsymbol{D}}}\}=\boldsymbol{0} \tag{4}$$

where ∇ denotes a frame indifferent material time derivative and $\overset{(n)}{\nabla}$ denotes the frame indifferent n^{th} time derivative. The Navier-Stokes, Maxwell, Oldroyd and Burgers models are special sub-classes of the above model.

Early implicit rate type theories for fluids are due to Maxwell, Burgers and Oldroyd. Implicit theories have been discussed in great generality including implicit theories wherein the material moduli depend on both the invariants of the stress and the velocity gradient has been carried out by Rajagopal (2003), (2006), (2007). A reasonably general implicit model (Prusa and Rajagopal (2012)):

$$\mathfrak{F}_{s=0}^{\infty}\{\rho(t-s), \theta(t-s), \boldsymbol{T}(t-s), \boldsymbol{F}(t-s)\} = \mathbf{0} \tag{3}$$

A special sub-class (Rajagopal 2012): $\overset{\overset{(n)}{\nabla}}{\boldsymbol{T}}$

$$\mathfrak{R}\{\rho, \theta, \boldsymbol{T}, \boldsymbol{D}, \overset{\nabla}{\boldsymbol{T}}, \overset{\nabla}{\boldsymbol{D}}, \cdots \overset{\overset{(n)}{\nabla}}{\boldsymbol{T}}, \overset{\overset{(n)}{\nabla}}{\boldsymbol{D}}\} = \mathbf{0} \tag{4}$$

where ∇ denotes a frame indifferent material time derivative and $\overset{(n)}{\nabla}$ denotes the frame indifferent n^{th} time derivative. The Navier-Stokes, Maxwell, Oldroyd and Burgers models are special sub-classes of the above model.

Why consider such implicit models

Consider the following generalization of the Navier-Stokes fluid:

$$\boldsymbol{T} = -p\mathbf{1} + 2\mu(p, \mathrm{tr}(\boldsymbol{D}^2))\boldsymbol{D}, \tag{5}$$

$$\mathrm{tr}(\boldsymbol{D}) = 0 \tag{6}$$

Since $p = -\frac{1}{3}\mathrm{tr}(\boldsymbol{T})$, it belongs to the class of implicit fluid models

$$\boldsymbol{h}(\boldsymbol{T}, \boldsymbol{D}) = \mathbf{0} \tag{7}$$

Why consider such implicit models

Consider the following generalization of the Navier-Stokes fluid:

$$\boldsymbol{T} = -p\boldsymbol{1} + 2\mu(p, \mathrm{tr}(\boldsymbol{D}^2))\boldsymbol{D}, \tag{5}$$

$$\mathrm{tr}(\boldsymbol{D}) = 0 \tag{6}$$

Since $p = -\frac{1}{3}\mathrm{tr}(\boldsymbol{T})$, it belongs to the class of implicit fluid models

$$\boldsymbol{h}(\boldsymbol{T}, \boldsymbol{D}) = \boldsymbol{0} \tag{7}$$

Fluids with pressure dependent viscosity have been studied by several persons: Bulicek, Gazzola, Hron, Kannan, Malek, Prusa, Rajagopal, Renardy, Saccomandi, Srinivasan, and others.

Consider the more general model:

$$\boldsymbol{h}(\rho, \boldsymbol{T}, \boldsymbol{D}) = \boldsymbol{0} \tag{8}$$

Since the fluid is isotropic,

$\boldsymbol{Q}\boldsymbol{h}(\rho, \boldsymbol{T}, \boldsymbol{D})\boldsymbol{Q}^T = \boldsymbol{h}(\rho, \boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^T, \boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}^T) \quad \forall\, \boldsymbol{Q} \in \mathcal{O}$

Thus,

$$\begin{aligned} &\alpha_0 \boldsymbol{1} + \alpha_1 \boldsymbol{T} + \alpha_2 \boldsymbol{D} + \alpha_3 \boldsymbol{T}^2 + \alpha_4 \boldsymbol{D}^2 \\ &+ \alpha_5(\boldsymbol{T}\boldsymbol{D} + \boldsymbol{D}\boldsymbol{T}) + \alpha_6(\boldsymbol{T}^2\boldsymbol{D} + \boldsymbol{D}^2\boldsymbol{T}) \\ &+ \alpha_7(\boldsymbol{T}\boldsymbol{D}^2 + \boldsymbol{D}\boldsymbol{T}^2) + \alpha_8(\boldsymbol{T}^2\boldsymbol{D}^2 + \boldsymbol{D}^2\boldsymbol{T}^2) = \boldsymbol{0} \end{aligned} \tag{9}$$

The material functions $\alpha_i, i = 0 \ldots 8$ depend on the density and the invariants.

$$\begin{aligned} &\mathrm{tr}\mathbf{T},\ \mathrm{tr}\mathbf{D}, \mathrm{tr}\mathbf{T}^2,\ \mathrm{tr}\mathbf{D}^2,\ \mathrm{tr}\mathbf{T}^3,\ \mathrm{tr}\mathbf{D}^3, \\ &\qquad \mathrm{tr}(\mathbf{T}\mathbf{D}),\ \mathrm{tr}(\mathbf{T}^2\mathbf{D}),\ \mathrm{tr}(\mathbf{D}^2\mathbf{T}),\ \mathrm{tr}(\mathbf{T}^2\mathbf{D}^2). \end{aligned}$$

Fluids with pressure dependent viscosity have been studied by several persons: Bulicek, Gazzola, Hron, Kannan, Malek, Prusa, Rajagopal, Renardy, Saccomandi, Srinivasan, and others.
Consider the more general model:

$$\boldsymbol{h}(\rho, \boldsymbol{T}, \boldsymbol{D}) = \boldsymbol{0} \tag{8}$$

Since the fluid is isotropic,
$\boldsymbol{Q}\boldsymbol{h}(\rho, \boldsymbol{T}, \boldsymbol{D})\boldsymbol{Q}^T = \boldsymbol{h}(\rho, \boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^T, \boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}^T) \quad \forall\, \boldsymbol{Q} \in \mathcal{O}$
Thus,

$$\begin{aligned}
&\alpha_0 \boldsymbol{1} + \alpha_1 \boldsymbol{T} + \alpha_2 \boldsymbol{D} + \alpha_3 \boldsymbol{T}^2 + \alpha_4 \boldsymbol{D}^2 \\
&+ \alpha_5 (\boldsymbol{T}\boldsymbol{D} + \boldsymbol{D}\boldsymbol{T}) + \alpha_6 (\boldsymbol{T}^2\boldsymbol{D} + \boldsymbol{D}^2\boldsymbol{T}) \\
&+ \alpha_7 (\boldsymbol{T}\boldsymbol{D}^2 + \boldsymbol{D}\boldsymbol{T}^2) + \alpha_8 (\boldsymbol{T}^2\boldsymbol{D}^2 + \boldsymbol{D}^2\boldsymbol{T}^2) = \boldsymbol{0}
\end{aligned} \tag{9}$$

The material functions $\alpha_i, i = 0 \ldots 8$ depend on the density and the invariants.

$$\mathrm{tr}\mathbf{T},\ \mathrm{tr}\mathbf{D},\ \mathrm{tr}\mathbf{T}^2,\ \mathrm{tr}\mathbf{D}^2,\ \mathrm{tr}\mathbf{T}^3,\ \mathrm{tr}\mathbf{D}^3,$$
$$\mathrm{tr}(\mathbf{T}\mathbf{D}),\ \mathrm{tr}(\mathbf{T}^2\mathbf{D}),\ \mathrm{tr}(\mathbf{D}^2\mathbf{T}),\ \mathrm{tr}(\mathbf{T}^2\mathbf{D}^2).$$

Fluids with pressure dependent viscosity have been studied by several persons: Bulicek, Gazzola, Hron, Kannan, Malek, Prusa, Rajagopal, Renardy, Saccomandi, Srinivasan, and others.
Consider the more general model:

$$\boldsymbol{h}(\rho, \boldsymbol{T}, \boldsymbol{D}) = \boldsymbol{0} \tag{8}$$

Since the fluid is isotropic,
$\boldsymbol{Q}\boldsymbol{h}(\rho, \boldsymbol{T}, \boldsymbol{D})\boldsymbol{Q}^T = \boldsymbol{h}(\rho, \boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^T, \boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}^T) \quad \forall\, \boldsymbol{Q} \in \mathcal{O}$
Thus,

$$\begin{aligned} &\alpha_0 \mathbf{1} + \alpha_1 \boldsymbol{T} + \alpha_2 \boldsymbol{D} + \alpha_3 \boldsymbol{T}^2 + \alpha_4 \boldsymbol{D}^2 \\ &+ \alpha_5(\boldsymbol{T}\boldsymbol{D} + \boldsymbol{D}\boldsymbol{T}) + \alpha_6(\boldsymbol{T}^2\boldsymbol{D} + \boldsymbol{D}^2\boldsymbol{T}) \\ &+ \alpha_7(\boldsymbol{T}\boldsymbol{D}^2 + \boldsymbol{D}\boldsymbol{T}^2) + \alpha_8(\boldsymbol{T}^2\boldsymbol{D}^2 + \boldsymbol{D}^2\boldsymbol{T}^2) = \boldsymbol{0} \end{aligned} \tag{9}$$

The material functions $\alpha_i, i = 0 \ldots 8$ depend on the density and the invariants.

$$\begin{aligned} &\mathrm{tr}\mathbf{T},\ \mathrm{tr}\mathbf{D}, \mathrm{tr}\mathbf{T}^2,\ \mathrm{tr}\mathbf{D}^2,\ \mathrm{tr}\mathbf{T}^3,\ \mathrm{tr}\mathbf{D}^3, \\ &\qquad \mathrm{tr}(\mathbf{T}\mathbf{D}),\ \mathrm{tr}(\mathbf{T}^2\mathbf{D}),\ \mathrm{tr}(\mathbf{D}^2\mathbf{T}),\ \mathrm{tr}(\mathbf{T}^2\mathbf{D}^2). \end{aligned}$$

- Unknowns:
  - Stress - six scalars since it is symmetric
  - Velocity - three scalars
  - Density - one scalar
- Total of 10 unknowns
- Constitutive relations: six scalar equations
- Balance of mass: one sclar equation
- Balance of linear momentum: three scalar equations
- Total number of equations: 10

- Unknowns:
  - Stress - six scalars since it is symmetric
  - Velocity - three scalars
  - Density - one scalar
- Total of 10 unknowns
- Constitutive relations: six scalar equations
- Balance of mass: one sclar equation
- Balance of linear momentum: three scalar equations
- Total number of equations: 10

- Unknowns:
  - Stress - six scalars since it is symmetric
  - Velocity - three scalars
  - Density - one scalar
- Total of 10 unknowns
- Constitutive relations: six scalar equations
- Balance of mass: one sclar equation
- Balance of linear momentum: three scalar equations
- Total number of equations: 10

- Unknowns:
    - Stress - six scalars since it is symmetric
    - Velocity - three scalars
    - Density - one scalar
- Total of 10 unknowns
- Constitutive relations: six scalar equations
- Balance of mass: one sclar equation
- Balance of linear momentum: three scalar equations
- Total number of equations: 10

- Unknowns:
  - Stress - six scalars since it is symmetric
  - Velocity - three scalars
  - Density - one scalar
- Total of 10 unknowns
- Constitutive relations: six scalar equations
- Balance of mass: one sclar equation
- Balance of linear momentum: three scalar equations
- Total number of equations: 10

- Unknowns:
  - Stress - six scalars since it is symmetric
  - Velocity - three scalars
  - Density - one scalar
- Total of 10 unknowns
- Constitutive relations: six scalar equations
- Balance of mass: one sclar equation
- Balance of linear momentum: three scalar equations
- Total number of equations: 10

- Unknowns:
  - Stress - six scalars since it is symmetric
  - Velocity - three scalars
  - Density - one scalar
- Total of 10 unknowns
- Constitutive relations: six scalar equations
- Balance of mass: one sclar equation
- Balance of linear momentum: three scalar equations
- Total number of equations: 10

- Unknowns:
  - Stress - six scalars since it is symmetric
  - Velocity - three scalars
  - Density - one scalar
- Total of 10 unknowns
- Constitutive relations: six scalar equations
- Balance of mass: one sclar equation
- Balance of linear momentum: three scalar equations
- Total number of equations: 10

- Unknowns:
    - Stress - six scalars since it is symmetric
    - Velocity - three scalars
    - Density - one scalar
- Total of 10 unknowns
- Constitutive relations: six scalar equations
- Balance of mass: one sclar equation
- Balance of linear momentum: three scalar equations
- Total number of equations: 10

The insidiuous effect of mathematics on physics

- When one has a constitutive expression for the stress in terms of either the density and the displacement (solids) or velocity (fluids), and substitues this expression into the balance of linear momentum, one has just the balance of linear momentum and the balance of mass (four equations) for the density and either displacement or velocity (four equations).
- However, one increased the order of the equation!
- Causality has been turned on its head!
- Hammer Syndrome!

The insidiuous effect of mathematics on physics

- When one has a constitutive expression for the stress in terms of either the density and the displacement (solids) or velocity (fluids), and substitues this expression into the balance of linear momentum, one has just the balance of linear momentum and the balance of mass (four equations) for the density and either displacement or velocity (four equations).
- However, one increased the order of the equation!
- Causality has been turned on its head!
- Hammer Syndrome!

The insidiuous effect of mathematics on physics

- When one has a constitutive expression for the stress in terms of either the density and the displacement (solids) or velocity (fluids), and substitues this expression into the balance of linear momentum, one has just the balance of linear momentum and the balance of mass (four equations) for the density and either displacement or velocity (four equations).
- However, one increased the order of the equation!
- Causality has been turned on its head!
- Hammer Syndrome!

The insidiuous effect of mathematics on physics

- When one has a constitutive expression for the stress in terms of either the density and the displacement (solids) or velocity (fluids), and substitues this expression into the balance of linear momentum, one has just the balance of linear momentum and the balance of mass (four equations) for the density and either displacement or velocity (four equations).
- However, one increased the order of the equation!
- Causality has been turned on its head!
- Hammer Syndrome!

Consider how an incompressible Navier-Stokes fluid is expressed:

$$\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D}, \tag{10}$$

$$\mathrm{tr}(\mathbf{D}) = 0. \tag{11}$$

We have introduced a "Lagrange multiplier" (constraint reaction) "p" which we know nothing about. We do not know the space which it belongs to, etc.

Consider how an incompressible Navier-Stokes fluid is expressed:

$$\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D}, \tag{10}$$

$$\mathrm{tr}(\mathbf{D}) = 0. \tag{11}$$

We have introduced a "Lagrange multiplier" (constraint reaction) "p" which we know nothing about. We do not know the space which it belongs to, etc.

Suppose we start with (Srinivasa and Rajagopal (2012))

$$\mathbf{L} := \frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \mathbf{f}(\rho, \mathbf{T}). \tag{12}$$

Balance of angular momentum (symmetry of stress) and Galilean Invariance leads to

$$\mathbf{D} = \mathbf{f}(\rho, \theta, \mathbf{T}). \tag{13}$$

Isotropy of the fluid leads to

$$\mathbf{f}(\rho, \theta, \mathbf{Q}\mathbf{T}\mathbf{Q}^T) = \mathbf{Q}\mathbf{f}(\rho, \theta, \mathbf{T})\mathbf{Q}^T \ \forall \mathbf{Q} \in \mathcal{O}, \tag{14}$$

and representation theorems lead to

$$\mathbf{D} = \gamma_1 \mathbf{1} + \gamma_2 \mathbf{T} + \gamma_3 \mathbf{T}^2, \tag{15}$$

where the $\gamma_i, \ i = 1, 2, 3$ depend on ρ, θ, $\mathrm{tr}\mathbf{T}$, $\mathrm{tr}\mathbf{T}^2$ and $\mathrm{tr}\mathbf{T}^3$.

Suppose we start with (Srinivasa and Rajagopal (2012))

$$\mathbf{L} := \frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \mathbf{f}(\rho, \mathbf{T}). \tag{12}$$

Balance of angular momentum (symmetry of stress) and Galilean Invariance leads to

$$\mathbf{D} = \mathbf{f}(\rho, \theta, \mathbf{T}). \tag{13}$$

Isotropy of the fluid leads to

$$\mathbf{f}(\rho, \theta, \mathbf{Q}\mathbf{T}\mathbf{Q}^T) = \mathbf{Q}\mathbf{f}(\rho, \theta, \mathbf{T})\mathbf{Q}^T \ \forall \mathbf{Q} \in \mathcal{O}, \tag{14}$$

and representation theorems lead to

$$\mathbf{D} = \gamma_1 \mathbf{1} + \gamma_2 \mathbf{T} + \gamma_3 \mathbf{T}^2, \tag{15}$$

where the $\gamma_i,\ i = 1, 2, 3$ depend on ρ, θ, $\mathrm{tr}\mathbf{T}$, $\mathrm{tr}\mathbf{T}^2$ and $\mathrm{tr}\mathbf{T}^3$.

Suppose we start with (Srinivasa and Rajagopal (2012))

$$\mathbf{L} := \frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \mathbf{f}(\rho, \mathbf{T}). \tag{12}$$

Balance of angular momentum (symmetry of stress) and Galilean Invariance leads to

$$\mathbf{D} = \mathbf{f}(\rho, \theta, \mathbf{T}). \tag{13}$$

Isotropy of the fluid leads to

$$\mathbf{f}(\rho, \theta, \mathbf{Q}\mathbf{T}\mathbf{Q}^T) = \mathbf{Q}\mathbf{f}(\rho, \theta, \mathbf{T})\mathbf{Q}^T \ \forall \mathbf{Q} \in \mathcal{O}, \tag{14}$$

and representation theorems lead to

$$\mathbf{D} = \gamma_1 \mathbf{1} + \gamma_2 \mathbf{T} + \gamma_3 \mathbf{T}^2, \tag{15}$$

where the $\gamma_i,\ i = 1, 2, 3$ depend on ρ, θ, $\mathrm{tr}\mathbf{T}$, $\mathrm{tr}\mathbf{T}^2$ and $\mathrm{tr}\mathbf{T}^3$.

Suppose we start with (Srinivasa and Rajagopal (2012))

$$\mathbf{L} := \frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \mathbf{f}(\rho, \mathbf{T}). \tag{12}$$

Balance of angular momentum (symmetry of stress) and Galilean Invariance leads to

$$\mathbf{D} = \mathbf{f}(\rho, \theta, \mathbf{T}). \tag{13}$$

Isotropy of the fluid leads to

$$\mathbf{f}(\rho, \theta, \mathbf{Q}\mathbf{T}\mathbf{Q}^T) = \mathbf{Q}\mathbf{f}(\rho, \theta, \mathbf{T})\mathbf{Q}^T \ \forall \mathbf{Q} \in \mathcal{O}, \tag{14}$$

and representation theorems lead to

$$\mathbf{D} = \gamma_1 \mathbf{1} + \gamma_2 \mathbf{T} + \gamma_3 \mathbf{T}^2, \tag{15}$$

where the $\gamma_i, \ i = 1, 2, 3$ depend on ρ, θ, $\mathrm{tr}\mathbf{T}$, $\mathrm{tr}\mathbf{T}^2$ and $\mathrm{tr}\mathbf{T}^3$.

Linearity in the stress leads to

$$\mathbf{D} = [\phi_0(\rho)]\mathbf{I} + [\phi_1(\rho)](\mathrm{tr}\mathbf{T})\mathbf{I} + [\phi_2(\rho)]\mathbf{T} \tag{16}$$

Starting with this model one can show that the Stokes assumption is incorrect.

To describe an incompressible fluid within the above context, the constitutive relation would be

$$\mathbf{D} = \alpha\left(\mathbf{T} - \frac{1}{3}(\mathrm{tr}\mathbf{T})\right). \tag{17}$$

Notice that

$$\mathrm{tr}\mathbf{D} = 0. \tag{18}$$

Linearity in the stress leads to

$$\mathbf{D} = [\phi_0(\rho)]\mathbf{I} + [\phi_1(\rho)](\mathrm{tr}\mathbf{T})\mathbf{I} + [\phi_2(\rho)]\mathbf{T} \tag{16}$$

Starting with this model one can show that the Stokes assumption is incorrect.

To describe an incompressible fluid within the above context, the constitutive relation would be

$$\mathbf{D} = \alpha\left(\mathbf{T} - \frac{1}{3}(\mathrm{tr}\mathbf{T})\right). \tag{17}$$

Notice that

$$\mathrm{tr}\mathbf{D} = 0. \tag{18}$$

Linearity in the stress leads to

$$\mathbf{D} = [\phi_0(\rho)]\mathbf{I} + [\phi_1(\rho)](\mathrm{tr}\mathbf{T})\mathbf{I} + [\phi_2(\rho)]\mathbf{T} \tag{16}$$

Starting with this model one can show that the Stokes assumption is incorrect.

To describe an incompressible fluid within the above context, the constitutive relation would be

$$\mathbf{D} = \alpha\left(\mathbf{T} - \frac{1}{3}(\mathrm{tr}\mathbf{T})\right). \tag{17}$$

Notice that

$$\mathrm{tr}\mathbf{D} = 0. \tag{18}$$

Linearity in the stress leads to

$$\mathbf{D} = [\phi_0(\rho)]\mathbf{I} + [\phi_1(\rho)](\mathrm{tr}\mathbf{T})\mathbf{I} + [\phi_2(\rho)]\mathbf{T} \tag{16}$$

Starting with this model one can show that the Stokes assumption is incorrect.

To describe an incompressible fluid within the above context, the constitutive relation would be

$$\mathbf{D} = \alpha\left(\mathbf{T} - \frac{1}{3}(\mathrm{tr}\mathbf{T})\right). \tag{17}$$

Notice that

$$\mathrm{tr}\mathbf{D} = 0. \tag{18}$$

Leads to exceedingly interesting models for solid behavior. We will only consider elastic response.

Classical Cauchy elastic body:

$$\mathbf{T} = \delta_1 \mathbf{1} + \delta_2 \mathbf{B} + \delta_3 \mathbf{B}^2, \tag{19}$$

where the $\delta_i,\ i = 1, 2, 3$ depend on ρ, θ, $\mathrm{tr}\mathbf{B}$, $\mathrm{tr}\mathbf{B}^2$ and $\mathrm{tr}\mathbf{B}^3$.

Let us consider an implicit constitutive relation of the form

$$\mathbf{f}(\mathbf{T}, \mathbf{B}) = 0. \tag{20}$$

Standard arguments in the case of isotropic bodies leads to

$$\begin{aligned} &\alpha_0 \mathbf{1} + \alpha_1 \mathbf{T} + \alpha_2 \mathbf{B} + \alpha_3 \mathbf{T}^2 + \alpha_4 \mathbf{B}^2 \\ &+ \alpha_5 (\mathbf{TB} + \mathbf{BT}) + \alpha_6 (\mathbf{T}^2 \mathbf{B} + \mathbf{B} \mathbf{T}^2) \\ &+ \alpha_7 (\mathbf{B}^2 \mathbf{T} + \mathbf{T} \mathbf{B}^2) + \alpha_8 (\mathbf{T}^2 \mathbf{B}^2 + \mathbf{B}^2 \mathbf{T}^2) = \mathbf{0}, \end{aligned} \tag{21}$$

where the material moduli $\alpha_i,\ i = 0, \ldots, 8$ depend upon

$$\rho,\ \mathrm{tr}\mathbf{T},\ \mathrm{tr}\mathbf{B}, \mathrm{tr}\mathbf{T}^2,\ \mathrm{tr}\mathbf{B}^2,\ \mathrm{tr}\mathbf{T}^3,\ \mathrm{tr}\mathbf{B}^3,$$
$$\mathrm{tr}(\mathbf{TB}),\ \mathrm{tr}(\mathbf{T}^2\mathbf{B}),\ \mathrm{tr}(\mathbf{B}^2\mathbf{T}),\ \mathrm{tr}(\mathbf{T}^2\mathbf{B}^2).$$

Leads to exceedingly interesting models for solid behavior. We will only consider elastic response.
Classical Cauchy elastic body:

$$\mathbf{T} = \delta_1 \mathbf{1} + \delta_2 \mathbf{B} + \delta_3 \mathbf{B}^2, \tag{19}$$

where the $\delta_i,\ i = 1, 2, 3$ depend on ρ, θ, $\mathrm{tr}\mathbf{B}$, $\mathrm{tr}\mathbf{B}^2$ and $\mathrm{tr}\mathbf{B}^3$.

Let us consider an implicit constitutive relation of the form

$$\mathbf{f}(\mathbf{T}, \mathbf{B}) = 0. \tag{20}$$

Standard arguments in the case of isotropic bodies leads to

$$\begin{aligned} &\alpha_0 \mathbf{1} + \alpha_1 \mathbf{T} + \alpha_2 \mathbf{B} + \alpha_3 \mathbf{T}^2 + \alpha_4 \mathbf{B}^2 \\ &+ \alpha_5(\mathbf{TB} + \mathbf{BT}) + \alpha_6(\mathbf{T}^2\mathbf{B} + \mathbf{BT}^2) \\ &+ \alpha_7(\mathbf{B}^2\mathbf{T} + \mathbf{TB}^2) + \alpha_8(\mathbf{T}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{T}^2) = \mathbf{0}, \end{aligned} \tag{21}$$

where the material moduli $\alpha_i,\ i = 0, \ldots, 8$ depend upon

$$\begin{aligned} &\rho,\ \mathrm{tr}\mathbf{T},\ \mathrm{tr}\mathbf{B}, \mathrm{tr}\mathbf{T}^2,\ \mathrm{tr}\mathbf{B}^2,\ \mathrm{tr}\mathbf{T}^3,\ \mathrm{tr}\mathbf{B}^3, \\ &\qquad \mathrm{tr}(\mathbf{TB}),\ \mathrm{tr}(\mathbf{T}^2\mathbf{B}),\ \mathrm{tr}(\mathbf{B}^2\mathbf{T}),\ \mathrm{tr}(\mathbf{T}^2\mathbf{B}^2). \end{aligned}$$

Leads to exceedingly interesting models for solid behavior. We will only consider elastic response.

Classical Cauchy elastic body:

$$\mathbf{T} = \delta_1 \mathbf{1} + \delta_2 \mathbf{B} + \delta_3 \mathbf{B}^2, \tag{19}$$

where the $\delta_i,\ i = 1, 2, 3$ depend on ρ, θ, $\mathrm{tr}\mathbf{B}$, $\mathrm{tr}\mathbf{B}^2$ and $\mathrm{tr}\mathbf{B}^3$.

Let us consider an implicit constitutive relation of the form

$$\mathbf{f}(\mathbf{T}, \mathbf{B}) = 0. \tag{20}$$

Standard arguments in the case of isotropic bodies leads to

$$\begin{aligned}
&\alpha_0 \mathbf{1} + \alpha_1 \mathbf{T} + \alpha_2 \mathbf{B} + \alpha_3 \mathbf{T}^2 + \alpha_4 \mathbf{B}^2 \\
&+ \alpha_5(\mathbf{TB} + \mathbf{BT}) + \alpha_6(\mathbf{T}^2\mathbf{B} + \mathbf{B}\mathbf{T}^2) \\
&+ \alpha_7(\mathbf{B}^2\mathbf{T} + \mathbf{T}\mathbf{B}^2) + \alpha_8(\mathbf{T}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{T}^2) = \mathbf{0},
\end{aligned} \tag{21}$$

where the material moduli $\alpha_i,\ i = 0, \ldots, 8$ depend upon

$$\rho,\ \mathrm{tr}\mathbf{T},\ \mathrm{tr}\mathbf{B},\ \mathrm{tr}\mathbf{T}^2,\ \mathrm{tr}\mathbf{B}^2,\ \mathrm{tr}\mathbf{T}^3,\ \mathrm{tr}\mathbf{B}^3,$$
$$\mathrm{tr}(\mathbf{TB}),\ \mathrm{tr}(\mathbf{T}^2\mathbf{B}),\ \mathrm{tr}(\mathbf{B}^2\mathbf{T}),\ \mathrm{tr}(\mathbf{T}^2\mathbf{B}^2).$$

Leads to exceedingly interesting models for solid behavior. We will only consider elastic response.
Classical Cauchy elastic body:

$$\mathbf{T} = \delta_1 \mathbf{1} + \delta_2 \mathbf{B} + \delta_3 \mathbf{B}^2, \tag{19}$$

where the $\delta_i,\ i = 1, 2, 3$ depend on ρ, θ, $\mathrm{tr}\mathbf{B}$, $\mathrm{tr}\mathbf{B}^2$ and $\mathrm{tr}\mathbf{B}^3$.

Let us consider an implicit constitutive relation of the form

$$\mathbf{f}(\mathbf{T}, \mathbf{B}) = 0. \tag{20}$$

Standard arguments in the case of isotropic bodies leads to

$$\begin{aligned} &\alpha_0 \mathbf{1} + \alpha_1 \mathbf{T} + \alpha_2 \mathbf{B} + \alpha_3 \mathbf{T}^2 + \alpha_4 \mathbf{B}^2 \\ &+ \alpha_5 (\mathbf{T}\mathbf{B} + \mathbf{B}\mathbf{T}) + \alpha_6 (\mathbf{T}^2 \mathbf{B} + \mathbf{B}\mathbf{T}^2) \\ &+ \alpha_7 (\mathbf{B}^2 \mathbf{T} + \mathbf{T}\mathbf{B}^2) + \alpha_8 (\mathbf{T}^2 \mathbf{B}^2 + \mathbf{B}^2 \mathbf{T}^2) = \mathbf{0}, \end{aligned} \tag{21}$$

where the material moduli $\alpha_i,\ i = 0, \ldots, 8$ depend upon

$$\rho,\ \mathrm{tr}\mathbf{T},\ \mathrm{tr}\mathbf{B},\ \mathrm{tr}\mathbf{T}^2,\ \mathrm{tr}\mathbf{B}^2,\ \mathrm{tr}\mathbf{T}^3,\ \mathrm{tr}\mathbf{B}^3,$$
$$\mathrm{tr}(\mathbf{T}\mathbf{B}),\ \mathrm{tr}(\mathbf{T}^2\mathbf{B}),\ \mathrm{tr}(\mathbf{B}^2\mathbf{T}),\ \mathrm{tr}(\mathbf{T}^2\mathbf{B}^2).$$

Notice a model of the form

$$\mathbf{B} = \bar{\alpha}_0 \mathbf{1} + \bar{\alpha}_1 \mathbf{T} + \bar{\alpha}_2 \mathbf{T}^2, \tag{22}$$

is also a subclass of the above implicit equation.

Suppose we require that

$$\max_{\mathbf{X} \in \kappa(B),\ t \in \mathbb{R}} ||\nabla_{\boldsymbol{x}} \mathbf{u}|| = O(\delta),\ \delta << 1, \tag{23}$$

where $||.||$ stands for the usual trace norm, induced through the scalar product.

It follows that

$$\mathbf{B} = \mathbf{1} + 2\boldsymbol{\epsilon} + \mathbf{O}(\delta^2). \tag{24}$$

In the case of a Cauchy elastic body we are inexorably led to

$$\mathbf{T} = \lambda(\text{tr}\boldsymbol{\epsilon})\mathbf{1} + 2\mu\boldsymbol{\epsilon}. \tag{25}$$

We have a linear relationship between the stress and the strain.

Notice a model of the form

$$\mathbf{B} = \bar{\alpha}_0 \mathbf{1} + \bar{\alpha}_1 \mathbf{T} + \bar{\alpha}_2 \mathbf{T}^2, \tag{22}$$

is also a subclass of the above implicit equation.

Suppose we require that

$$\max_{\mathbf{X} \in \kappa(B),\ t \in \mathbb{R}} ||\nabla_{\boldsymbol{x}} \mathbf{u}|| = O(\delta),\ \delta << 1, \tag{23}$$

where $||.||$ stands for the usual trace norm, induced through the scalar product.

It follows that

$$\mathbf{B} = \mathbf{1} + 2\boldsymbol{\epsilon} + \mathbf{O}(\delta^2). \tag{24}$$

In the case of a Cauchy elastic body we are inexorably led to

$$\mathbf{T} = \lambda(\mathrm{tr}\boldsymbol{\epsilon})\mathbf{1} + 2\mu\boldsymbol{\epsilon}. \tag{25}$$

We have a linear relationship between the stress and the strain.

Notice a model of the form

$$\mathbf{B} = \bar{\alpha}_0\mathbf{1} + \bar{\alpha}_1\mathbf{T} + \bar{\alpha}_2\mathbf{T}^2, \tag{22}$$

is also a subclass of the above implicit equation.

Suppose we require that

$$\max_{\mathbf{X}\in\kappa(B),\ t\in\mathbb{R}} ||\nabla_{\boldsymbol{x}}\mathbf{u}|| = O(\delta),\ \delta << 1, \tag{23}$$

where $||.||$ stands for the usual trace norm, induced through the scalar product.

It follows that

$$\mathbf{B} = \mathbf{1} + 2\boldsymbol{\epsilon} + \mathbf{O}(\delta^2). \tag{24}$$

In the case of a Cauchy elastic body we are inexorably led to

$$\mathbf{T} = \lambda(\text{tr}\boldsymbol{\epsilon})\mathbf{1} + 2\mu\boldsymbol{\epsilon}. \tag{25}$$

We have a linear relationship between the stress and the strain.

Notice a model of the form

$$\mathbf{B} = \bar{\alpha}_0 \mathbf{1} + \bar{\alpha}_1 \mathbf{T} + \bar{\alpha}_2 \mathbf{T}^2, \tag{22}$$

is also a subclass of the above implicit equation.

Suppose we require that

$$\max_{\mathbf{X} \in \kappa(B),\ t \in \mathbb{R}} ||\nabla_{\boldsymbol{x}} \mathbf{u}|| = O(\delta),\ \delta << 1, \tag{23}$$

where $||.||$ stands for the usual trace norm, induced through the scalar product.

It follows that

$$\mathbf{B} = \mathbf{1} + 2\boldsymbol{\epsilon} + \mathbf{O}(\delta^2). \tag{24}$$

In the case of a Cauchy elastic body we are inexorably led to

$$\mathbf{T} = \lambda(\mathrm{tr}\boldsymbol{\epsilon})\mathbf{1} + 2\mu\boldsymbol{\epsilon}. \tag{25}$$

We have a linear relationship between the stress and the strain.

Notice a model of the form

$$\mathbf{B} = \bar{\alpha}_0 \mathbf{1} + \bar{\alpha}_1 \mathbf{T} + \bar{\alpha}_2 \mathbf{T}^2, \tag{22}$$

is also a subclass of the above implicit equation.

Suppose we require that

$$\max_{\mathbf{X} \in \kappa(B),\ t \in \mathbb{R}} ||\nabla_{\boldsymbol{x}} \mathbf{u}|| = O(\delta),\ \delta << 1, \tag{23}$$

where $||.||$ stands for the usual trace norm, induced through the scalar product.

It follows that

$$\mathbf{B} = \mathbf{1} + 2\boldsymbol{\epsilon} + \mathbf{O}(\delta^2). \tag{24}$$

In the case of a Cauchy elastic body we are inexorably led to

$$\mathbf{T} = \lambda(\mathrm{tr}\boldsymbol{\epsilon})\mathbf{1} + 2\mu\boldsymbol{\epsilon}. \tag{25}$$

We have a linear relationship between the stress and the strain.

Linearization of the implicit model leads to

$$\boldsymbol{\epsilon} + \hat{\alpha_1}\mathbf{1} + \hat{\alpha_2}\mathbf{T} + \hat{\alpha_3}\mathbf{T}^2 + \hat{\alpha_4}[\mathbf{T}\boldsymbol{\epsilon} + \boldsymbol{\epsilon}\mathbf{T}] + \hat{\alpha_5}[\mathbf{T}^2\boldsymbol{\epsilon} + \boldsymbol{\epsilon}\mathbf{T}^2] = \mathbf{0}. \quad (26)$$

- We have a non-linear relationship between the linearized strain and the stress!!
- Has tremendous applications in Fracture Mechanics.
- Even when we linearize (*) we obtain

$$\boldsymbol{\epsilon} = \bar{\beta}_0\mathbf{1} + \bar{\beta}_1\mathbf{T} + \bar{\beta}_2\mathbf{T}^2 \quad (27)$$

which is yet a non-linear relationship between the linearized strain and the stress!

Linearization of the implicit model leads to

$$\boldsymbol{\epsilon} + \hat{\alpha_1}\mathbf{1} + \hat{\alpha_2}\mathbf{T} + \hat{\alpha_3}\mathbf{T}^2 + \hat{\alpha_4}[\mathbf{T}\boldsymbol{\epsilon} + \boldsymbol{\epsilon}\mathbf{T}] + \hat{\alpha_5}[\mathbf{T}^2\boldsymbol{\epsilon} + \boldsymbol{\epsilon}\mathbf{T}^2] = \mathbf{0}. \quad (26)$$

- We have a non-linear relationship between the linearized strain and the stress!!
- Has tremendous applications in Fracture Mechanics.
- Even when we linearize (*) we obtain

$$\boldsymbol{\epsilon} = \bar{\beta}_0\mathbf{1} + \bar{\beta}_1\mathbf{T} + \bar{\beta}_2\mathbf{T}^2 \quad (27)$$

which is yet a non-linear relationship between the linearized strain and the stress!

Linearization of the implicit model leads to

$$\boldsymbol{\epsilon} + \hat{\alpha_1}\mathbf{1} + \hat{\alpha_2}\mathbf{T} + \hat{\alpha_3}\mathbf{T}^2 + \hat{\alpha_4}[\mathbf{T}\boldsymbol{\epsilon} + \boldsymbol{\epsilon}\mathbf{T}] + \hat{\alpha_5}[\mathbf{T}^2\boldsymbol{\epsilon} + \boldsymbol{\epsilon}\mathbf{T}^2] = \mathbf{0}. \quad (26)$$

- We have a non-linear relationship between the linearized strain and the stress!!
- Has tremendous applications in Fracture Mechanics.
- Even when we linearize (*) we obtain

$$\boldsymbol{\epsilon} = \bar{\beta}_0\mathbf{1} + \bar{\beta}_1\mathbf{T} + \bar{\beta}_2\mathbf{T}^2 \quad (27)$$

which is yet a non-linear relationship between the linearized strain and the stress!

Linearization of the implicit model leads to

$$\boldsymbol{\epsilon} + \hat{\alpha_1}\mathbf{1} + \hat{\alpha_2}\mathbf{T} + \hat{\alpha_3}\mathbf{T}^2 + \hat{\alpha_4}[\mathbf{T}\boldsymbol{\epsilon} + \boldsymbol{\epsilon}\mathbf{T}] + \hat{\alpha_5}[\mathbf{T}^2\boldsymbol{\epsilon} + \boldsymbol{\epsilon}\mathbf{T}^2] = \mathbf{0}. \quad (26)$$

- We have a non-linear relationship between the linearized strain and the stress!!
- Has tremendous applications in Fracture Mechanics.
- Even when we linearize (*) we obtain

$$\boldsymbol{\epsilon} = \bar{\beta}_0\mathbf{1} + \bar{\beta}_1\mathbf{T} + \bar{\beta}_2\mathbf{T}^2 \quad (27)$$

which is yet a non-linear relationship between the linearized strain and the stress!

- Can show that strains can be bounded at crack tip for the anti-plane stress problem (Rajagopal and Walton (2011)).
- Can show the strain is bounded at the tip of a V-notch (Kulvait, Malek and Rajagopal (2012)).

- Can show that strains can be bounded at crack tip for the anti-plane stress problem (Rajagopal and Walton (2011)).
- Can show the strain is bounded at the tip of a V-notch (Kulvait, Malek and Rajagopal (2012)).

- Implicit constitutive theories are the best way in which to present constrained materials.
- For instance if one wants to model a fluid that is incompressible to mechanical stimuli but is expansible or compressible to thermal stimuli, it can be represented, in the case of the Navier-Stokes Fourier fluid, very simply as (Rajagopal (2012))

$$\mathbf{D} = \beta^f(\theta)\left(\mathbf{T} - \frac{1}{3}\mathrm{tr}(\mathbf{T})\mathbf{1}\right) + \frac{1}{3}\alpha^f(\theta)\dot{\theta}\mathbf{1} \qquad (28)$$

The celebrated Oberbeck-Boussinesq approximation follows very cleanly from the above representation. Practically all the justifications using the classical approach are either wrong or most convoluted.

- Implicit constitutive theories are the best way in which to present constrained materials.
- For instance if one wants to model a fluid that is incompressible to mechanical stimuli but is expansible or compressible to thermal stimuli, it can be represented, in the case of the Navier-Stokes Fourier fluid, very simply as (Rajagopal (2012))

$$\mathbf{D} = \beta^f(\theta)\left(\mathbf{T} - \frac{1}{3}\mathrm{tr}(\mathbf{T})\mathbf{1}\right) + \frac{1}{3}\alpha^f(\theta)\dot{\theta}\mathbf{1} \tag{28}$$

  The celebrated Oberbeck-Boussinesq approximation follows very cleanly from the above representation. Practically all the justifications using the classical approach are either wrong or most convoluted.

- The above approach can be extended to solids. In the case of an elastic solid that is incompressible with regard to mechanical stimuli but can expand or contract due to thermal stimuli, and which can undergo only small displacement gradients, the model becomes once again simple and elegant:

$$\boldsymbol{\epsilon} = \gamma_1^s(\rho, \theta, I_1, I_2, I_3)\mathbf{T}_d + \gamma_2^s(\rho, \theta, I_1, I_2, I_3)\mathbf{T}_d^2 + \frac{1}{3}\alpha^f(\theta)\dot{\theta}\mathbf{1} \tag{29}$$

where $\mathbf{T}_d$ and $\mathbf{T}_d^2$ are the deviatoric parts of the tensors.

- The above constraint can be generalized to non-linear elastic solids, but the model is more complicated.

- The above approach can be extended to solids. In the case of an elastic solid that is incompressible with regard to mechanical stimuli but can expand or contract due to thermal stimuli, and which can undergo only small displacement gradients, the model becomes once again simple and elegant:

$$\boldsymbol{\epsilon} = \gamma_1^s(\rho, \theta, I_1, I_2, I_3)\mathbf{T}_d + \gamma_2^s(\rho, \theta, I_1, I_2, I_3)\mathbf{T}_d^2 + \frac{1}{3}\alpha^f(\theta)\dot{\theta}\mathbf{1} \tag{29}$$

where $\mathbf{T}_d$ and $\mathbf{T}_d^2$ are the deviatoric parts of the tensors.

- The above constraint can be generalized to non-linear elastic solids, but the model is more complicated.

- *Aristotle has said that 'the sweetest of all things is knowledge. And he is right. But if you were to suppose that the publication of a new view were productive of unbounded sweetness, you would be highly mistaken. No one disturbs his fellow man with a new view unpunished.*

– E. Mach

- *Most people would rather die than think. Most do.*

– B. Russell

- *Everything of importance has been said by somebody who did not discover it.*

– A. N. Whitehead

- *Aristotle has said that 'the sweetest of all things is knowledge. And he is right. But if you were to suppose that the publication of a new view were productive of unbounded sweetness, you would be highly mistaken. No one disturbs his fellow man with a new view unpunished.*

  – E. Mach

- *Most people would rather die than think. Most do.*

  – B. Russell

- *Everything of importance has been said by somebody who did not discover it.*

  – A. N. Whitehead

• *Aristotle has said that 'the sweetest of all things is knowledge. And he is right. But if you were to suppose that the publication of a new view were productive of unbounded sweetness, you would be highly mistaken. No one disturbs his fellow man with a new view unpunished.*

– E. Mach

• *Most people would rather die than think. Most do.*

– B. Russell

• *Everything of importance has been said by somebody who did not discover it.*

– A. N. Whitehead