Nečas Center for Mathematical Modeling

Geophysical models of heat and fluid flow in damageable poro-elastic continua

Tomáš Roubíček

Preprint no. 2016-012

Research Team 1
Mathematical Institute of the Charles University
Sokolovská 83, 186 75 Praha 8
http://ncmm.karlin.mff.cuni.cz/

Geophysical models of heat and fluid flow in damageable poro-elastic continua

TOMÁŠ ROUBÍČEK

Mathematical Institute, Charles University, Sokolovská 83,
CZ-186 75 Praha 8, Czech Republic.

Abstract: A rather general model for fluid and heat transport in poro-elastic continua undergoing possibly also plastic-like deformation and damage is developed with the goal to cover various specific models of rock rheology used in geophysics of Earth's crust. Nonconvex free energy at small elastic strains, gradient theories (in particular the concept of 2nd-grade nonsimple continua), and Biot poro-elastic model are employed, together with possible large displacement due to large plastic-like strains evolving during long time periods. Also the additive splitting is justified in stratified situations which are of interest in modelling of lithospheric crust faults. Thermodynamically-based formulation includes entropy balance (in particular the Clausius-Duhem inequality) and an explicit global energy balance. It is further outlined that the energy balance can be used to ensure, under suitable data qualification, existence of a weak solution and stability and convergence of suitable approximation schemes at least in some particular situations.

AMS Subject Classification: 35Q74, 35Q79, 35Q86, 74A15, 74A30, 74L05, 74R20, 80A20, 86A17.

1 Introduction

Amazing and rapidly developing application of *Continuum Mechanics and Thermodynamics* is in *geophysical modelling*, both as far as fluids and solids concern, depending particularly on the time/space scale. Here we focus on models applicable in upper Earth mantle (specifically in shallow depths of the crust, up to 30km or even less) which are therefore based on solid mechanics, although some fluid (water) transport in porous media (rocks) considered usually in a simplified way (in contrast to the full Navier-Stoke-type flow) plays typically an important role, too. The *heat and water transport* processes are intimately coupled with mechanical properties and possibly also with *evolution of porosity and of damage*. Damage is microscopically understood as density of microcracks and may evolve in bi-directional way (admitting also healing) and allows for an interpretation of an aging. Simultaneously to such (usually relatively very slow) transport processes, damage may concentrate in fault zones, leading to fast rupture of these lithospheric faults and causing *tectonic earthquakes* with emission of *elastic waves*. Needless to say, modelling of such complex response of poro-visco-elastic rocks is extremely complicated and a many-decade-lasting effort has been done towards computational models in geophysics, combining basic rational continuum-mechanical and thermodynamical machinery with sound concepts of the generalized standard materials [29] employing the extremely fruitful concept of internal variables [21, 12], here specifically damage in the sense of the pioneering work of L.M. Kachanov [41] (cf. also [24, 42, 45, 60]) and sometimes also porosity or some other variables (describing granular or other phase transitions), various rheological and plasticity theories (although not always used consistently), Darcy flow [15], Biot theory of saturated flow through porous media [6], Korteweg-type stresses [43], sometimes also various gradient theories (although usually limited to only some internal variables and avoiding nonsimple materials), etc.

Actually, instead of rocks, a concrete (as a damageable poro-elastic material) can be considered and then applications in civil engineering, cf. e.g. [40], although some aspects are naturally suppressed due to shorter time/space scales considered, e.g. damage is irreversible (without healing) and displacements are indeed small.

The state of art in geophysical modelling is that (often) the above mentioned physical principles and concepts are not fully respected, (often) unclarified simplifications are made, (mostly) models which are not analytically justified (possibly with no solutions in any reasonable sense) are routinely used, and (mostly) nonconvergent approximations which may easily be numerically unstable in some regimes are used for computational simulations of such models (that, as said, even may not possess any solutions), even though they can be nicely visualized and nicely fitted to some experimental or real-world observations in particular cases. This is euphemistically called computational modelling and, unfortunately, is becoming a more and more dominant trend in some other parts of science and engineering, too. In mathematics, these trends are occasionally manifested in a rather opposite way by attempts of casting nonphysically weak concepts of solutions to facilitate a rather junk analysis, cf. e.g. [74] for discussion of these issues. On the other hand, fortunately, in combination with ever more interesting continuum-mechanical problems with real-world applications addressed, this gives a big challenge and inspiration to applied mathematics where ever better applicable tools are developing during recent decades.

The goal of this article is to device a relatively general model that would possesses in particular the following attributes:

α) main features of the models previously considered in geophysical literature are involved, especially a free energy loosing its convexity if damage develops (together with a specific flow-rule for damage evolution as devised in [55] and later used in numerous articles as e.g. [31, 53, 54]), water flow coupled with porosity propagation as in [53], and a combination of small elastic strain (relevant in modelling of poro-elastic rocks in upper lithospheric mantle) with large displacement (relevant in geophysically large time scales),

β) thermodynamical consistency in the sense that the model possesses a clear energetics and comply with the Clausius-Duhem inequality as well as with nonnegativity of temperature,

γ) a possibility to set length-scale separately for the (typically rather narrow) core of a fault and the (typically wider) damage zone around it,

δ) validity of the model globally in time (not only till a unspecified short time [49] which, due to spatial stress concentration, may even be zero), cf. also Remark 4.4 below,

ϵ) amenability for mathematical and numerical analysis (at least in some special cases) which, in particular, would allow us to devise a numerically stable approximation scheme that would converge to solutions of the original continuum-mechanical model.

Due to the nonconvexity mentioned in (α) which typically causes loss of existence of any solutions and limits validity of the model in time, cf. the explicit discussion in [52, Sect. 4.1], to coupe with (δ) we implement the concept of the 2nd-grade nonsimple materials, cf. Remark 2.2 below. As to the attribute (γ), we use a gradient theory for damage similarly like in [54, 51] to control a width of the damage zone but accompany it with another gradient theory for the inelastic strain to control the width of the fault core. All these attributes will qualitatively improve the models formulated and used in literature and provide a more thorough understanding.

We will consider inertia so that seismic waves typically emitted during fast damage and inelastic shift are not excluded from the model, although their proper computational implementation within the nonlinear model requires special time-integration energy-conserving numerical techniques (like Newmark or Hilber-Hughes-Taylor formulas [32], cf. e.g. [73]). In other words, beside slower transport processes and compaction and healing, the model potentially captures also earthquakes during fast rupturing of lithospheric faults. On the other hand, as usual, various simplifications are adopted. In particular, we consider small elastic strains (but not necessarily small inelastic strain and small displacements), isotropic material, and neglect the convective heat flux due to the fluid flow and also thermal expansion both of the solid and water. Also density variations will be neglected.

The plan of the paper is the following: in Section 2 we formulate a rather general model whose energetics and thermodynamic consistency is then revealed in Section 3. Then, in Sect. 4, we consider a more specific ansatz of the free energy which is used in geophysics, based on a damageable poro-elastic rock model and the Biot theory of saturated flow through porous media. Due to keeping the paper reasonably short and readable for a broader community, mathematical technicalities are suppress and rather only a mathematical strategy to read necessary a-priori estimates from the obtained energy balance and their usage for convergence of suitable approximation schemes that can potentially be used for numerical simulations are only sketched in the Appendix (Sect. 5).

2 A general model of damageable poro-visco-elastic continua with heat and water transport

We will formulate the model in the Euler coordinates, $\Omega \subset \mathbb{R}^3$ being a bounded smooth (fixed) domain. Distinguish intensive and extensive variables, cf. Remark 3.2 below, we consider the basic variables of our general model as follows:

u displacement (valued in $\mathbb{R}^3$),
α damage,
ϕ porosity (effectively understood as the volumetric part of the inelastic strain),
ζ water content (as a volume fraction),
π plastic-like part of the inelastic strain (valued in $\mathbb{R}^{3\times3}_{\rm dev}$),
θ temperature,

where $\mathbb{R}^{3\times3}_{\rm dev} := \{A \in \mathbb{R}^{3\times3};\ {\rm tr}A = 0\}$; let us emphasize that π need not be symmetric. The variables α, ζ, and ϕ are scalar valued assumed to range over the interval $[0, 1]$. We count with a saturated flow, which will be reflected by that $\zeta \geq \phi$ which, however, will not be counted as a constraint but will be only involved in a "soft" way in the free energy through the so-called Biot term, see (4.2). For readers' convenience, Table 2 summarizes the main nomenclature used through the paper, some of them introduced later in the subsequent sections.

$\sigma_{\rm el}$ elastic stress,	$\lambda = \lambda(\alpha, \phi)$, $\mu = \mu(\alpha, \phi)$ Lamé coefficients,
g gravity force,	$\gamma = \gamma(\alpha, \phi)$ a non-Hookean coefficient,
ϱ mass density,	$e(u) = \frac{1}{2}\left(\nabla u + \nabla u^\top\right)$ total-strain tensor,
ψ free energy,	p water pressure (as a chemical potential),
$\psi_{\rm Mech}$ mechanical part of ψ,	r the dissipated mechanical energy rate,
$\psi_{\rm T}$ thermal part of ψ,	$\psi_{\rm M}$ mechanical part of ψ without gradients,
δ potential of dissipative forces,	$\mathfrak{m} = \mathfrak{m}(\alpha, \phi)$ the hydraulic conductivity,
$\varepsilon_{\rm e}$ elastic strain (assumed small),	$M = M(\alpha, \phi)$ Biot modulus,
$\varepsilon_{\rm i}$ inelastic strain,	β Biot coefficient,
$c_{\rm v}(\theta)$ heat capacity,	$h_{\rm ext}$ is the prescribed external traction,
$\mathfrak{k}(\zeta, \theta)$ heat-conductivity coefficient,	$f_{\rm ext}$ the external water flux,
ϑ heat content,	$j_{\rm ext}$ is the prescribed heat flux,
η entropy,	χ specific stored energy of damage,
j heat flux,	κ_0, κ_1, κ_2, κ_3, length-scale coefficients.

Table 1. *Summary of the basic notation used through the paper.*

As in Maxwell viscoelasticity, the total strain tensor is a sum of the elastic and the inelastic components of deformation:

$$e(u) = \varepsilon_{\rm e} + \varepsilon_{\rm i} \quad \text{where} \quad \varepsilon_{\rm i} = \phi\mathbb{I} + \pi. \tag{2.1}$$

where $\mathbb{I} \in \mathbb{R}^{3\times3}$ is the identity matrix, cf. Proposition 2.1 for a certain justification of this decomposition. Note that, like π, also $\varepsilon_{\rm e}$ and $\varepsilon_{\rm i}$ need not be symmetric but the free energies used in Section 4 will eventually ignore the antisymmetric part of $\varepsilon_{\rm e}$. In fact, the mentioned nonsymmetry is related with the natural interpretation of π as ∇u up to lower-order terms, cf. (2.11), but actually a symmetric part of ∇u can be taken with the same effect, which is likely an implicit understanding in geophysical literature. Also, any "off-set" in ϕ is, for notational simplicity, not considered and can be captured by an initial condition $\phi(0) = \phi_0$. Note that we used the decomposition of the inelastic strain $\varepsilon_{\rm i}$ to the spherical and the deviatoric parts, which reflects different physical mechanisms and allows for specifying different flow rules that will govern their evolution. In particular, the deviatoric part will be handled in reference configuration while the porosity will be displaced according to u.

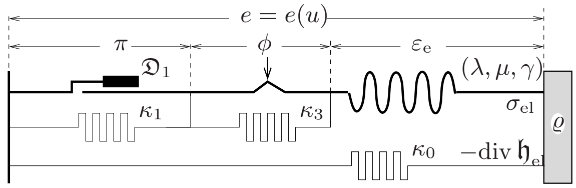

Fig. 1 *Schematic rheological model used in (2.6a,b) and part of (2.6c): nonlinear elastic material (=nonlinear elastic spring with parameters λ, μ, γ depending on damage α, porosity ϕ, and water content ζ) combined plastic-like element $\mathfrak{D}_1$ (depending on α, ϕ, and temperature θ). Moreover, the hyper-stresses with the constants κ_0, κ_1, and κ_3 are depicted, too. (The structural stresses $\mathfrak{s}_{\rm el}$ from* (2.4b) *are not depicted, however.)*

Further, we consider a rather general free energy $\psi = \psi(\varepsilon_{\rm e}, \alpha, \phi, \zeta, \theta, \nabla\varepsilon_{\rm e}, \nabla\pi, \nabla\alpha, \nabla\phi)$ but we confine ourselves to linear and decoupled gradient theories (i.e. ψ is quadratic and additive in the gradients) and to a separated thermal and mechanical parts (i.e. e.g. thermal expansion and related adiabatic effects and possible dependence of heat capacity on mechanical variables are not considered):

$$\begin{aligned}\psi &= \psi(\varepsilon_{\rm e}, \alpha, \phi, \zeta, \theta, \nabla e, \nabla\pi, \nabla\alpha, \nabla\phi) = \psi_{_{\rm Mech}}(\varepsilon_{\rm e}, \alpha, \phi, \zeta, \nabla e, \nabla\pi, \nabla\alpha, \nabla\phi) + \psi_{_{\rm T}}(\theta) \\ &= \psi_{_{\rm M}}(\varepsilon_{\rm e}, \alpha, \phi, \zeta) + \psi_{_{\rm T}}(\theta) + \frac{1}{2}\kappa_0|\nabla e|^2 + \frac{1}{2}\kappa_1|\nabla\pi|^2 + \frac{1}{2}\kappa_2|\nabla\alpha|^2 + \frac{1}{2}\kappa_3|\nabla\phi|^2. \end{aligned} \tag{2.2}$$

In fact, we rather consider the free energy in terms of the total-strain tensor and inelastic/ductile strains results after substitution $\varepsilon_{\rm e} = e{-}\pi{-}\phi\mathbb{I}$, cf. (2.1):

$$\begin{aligned}\tilde\psi &= \tilde\psi(e, \pi, \alpha, \phi, \zeta, \theta, \nabla e, \nabla\pi, \nabla\alpha, \nabla\phi) \\ &:= \psi_{_{\rm M}}(e{-}\pi{-}\phi\mathbb{I}, \alpha, \phi, \zeta) + \psi_{_{\rm T}}(\theta) + \frac{1}{2}\kappa_0|\nabla e|^2 + \frac{1}{2}\kappa_1|\nabla\pi|^2 + \frac{1}{2}\kappa_2|\nabla\alpha|^2 + \frac{1}{2}\kappa_3|\nabla\phi|^2\,. \end{aligned} \tag{2.3}$$

The partial derivatives of $\tilde\psi$ then determine corresponding driving forces. In particular, we define the elastic stress $\sigma_{\rm el}$ and the so-called elastic hyper-stress $\mathfrak{h}_{\rm el}$ respectively by

$$\sigma_{\rm el} = \tilde\psi'_e, \quad \mathfrak{h}_{\rm el} = \tilde\psi'_{\nabla e} = \kappa_0\nabla e, \tag{2.4a}$$

and the so-called structural stress (cf. Remark 3.3 below)

$$\begin{aligned}\mathfrak{s}_{\rm el} = \Big(\psi_{_{\rm M}}(e{-}\pi{-}\phi\mathbb{I}, \alpha, \phi, \zeta) + \psi_{_{\rm T}}(\theta) + \frac{1}{2}\kappa_1|\nabla\pi|^2 + \frac{1}{2}\kappa_2|\nabla\alpha|^2 + \frac{1}{2}\kappa_3|\nabla\phi|^2\Big)\mathbb{I} \\ - \kappa_1\nabla\pi\otimes\nabla\pi - \kappa_2\nabla\alpha\otimes\nabla\alpha - \kappa_3\nabla\phi\otimes\nabla\phi, \end{aligned} \tag{2.4b}$$

and the resulting total stress $\sigma_{\rm el,tot}$ as:

$$\sigma_{\rm el,tot} = \sigma_{\rm el} - {\rm div}\,\mathfrak{h}_{\rm el} + \mathfrak{s}_{\rm el}. \tag{2.4c}$$

Furthermore, we define the driving stress for the plastification $\sigma_{\rm i}$ and the corresponding hyper-stress $\mathfrak{h}_{\rm i}$ as well as the resulting total stress $\sigma_{\rm i,tot}$:

$$\sigma_{\rm i} = \tilde\psi'_\pi, \quad \mathfrak{h}_{\rm i} = \tilde\psi'_{\nabla\pi} = \kappa_1\nabla\pi, \quad \sigma_{\rm i,tot} = \sigma_{\rm i} - {\rm div}\,\mathfrak{h}_{\rm i}, \tag{2.4d}$$

driving "stress" for porosity-evolution (a so-called effective pressure) $p_{\rm eff}$ and the corresponding hyper-stress $\mathfrak{h}_{\rm eff}$ as well as the total stress $p_{\rm eff,tot}$:

$$p_{\rm eff} = \tilde\psi'_\phi, \quad \mathfrak{h}_{\rm eff} = \tilde\psi'_{\nabla\phi} = \kappa_2\nabla\phi, \quad p_{\rm eff,tot} = p_{\rm eff} - {\rm div}\,\mathfrak{h}_{\rm eff}, \tag{2.4e}$$

further we define the driving force for damage $\sigma_{\rm dam}$ and the corresponding hyper-force $\mathfrak{h}_{\rm dam}$ as well as the total driving force $\sigma_{\rm dam,tot}$

$$\sigma_{\rm dam}=\tilde\psi'_\alpha,\quad \mathfrak{h}_{\rm dam}=\tilde\psi'_{\nabla\alpha}=\kappa_3\nabla\alpha,\quad \sigma_{\rm dam,tot}=\sigma_{\rm dam}-{\rm div}\,\mathfrak{h}_{\rm dam},\tag{2.4f}$$

and eventually the pore pressure p, entropy η, and the heat capacity $c_{\rm v}$ respectively as:

$$p=\tilde\psi'_\zeta,\quad \eta=-\tilde\psi'_\theta=-\psi'_{\rm T}(\theta),\quad\text{and}\quad c_{\rm v}=-\theta\tilde\psi''_{\theta\theta}=-\theta\psi''_{\rm T}(\theta).\tag{2.4g}$$

An important aspect is occurrence of the structural stress $\mathfrak{s}_{\rm el}$ in (2.4a) which is related with the proper energy balance in the case of gradient theories for the internal variables, cf. Remark 3.3 below.

Another important ingredient is a choice of a (pseudo)potential δ of dissipative forces for the evolution of π, α, and ϕ in the form

$$\delta(\alpha,\phi,\theta;\dot\pi,\dot\alpha,\dot\phi)=\delta_1(\alpha,\phi,\theta;\dot\pi)+\delta_2(\alpha,\phi,\theta;\dot\alpha,\dot\phi)\tag{2.5}$$

with $\delta_1(\alpha,\phi,\theta;\cdot):\mathbb{R}^{3\times3}_{\rm dev}\to\mathbb{R}^+$ and $\delta_2(\alpha,\phi,\theta;\cdot,\cdot):\mathbb{R}\times\mathbb{R}\to\mathbb{R}^+$ a convex of the rates, vanishing at 0. Then their subdifferentials will define the (set-valued) mappings $\mathfrak{D}_1(\alpha,\phi,\theta;\cdot):=\partial\delta_1(\alpha,\phi,\theta;\cdot):\mathbb{R}^{3\times3}_{\rm dev}\rightrightarrows\mathbb{R}^{3\times3}_{\rm dev}$ for inelastic flow rule, and $\mathfrak{D}_2(\alpha,\phi,\theta;\cdot):=\partial\delta_2(\alpha,\phi,\theta;\dot\alpha,\dot\phi):\mathbb{R}^2\rightrightarrows\mathbb{R}^2$ for the porosity-damage flow rule.

We assume rather for simplicity that $c_{\rm v}$ is not dependent on any other variables than at most temperature. The governing equations/inclusions are then considered as:

$$\varrho\ddot u={\rm div}\,\sigma_{\rm el,tot}+g,\qquad\text{(momentum equilibrium)}\tag{2.6a}$$

$$\mathfrak{D}_1\Big(\alpha,\phi,\theta;\frac{{\rm D}\pi}{{\rm D}t}\Big)+{\rm dev}\,\sigma_{\rm i,tot}\ni0,\qquad\text{(flow rule for inelastic strain)}\tag{2.6b}$$

$$\mathfrak{D}_2\Big(\alpha,\phi,\theta;\frac{\rm D}{{\rm D}t}\Big(\begin{array}{c}\alpha\\ \phi\end{array}\Big)\Big)+\Big(\begin{array}{c}\sigma_{\rm dam,tot}+N_{[0,1]}(\alpha)\\ p_{\rm eff,tot}+N_{[0,1]}(\phi)\end{array}\Big)\ni0,\qquad\text{(flow rule for damage/porosity)}\tag{2.6c}$$

$$\frac{{\rm D}\zeta}{{\rm D}t}\in{\rm div}\big(\mathfrak{m}(\alpha,\phi)\nabla p\big)-N_{[0,1]}(\zeta),\qquad\text{(water-transport equation)}\tag{2.6d}$$

$$c_{\rm v}(\theta)\frac{{\rm D}\theta}{{\rm D}t}={\rm div}\big(\mathfrak{k}(\zeta,\theta)\nabla\theta\big)+r+\theta\eta\,{\rm div}\dot u\qquad\text{(heat-transfer equation)}\tag{2.6e}$$

$$\text{with }\ r=r\Big(\alpha,\phi,\theta;\frac{{\rm D}\pi}{{\rm D}t},\frac{{\rm D}\alpha}{{\rm D}t},\frac{{\rm D}\phi}{{\rm D}t}\Big)=\mathfrak{D}_1\Big(\alpha,\phi,\theta;\frac{{\rm D}\pi}{{\rm D}t}\Big){:}\frac{{\rm D}\pi}{{\rm D}t}+\mathfrak{D}_2\Big(\alpha,\phi,\theta;\frac{\rm D}{{\rm D}t}\Big(\begin{array}{c}\alpha\\ \phi\end{array}\Big)\Big){\cdot}\frac{\rm D}{{\rm D}t}\Big(\begin{array}{c}\alpha\\ \phi\end{array}\Big)+\mathfrak{m}(\alpha,\phi)|\nabla p|^2,\qquad\text{(heat-production rate)}\tag{2.6f}$$

where "dev" denotes the deviatoric part of the involved tensor, $\frac{\rm D}{{\rm D}t}=\frac{\partial}{\partial t}+\dot u\cdot\nabla$ denotes the material derivative counting moving continuum with the velocity $\dot u$, and $N_{[0,1]}$ denotes the normal cone to the interval $[0,1]$ where both α, ϕ, and ζ are valued during their evolution.

The system (2.6) deserves certainly some comments. We use description in the Euler coordinates. The displacement u as well as the total and inelastic strain tensors are counted in this fixed configuration but need not be small. In contrast, the elastic strain, $\varepsilon_{\rm e}$, is however assumed small. The other fields can thus be transported within the (possibly) large displacement. The inertial term in the momentum equation (2.6a) is sometimes considered as $\varrho\frac{{\rm D}v}{{\rm D}t}$ for $v=\dot u$, which means $\varrho\ddot u+\varrho\dot u{\cdot}\nabla\dot u$ and reveals that we neglected the second order velocity term, $\dot u{\cdot}\nabla\dot u$ as this term is supposed to be small under slow motion concept. The diffusion-type equation (2.6d) is a combination of the *Darcy law* for a fluid flux with respect to a solid matrix being $-\mathfrak{m}(\alpha,\phi)\nabla p$

and the equation for fluid mass conservation. In the heat-transfer equation (2.6e), we simplified the convective heat flux by neglecting convection due to the water flow, i.e. an extra term of the type $\zeta\mathfrak{m}\vartheta_{\rm water}\nabla p$ in (2.6e) with $\vartheta_{\rm water}$ denoting the heat content corresponding to water; such a term seems not to be amenable to a-priori estimation and related analysis. The last term in (2.6e) is related with the adiabatic-heat effects in the heat-transfer equation and balances the entropy so that both the Clausius-Duhem inequality and the energy balance hold, cf. (3.12) and (3.4) vs. (3.11), respectively; in literature, this term is often considered and may have an important quantitative contribution, cf. e.g. [58, Eq. (1.41)].

It is quite natural to assume $\mathfrak{D}_1(\alpha,\phi,\theta;\frac{{\rm D}\pi}{{\rm D}t}){:}\frac{{\rm D}\pi}{{\rm D}t}$ single-valued although $\mathfrak{D}_1(\alpha,\phi,\theta;\cdot)$ itself may be set-valued at 0, and similarly for $\mathfrak{D}_2$, too. The water-transport equation (2.6d) can also be viewed as a Nernst equation governed by the gradient of the chemical potential ψ'_ζ.

The system (2.6) is to be accompanied by suitable conditions on the boundary Γ of the reference domain Ω. Without going into unnecessary technicalities, we can consider

$$\sigma_{\rm el,tot}\nu - {\rm div}_{\rm s}(\mathfrak{h}_{\rm el}\nu) + k_{\rm el}u = h_{\rm ext}, \qquad \mathfrak{h}_{\rm el}{:}(\nu\otimes\nu) = 0, \tag{2.7a}$$

$$\nabla\pi\nu = 0, \qquad \nabla\alpha{\cdot}\nu = 0, \qquad \nabla\phi{\cdot}\nu = 0, \tag{2.7b}$$

$$\mathfrak{m}(\alpha,\phi)\nabla p{\cdot}\nu = f_{\rm ext}, \tag{2.7c}$$

$$\mathfrak{k}(\zeta,\theta)\nabla\theta{\cdot}\nu = j_{\rm ext} \tag{2.7d}$$

where ν is the unit outward normal to the boundary Γ, "${\rm div}_{\rm s}$" denotes the surface divergence, and $k_{\rm el}$ describes an elastic support of the boundary, and where $h_{\rm ext}$ is the prescribed external traction, $f_{\rm ext}$ the external water flux, and $j_{\rm ext}$ is the prescribed heat flux. The external traction can be considered as $t_{\rm ext} = k_{\rm el}u_{\rm ext}$ with $u_{\rm ext}$ a prescribed displacement, cf. e.g. [31, 50], which overcomes technicalities with prescribing the boundary displacement directly by Dirichlet conditions. For the (rather technical) justification of the conditions (2.7a) as the truly natural for the nonsimple material we refer e.g. to [76, 77] or also [71, Sect. 2.4.4]. In addition, the system (2.6) should be accompanied by some conditions on its state at particular times, e.g. some periodicity condition or, as considered below in (5.2), initial conditions.

The Green-Naghdi additive decomposition [28] used in (2.1) also deserves explanation. Generally, large displacements lead to large strains and, instead of the additive decomposition, to the Kröner (also called Lee-Liu) multiplicative decomposition $F = F_{\rm el}F_{\rm pl}$ with the deformation gradient $F = \nabla y$ where y is the deformation, $F_{\rm el}$ denotes the elastic strain and $F_{\rm pl}$ the plastic strain, cf. [47, 44]. Standardly, the small-strain tensor $e(u)$ used in (2.1) arises, when introducing the displacement

$$u = y - {\rm identity}, \tag{2.8}$$

from the *Green-Lagrange strain* tensor $E = \frac12(F^\top F - \mathbb{I}) = \frac12(\nabla u)^\top + \frac12\nabla u + \frac12(\nabla u)^\top\nabla u$ with $\mathbb{I}$ the identity matrix by neglecting the higher-order term $\frac12(\nabla u)^\top\nabla u$. This is legitimate if ∇u is small. In combination with plasticity, a rigorous passage from the multiplicative to the additive decompositions in the quasistatic case under small deformation was performed in [62] by using the ansatz

$$F_{\rm pl} = \mathbb{I} + P \tag{2.9}$$

while considering the so-called linearized plastic strain P small. However, in the geophysical applications we have in mind situations where two (poro)elastic blocks can mutually move along flat fault region (say, in this direction of x_1-axis, while x_2-axis is the normal to this fault) the displacement u need not be small, cf. e.g. [50, 54]. In such *stratified configuration*, only u_1 and $\frac{\partial u_1}{\partial x_2}$ are really large. Then also $[(\nabla u)^\top\nabla u]_{ij} = \sum_{k=1}^d \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}$ may be large for $i = 2 = j$. This might (but may not!) be compensated by the combination with plastic slip. It seems that the underlying assumptions were never really identified in the geophysical literature. In the following assertion we consider a mere homogeneous shift in x_1-direction together with the elastic strain not far from identity:

Proposition 2.1 (Green-Naghdi-type additive decomposition (2.1).**)** *Let us assume* (2.8) *and* (2.9) *with*

$$u_2,\ u_3,\ P_{31},\ P_{i2},\ \text{and } P_{i3} \text{ for } i=1,2,3 \text{ and } \det F_{\rm pl}-1 \ \text{ vanish and } \ F_{\rm el}-\mathbb{I} \ \text{ is small.} \tag{2.10}$$

Then the elastic Green-Lagrange tensor $E_{\rm el}=\frac12(F_{\rm el}^\top F_{\rm el}-\mathbb{I})$ *equals up to higher-order terms to the symmetric part of* $e(u){-}\varepsilon_{\rm i}$ *provided* $F=\mathbb{I}+\nabla u$ *and* $e_{\rm i}=P$.

Proof. The assumption $\det F_{\rm pl}=1$ in (2.10) causes that also P_{11} vanishes, which further causes also that also $\frac{\partial u_1}{\partial x_1}$, together with $\frac{\partial u_1}{\partial x_2}-P_{12}$ and $\frac{\partial u_1}{\partial x_3}$, are small because $F_{\rm el}=(\mathbb{I}{+}\nabla u)(\mathbb{I}{+}P)^{-1}\sim\mathbb{I}$ implies that

$$\nabla u\sim P\,; \tag{2.11}$$

note that $\mathbb{I}{+}P$ is indeed invertible because of the special form of P. The additive splitting (2.1) can be seen when considering all small variables as zero and then to calculate

$$\begin{aligned}
E_{\rm el}&=\frac12\big(F_{\rm el}^\top F_{\rm el}-\mathbb{I}\big)=\frac12\big((\mathbb{I}{+}P)^{-\top}(\mathbb{I}{+}\nabla u)^\top(\mathbb{I}{+}\nabla u)(\mathbb{I}{+}P)^{-1}-\mathbb{I}\big)\\
&=\frac12\begin{pmatrix}1&P_{12}&0\\0&1&0\\0&0&1\end{pmatrix}^{-\top}\begin{pmatrix}1{+}\frac{\partial u_1}{\partial x_1}&\frac{\partial u_1}{\partial x_2}&\frac{\partial u_1}{\partial x_3}\\0&1&0\\0&0&1\end{pmatrix}^{\top}\begin{pmatrix}1{+}\frac{\partial u_1}{\partial x_1}&\frac{\partial u_1}{\partial x_2}&\frac{\partial u_1}{\partial x_3}\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&P_{12}&0\\0&1&0\\0&0&1\end{pmatrix}^{-1}-\frac12\mathbb{I}\\
&=\frac12\begin{pmatrix}1&0&0\\-P_{12}&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1{+}\frac{\partial u_1}{\partial x_1}&0&0\\\frac{\partial u_1}{\partial x_2}&1&0\\\frac{\partial u_1}{\partial x_3}&0&1\end{pmatrix}\begin{pmatrix}1{+}\frac{\partial u_1}{\partial x_1}&\frac{\partial u_1}{\partial x_2}&\frac{\partial u_1}{\partial x_3}\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&-P_{12}&0\\0&1&0\\0&0&1\end{pmatrix}-\frac12\mathbb{I}\\
&=\frac12\begin{pmatrix}(1{+}\frac{\partial u_1}{\partial x_1})^2&(\frac{\partial u_1}{\partial x_2}{-}P_{12})(1{+}\frac{\partial u_1}{\partial x_1})&(1{+}\frac{\partial u_1}{\partial x_1})\frac{\partial u_1}{\partial x_3}\\(\frac{\partial u_1}{\partial x_2}{-}P_{12})(1{+}\frac{\partial u_1}{\partial x_1})&1{+}(\frac{\partial u_1}{\partial x_2}{-}P_{12})^2&(\frac{\partial u_1}{\partial x_2}{-}P_{12})\frac{\partial u_1}{\partial x_3}\\(1{+}\frac{\partial u_1}{\partial x_1})\frac{\partial u_1}{\partial x_3}&(\frac{\partial u_1}{\partial x_2}{-}P_{12})\frac{\partial u_1}{\partial x_3}&1{+}(\frac{\partial u_1}{\partial x_3})^2\end{pmatrix}-\frac12\mathbb{I}\\
&=\frac12\begin{pmatrix}2\frac{\partial u_1}{\partial x_1}&\frac{\partial u_1}{\partial x_2}&\frac{\partial u_1}{\partial x_3}\\\frac{\partial u_1}{\partial x_2}&0&0\\\frac{\partial u_1}{\partial x_3}&0&0\end{pmatrix}-\frac12\begin{pmatrix}0&P_{12}&0\\P_{12}&0&0\\0&0&0\end{pmatrix}+\frac12\begin{pmatrix}(\frac{\partial u_1}{\partial x_1})^2&(\frac{\partial u_1}{\partial x_2}{-}P_{12})\frac{\partial u_1}{\partial x_1}&\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_3}\\(\frac{\partial u_1}{\partial x_2}{-}P_{12})\frac{\partial u_1}{\partial x_1}&(\frac{\partial u_1}{\partial x_2}{-}P_{12})^2&(\frac{\partial u_1}{\partial x_2}{-}P_{12})\frac{\partial u_1}{\partial x_3}\\\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_3}&(\frac{\partial u_1}{\partial x_2}{-}P_{12})\frac{\partial u_1}{\partial x_3}&(\frac{\partial u_1}{\partial x_3})^2\end{pmatrix}.
\end{aligned}$$

The last term is of a higher order since $\frac{\partial u_1}{\partial x_1}$, $\frac{\partial u_1}{\partial x_2}{-}P_{12}$, and $\frac{\partial u_1}{\partial x_3}$ are small. Altogether, this yields $E_{\rm el}\sim e(u)-\frac12(P^\top{+}P)$, which justifies usage of the small strain and the additive splitting in (2.1) provided $P=\varepsilon_{\rm i}$. $\square$

Let us still comment that the assumption about incompressibility of the plastic strain, i.e. $\det F_{\rm pl}=1$ in (2.10), is quite usual in plasticity and here compatible with the concept (or an assumption, not explicitly articulated in geophysical literature, however) of only small volumetric variations of porosity. Let us also note that the standard frame-indifference qualification of the stored energy implies that it depends on the right Cauchy-Green tensor $F^\top F$ and therefore on the symmetric Green-Lagrange tensor $E_{\rm el}$, ignoring thus the antisymmetric part of $e(u){-}e_{\rm i}$, which explains the assertion of Proposition 2.1. Let us also remark that, when taking into account that the original $\psi_{\rm M}$ depends on the Green-Lagrange tensor $E_{\rm el}$ rather than on the symmetric part of $\varepsilon_{\rm e}$, the ansatz (4.2) used below is known as a *St. Venant-Kirchhoff material* (generalized here towards possible nonconvexity for $\gamma\neq0$).

Remark 2.1 (Constraints in (2.6).) The relations (2.6c,d) are inclusions due to the set-valued mapping $N_{[0,1]}(\cdot)$ rather than equalities. In particular, $N_{[0,1]}(\cdot)=\emptyset$ if its argument is out of the interval $[0,1]$, which eventually ensures that any (weak) solution $(u,\pi,\alpha,\phi,\zeta,\theta)$ to this system must be composed from α, ϕ, and ζ valued in $[0,1]$. In fact, the weak formulation leads to suitable variational inequalities which explicitly involve these constraints, cf. e.g. [14, 22, 33, 71] for an introduction of this rather standard technique. As we use the 1st-order gradient theory, an alternative model might avoid $N_{[0,1]}$ and consider (2.6c,d) as equalities provided the corresponding driving forces vanishes out of $[0,1]$, i.e. $\tilde\psi'_\alpha|_{\alpha\in\mathbb{R}\setminus[0,1]}=0$, $\tilde\psi'_\phi|_{\phi\in\mathbb{R}\setminus[0,1]}=0$, and $\tilde\psi'_\zeta|_{\zeta\in\mathbb{R}\setminus[0,1]}=0$.

As far as damage concerns, such alternative is sometimes considered in unidirectional damage in a famous Ambrosio-Tortorelli approximation of cracks (see [2, 10, 73] and references therein) but is usually not applied in geophysical models, cf. (4.4) below.

Remark 2.2 (Concept of nonsimple materials.) The concept of gradient-theories for strains to describe materials referred as nonsimple, or also multipolar or complex, has been invented long time ago, cf. [81] or also [27, 45, 59, 65, 66, 80, 82]. Here, rather for notational simplicity, we used the potential in the quadratic form with only one coefficient, namely $\frac{1}{2}\kappa_0|\nabla e|^2$; for a more realistic form see also e.g. [79]. Moreover, let us note that, in (2.2), we have used ∇e but not $\nabla\varepsilon_{\rm e}$ which would yield terms like $\kappa_0\nabla e(u)\,\vdots\,\nabla\frac{{\rm D}\pi}{{\rm D}t}$ and $\kappa_0\nabla e(u)\,\vdots\,\nabla\frac{{\rm D}\phi}{{\rm D}t}$ in the energy balance (3.1) and (3.10) which do no seem amenable to estimation within the considered model.

3 Energetics of the general model

An important attribute of the model is a possibility to articulate the energy balance in a sense that can be effectively used for a rigorous analysis and designing numerically stable computational schemes. It needs not only local energy balances but an energy on the whole considered domain $\Omega\subset\mathbb{R}^3$. Here, the energetics of the system (2.6) can be revealed by testing the particular equations/inclusions in (2.6a-d) successively by $\dot{u}$, $\frac{\rm D}{{\rm D}t}\pi$, $\frac{\rm D}{{\rm D}t}\phi$, $\frac{\rm D}{{\rm D}t}\alpha$, and p; here "testing" means (as usual in mathematical jargon) a multiplication by the indicated field and integration over the domain Ω and, whenever needed, usage of the Green formula and boundary conditions.

Proposition 3.1 (Mechanical-energy balance.) *Let the evolutionary boundary-value problem (2.6)–(2.7) with (2.4) possess a smooth solution $(u,\pi,\alpha,\phi,\zeta,\theta)$. Then the following energy balance holds:*

$$\frac{\rm d}{{\rm d}t}\bigg(\int_\Omega\frac{\varrho}{2}|\dot{u}|^2+\psi_{_{\rm Mech}}\bigg)\,{\rm d}x+\int_\Gamma\frac{k_{\rm el}}{2}|u|^2\,{\rm d}S\bigg)+\int_\Omega r\,{\rm d}x \\ =\int_\Gamma h_{\rm ext}\cdot\dot{u}+pf_{\rm ext}\cdot\nu-\psi_{_{\rm M}}\dot{u}\cdot\nu\,{\rm d}S+\int_\Omega(\dot{u}\otimes\sigma_{\rm el})\,\vdots\,\nabla e(u)-\psi_{_{\rm T}}(\theta){\rm div}\,\dot{u}\,{\rm d}x, \tag{3.1}$$

where $\psi_{_{\rm Mech}}$ is the total mechanical energy from (2.2), r is the specific dissipation rate from (2.6f), and where the last term (3.1) means componentwise $\sum_{i,j,k=1}^d\dot{u}_k[\sigma_{\rm el}]_{ij}\frac{\partial e_{ij}}{\partial x_k}$.

Proof. First, as already mentioned, a multiple usage of the Green formula will be the standard ingredient. In particular, denoting a generic scalar field z with $\nabla z\cdot\nu=0$ on Γ, we will repeatedly use the calculus

$$\int_\Omega\Delta z(\dot{u}\cdot\nabla z)\,{\rm d}x=-\int_\Omega\nabla z\cdot\nabla(\dot{u}\cdot\nabla z)\,{\rm d}x=-\int_\Omega(\nabla z{\otimes}\nabla z){:}e(\dot{u})+(\nabla z{\otimes}\dot{u}){:}\nabla^2z\,{\rm d}x \tag{3.2}$$

and also

$$\int_\Omega(\nabla z{\otimes}\dot{u}){:}\nabla^2z\,{\rm d}x=-\int_\Omega{\rm div}(\nabla z{\otimes}\dot{u}){\cdot}\nabla z\,{\rm d}x=-\int_\Omega|\nabla z|^2{\rm div}\,\dot{u}+(\nabla z{\otimes}\dot{u}){:}\nabla^2z\,{\rm d}x$$

so that (3.2) can rather be written as

$$\int_\Omega\Delta z(\dot{u}\cdot\nabla z)\,{\rm d}x=\int_\Omega\frac{1}{2}|\nabla z|^2{\rm div}\,\dot{u}-(\nabla z{\otimes}\nabla z){:}e(\dot{u})\,{\rm d}x. \tag{3.3}$$

This last formula is important because it reveals that the "optically" high order term $\Delta z\dot{u}\cdot\nabla z$ is actually the desired structural-stress-type term $(\nabla z{\otimes}\nabla z-\frac{1}{2}|\nabla z|^2\mathbb{I}){:}e(\dot{u})$.

The test of (2.6a) leading to (a part of) the mechanical energy balance is standardly by the velocity $\dot{u}$. This test yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\int_\Omega \frac{\varrho}{2}|\dot{u}|^2\,\mathrm{d}x + \int_\Gamma \frac{k_{\rm el}}{2}|u|^2\,\mathrm{d}S\bigg) + \int_\Omega \Big(\tilde\psi_e' - \kappa_1\nabla\pi\otimes\nabla\pi - \kappa_2\nabla\alpha\otimes\nabla\alpha - \kappa_3\nabla\phi\otimes\nabla\phi\Big){:}e(\dot{u})$$
$$+\Big(\psi_{\rm M}+\psi_{\rm T}+\frac{\kappa_1}{2}|\nabla\pi|^2+\frac{\kappa_2}{2}|\nabla\alpha|^2+\frac{\kappa_3}{2}|\nabla\phi|^2\Big)\mathrm{div}\,\dot{u} + \kappa_0\nabla e(u)\,\vdots\,\nabla e(\dot{u})\,\mathrm{d}x = \int_\Gamma h_{\rm ext}{\cdot}\dot{u}\,\mathrm{d}S. \tag{3.4}$$

The test of (2.6b) by $\frac{\mathrm{D}\pi}{\mathrm{D}t}$ yields

$$\int_\Omega \tilde\psi_\pi'{:}\dot\pi + \kappa_1\nabla\pi\,\vdots\,\nabla\dot\pi + \mathfrak{D}_1\Big(\alpha,\phi,\theta;\frac{\mathrm{D}\pi}{\mathrm{D}t}\Big){:}\frac{\mathrm{D}\pi}{\mathrm{D}t}\,\mathrm{d}x$$
$$= \int_\Omega \frac{1}{2}\kappa_1|\nabla\pi|^2\mathrm{div}\,\dot{u} - \kappa_1(\nabla\pi\otimes\nabla\pi){:}e(\dot{u}) - \tilde\psi_\pi'{:}(\dot{u}{\cdot}\nabla\pi)\,\mathrm{d}x; \tag{3.5}$$

here we used the formula (3.3) with $z=\pi$ and "$|\cdot|$" in (3.5) denotes the Frobenius norm and "$\otimes$" means also the summation along the indices of the $3{\times}3$-tensor π. Similarly, the test of (2.6c) by $\frac{\mathrm{D}\phi}{\mathrm{D}t}$ and $\frac{\mathrm{D}\alpha}{\mathrm{D}t}$ gives

$$\int_\Omega \tilde\psi_\phi'\dot\phi + \tilde\psi_\alpha'\dot\alpha + \kappa_2\nabla\alpha{\cdot}\nabla\dot\alpha + \kappa_3\nabla\phi{\cdot}\nabla\dot\phi + \mathfrak{D}_2\Big(\alpha,\phi,\theta;\frac{\mathrm{D}}{\mathrm{D}t}\begin{pmatrix}\alpha\\ \phi\end{pmatrix}\Big){\cdot}\frac{\mathrm{D}}{\mathrm{D}t}\begin{pmatrix}\alpha\\ \phi\end{pmatrix}\mathrm{d}x$$
$$= \int_\Omega \big(\kappa_2\Delta\alpha - \tilde\psi_\alpha'\big)(\dot{u}{\cdot}\nabla\alpha) + \big(\kappa_3\Delta\phi - \tilde\psi_\phi'\big)(\dot{u}{\cdot}\nabla\phi)\,\mathrm{d}x$$
$$= \int_\Omega \frac{1}{2}\kappa_2|\nabla\alpha|^2\mathrm{div}\,\dot{u} - \kappa_2(\nabla\alpha\otimes\nabla\alpha){:}e(\dot{u}) - \tilde\psi_\alpha'(\dot{u}{\cdot}\nabla\alpha)$$
$$+\frac{1}{2}\kappa_3|\nabla\phi|^2\mathrm{div}\,\dot{u} - \kappa_3(\nabla\phi\otimes\nabla\phi){:}e(\dot{u}) - \tilde\psi_\phi'(\dot{u}{\cdot}\nabla\phi)\,\mathrm{d}x; \tag{3.6}$$

here we used (3.3) with $z=\phi$ and $z=\alpha$. The mentioned test of (2.6d) by $p=\tilde\psi_\zeta'$ yields

$$\int_\Omega \tilde\psi_\zeta'\dot\zeta\,\mathrm{d}x = \int_\Omega \mathfrak{m}(\alpha,\phi)|\nabla p|^2 - \tilde\psi_\zeta'(\dot{u}{\cdot}\nabla\zeta)\,\mathrm{d}x + \int_\Gamma p f_{\rm ext}{\cdot}\nu\,\mathrm{d}S. \tag{3.7}$$

By summing (3.4)–(3.7), we enjoy the cancellation of the structural stresses and eventually obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\int_\Omega \frac{\varrho}{2}|\dot{u}|^2 + \psi_{\rm Mech}\,\mathrm{d}x + \int_\Gamma \frac{k_{\rm el}}{2}|u|^2\,\mathrm{d}S\bigg) + \int_\Omega r\,\mathrm{d}x$$
$$= \int_\Omega \Big(\frac{\partial}{\partial t}\frac{\varrho}{2}|\dot{u}|^2 + \tilde\psi_e'{:}e(\dot{u}) + \tilde\psi_\pi'{:}\dot\pi + \tilde\psi_\alpha'\dot\alpha + \tilde\psi_\phi'\dot\phi + \tilde\psi_\zeta'\dot\zeta$$
$$+\kappa_0\nabla e(u)\,\vdots\,\nabla e(\dot{u}) + \kappa_1\nabla\pi\,\vdots\,\nabla\dot\pi + \kappa_2\nabla\alpha{\cdot}\nabla\dot\alpha + \kappa_3\nabla\phi{\cdot}\nabla\dot\phi + r\Big)\,\mathrm{d}x$$
$$= \int_\Gamma h_{\rm ext}{\cdot}\dot{u} + p f_{\rm ext}{\cdot}\nu\,\mathrm{d}S$$
$$-\int_\Omega \big(\psi_{\rm M}+\psi_{\rm T}\big)\,\mathrm{div}\,\dot{u} + \tilde\psi_\pi'(\dot{u}{\cdot}\nabla\pi) + \tilde\psi_\alpha'(\dot{u}{\cdot}\nabla\alpha) + \tilde\psi_\phi'(\dot{u}{\cdot}\nabla\phi) + \tilde\psi_\zeta'(\dot{u}{\cdot}\nabla\zeta)\,\mathrm{d}x$$
$$= \int_\Gamma h_{\rm ext}{\cdot}\dot{u} + p f_{\rm ext}{\cdot}\nu - \psi_{\rm M}\dot{u}{\cdot}\nu\,\mathrm{d}S + \int_\Omega (\dot{u}\otimes\sigma_{\rm el})\,\vdots\,\nabla e(u) - \psi_{\rm T}\,\mathrm{div}\,\dot{u}\,\mathrm{d}x, \tag{3.8}$$

which is just the balance of mechanical energy (3.1).

The last equality in (3.8) have used the identity $\int_\Omega \psi_{\rm M}\,\mathrm{div}\,\dot{u}\,\mathrm{d}x = \int_\Gamma \psi_{\rm M}\dot{u}{\cdot}\nu\,\mathrm{d}S - \int_\Omega \nabla\psi_{\rm M}{\cdot}\dot{u}\,\mathrm{d}x = \int_\Gamma \psi_{\rm M}\dot{u}{\cdot}\nu\,\mathrm{d}S - \int_\Omega \tilde\psi_e'(\dot{u}{\cdot}\nabla e(u)) + \tilde\psi_\pi'(\dot{u}{\cdot}\nabla\pi) + \tilde\psi_\alpha'(\dot{u}{\cdot}\nabla\alpha) + \tilde\psi_\phi'(\dot{u}{\cdot}\nabla\phi) + \tilde\psi_\zeta'(\dot{u}{\cdot}\nabla\zeta)\,\mathrm{d}x$. $\square$

We still rewrite the heat-transfer equation (2.6e) in terms of the heat content $\vartheta := C_{\rm v}(\theta)$ with $C_{\rm v}$ denoting a primitive function to the heat capacity $c_{\rm v}$. Then (2.6e) transforms to

$$\dot{\vartheta} + {\rm div}(\dot{u}\vartheta) = {\rm div}\big(\mathfrak{k}(\zeta,\theta)\nabla\theta\big) + r\Big(\alpha,\phi,\theta;\frac{{\rm D}\pi}{{\rm D}t},\frac{{\rm D}\alpha}{{\rm D}t},\frac{{\rm D}\phi}{{\rm D}t}\Big) + (\vartheta{-}\theta\eta){\rm div}\,\dot{u}. \tag{3.9}$$

Sometimes, the rescaling temperature $\vartheta = C_{\rm v}(\theta)$ and thus the transition from (2.6e) to (3.9) is called an *enthalpy transformation*, here presented under advection in compressible continuum. To see that (3.9) is indeed equivalent to (2.6e), it suffices to substitute ϑ and use the calculus

$$\dot{\vartheta} + {\rm div}(\dot{u}\vartheta) + (\theta\eta{-}\vartheta){\rm div}\,\dot{u} = \dot{\vartheta} + \dot{u}{\cdot}\nabla\vartheta + ({\rm div}\,\dot{u})\vartheta + (\theta\eta{-}\vartheta){\rm div}\,\dot{u} = c_{\rm v}(\theta)\frac{{\rm D}\theta}{{\rm D}t} - \theta\eta\,{\rm div}\,\dot{u}.$$

Proposition 3.2 (Total-energy balance.) *Assume again that the evolutionary boundary-value problem (2.6)–(2.7) with (2.4) possesses a smooth solution $(u,\pi,\alpha,\phi,\zeta,\theta)$. Then the following the total energy balance holds:*

$$\frac{\rm d}{{\rm d}t}\bigg(\int_\Omega\Big(\underbrace{\frac{\varrho}{2}|\dot{u}|^2 + \psi_{\rm M} + \vartheta}_{\substack{\text{kinetic, mechanical and}\\ \text{heat energies in the bulk}}}\Big){\rm d}x + \int_\Gamma \underbrace{\frac{k_{\rm el}}{2}|u|^2}_{\substack{\text{elastic-support}\\ \text{energy on }\Gamma}}\,{\rm d}S\bigg)$$
$$= \int_\Gamma \underbrace{h_{\rm ext}{\cdot}\dot{u}}_{\substack{\text{power of}\\ \text{external force}}} + \underbrace{pf_{\rm ext}{\cdot}\nu}_{\substack{\text{flux of energy due}\\ \text{to water flow thru }\Gamma}} + \underbrace{j_{\rm ext}}_{\substack{\text{heat flux}\\ \text{thru }\Gamma}} - \underbrace{(\psi_{\rm M}+\vartheta)\dot{u}{\cdot}\nu}_{\substack{\text{internal energy}\\ \text{flux thru }\Gamma}}\;{\rm d}S + \int_\Omega(\dot{u}\otimes\sigma_{\rm el})\vdots\nabla e(u)\,{\rm d}x. \tag{3.10}$$

Proof. Now we can test the transformed heat equation (3.9) by 1 and we obtain the balance of the overall heat-energy balance

$$\frac{\rm d}{{\rm d}t}\int_\Omega\vartheta\,{\rm d}x = \int_\Omega r + {\rm div}(j{-}\dot{u}\vartheta) + (\vartheta{-}\theta\eta){\rm div}\dot{u}\,{\rm d}x = \int_\Omega r + \psi_{\rm T}(\theta){\rm div}\dot{u}\,{\rm d}x + \int_\Gamma j_{\rm ext}{-}\vartheta\dot{u}{\cdot}\nu\,{\rm d}S, \tag{3.11}$$

where $j = \mathfrak{k}(\zeta,\theta)\nabla\theta$ is the heat flux. Adding it to (3.1), we can see cancellation of a lot of terms, in particular also $\pm(r{+}\psi_{\rm T}(\theta){\rm div}\dot{u})$, and we obtain (3.10). $\square$

Proposition 3.3 (Entropy balance.) *Let us again assume that the boundary-value problem (2.6)–(2.7) with (2.4) possesses a smooth solution and moreover let $\theta > 0$. Then:*

$$\underbrace{\frac{\rm d}{{\rm d}t}\int_\Omega\eta\,{\rm d}x}_{\text{total entropy}} = \int_\Omega \underbrace{\frac{r}{\theta}{+}\frac{\mathfrak{k}|\nabla\theta|^2}{\theta^2}}_{\substack{\text{entropy production}\\ \text{in the bulk }\Omega}}{\rm d}x - \int_\Gamma \underbrace{\frac{j_{\rm ext}}{\theta} + \eta\dot{u}{\cdot}\nu}_{\substack{\text{entropy flux thru}\\ \text{the boundary }\Gamma}}{\rm d}S\,. \tag{3.12}$$

Proof. Realizing the Gibbs relation $\vartheta = \psi_{\rm T}(\theta) + \theta\eta = \psi_{\rm T}(\theta) - \theta\psi_{\rm T}'(\theta)$ and the calculus

$$\begin{aligned}\dot{\vartheta} + {\rm div}(\dot{u}\vartheta) - (\vartheta{-}\theta\eta){\rm div}\,\dot{u} &= \big(\psi_{\rm T}(\theta){-}\theta\psi_{\rm T}'(\theta)\big)^{\bullet} + {\rm div}(\dot{u}\psi_{\rm T}(\theta){-}\dot{u}\theta\psi_{\rm T}'(\theta)) - \psi_{\rm T}(\theta){\rm div}\,\dot{u}\\ &= \theta\dot{\eta} + {\rm div}(\dot{u}\psi_{\rm T}(\theta)) - \theta{\rm div}(\dot{u}\psi_{\rm T}'(\theta)) - \psi_{\rm T}'(\theta)\dot{u}{\cdot}\nabla\theta - \psi_{\rm T}(\theta){\rm div}\,\dot{u}\\ &= \theta\big(\dot{\eta}{+}{\rm div}(\dot{u}\eta)\big) + {\rm div}(\dot{u}\psi_{\rm T}(\theta)) - \dot{u}{\cdot}\nabla\psi_{\rm T}(\theta) - \psi_{\rm T}(\theta){\rm div}\,\dot{u}\\ &= \theta\big(\dot{\eta}{+}{\rm div}(\dot{u}\eta)\big),\end{aligned} \tag{3.13}$$

the heat equation in the entropy formulation (3.9) can be written in the form of a so-called entropy equation as:

$$\theta\big(\dot{\eta}{+}{\rm div}(\dot{u}\eta)\big) = r - {\rm div}\,j \tag{3.14}$$

with the heat flux $j = -\mathfrak{k}\nabla\theta$. Hence, we obtain (3.12) by the following calculus:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \eta\,\mathrm{d}x = \int_\Omega \dot{\eta}\,\mathrm{d}x = \int_\Omega \frac{r - \mathrm{div}\, j}{\theta} - \mathrm{div}(\dot{u}\eta)\,\mathrm{d}x = \int_\Omega \frac{r}{\theta} + \frac{\mathfrak{k}|\nabla\theta|^2}{\theta^2}\,\mathrm{d}x - \int_\Gamma \frac{j_{\rm ext}}{\theta} + \eta\dot{u}{\cdot}\nu\,\mathrm{d}S. \qquad \square$$

Noteworthy, for the isolated system the last boundary integral vanishes in (3.12) and, counting that the heat-production rate r and the heat-transfer coefficient $\mathfrak{k}$ are non-negative, we obtain the *Clausius-Duhem inequality* $\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \eta\,\mathrm{d}x \geq 0$ representing the *2nd Law of thermodynamics*.

The positivity of temperature assumed in Proposition 3.3 can be seen from the original form of the heat-transfer equation (2.6e) by using usual arguments relying on physically relevant initial/boundary conditions in cooperation with the last (adiabatic) term in (2.6e) which may alternate sign (and hence cause both heating and cooling) but which is "switched off" if temperature approaches zero. In other words, our system is (at least formally) consistent also with the *3rd Law of thermodynamics*. The rigorous proof of the non-negativity of temperature is usually executed by the test by $\theta^- = \min(0,\theta)$. Let us emphasize that, however, the proof of positivity of θ is much more difficult and requires much stronger data qualification, cf. [25, Sect. 4.2.1], or [60, Remark 5.3.13] or also [71, Remark 12.10].

Note also the difference between (2.6e) and (3.9) expressing the advection of temperature θ as an intensive variable in contrast to the heat content ϑ as an extensive variable, cf. also Remark 3.3.

Remark 3.1 (Gradient theory used in the model (2.2).) The gradient theory for elastic strain (related to the κ_0-term, discussed already in Remark 2.2) as well as for the internal variables (related to the κ_i-terms with $i = 1, 2, 3$) is quite standard concept in literature and are used for decades and the related literature is very wide. For gradient plasticity, we refer to e.g. [1, 18, 39] or also [34, Sec. 4.3 and 7.3], while for the damage gradient to e.g. [5, Chap. 13] or [24, Chap. 12]. All these gradients are natural in the model for various reasons. The κ-coefficients yields in a way a typical internal length scaling:

- κ_0 is responsible for controlling a scale of a possible fine microstructure that may possibly occur due to nonconvexity of ψ in terms of $\varepsilon_{\rm e}$, like it is routinely observed in ferroelastic materials, cf. e.g. [3] or also e.g. [70],
- κ_1 "controls" the width of the shear (=slip) bands that can inelastic strain create,
- κ_2 controls the width of the damage zone, cf. e.g. [39], and
- κ_3 controls the internal length scale of the porosity.

The more specific quantitative influence of these κ-coefficients on the particular length scale may however be not simple, cf. e.g. [4, 5]. In this model, the coefficient $\kappa_1 \geq 0$ fully influences the internal scale of the plastic deformation π. In particular, for $\kappa_1 = 0$, one gets the fully rate-independent Prandtl-Reuss perfect plasticity which allows for developing an infinitesimally narrow shear bands, i.e. modelling of a fault core of zero thickness as a crack, surrounded by the damage zone whose width is controlled independently by $\kappa_2 > 0$. Cf. Remark 5.3 below. All the gradients $\nabla e(u)$, $\nabla\alpha$, and $\nabla\phi$ yield also a natural control of the gradient of the water content through (5.7) which facilitates convergence and existence of solutions, cf. the Appendix below.

Remark 3.2 (Intensive vs extensive physical quantities.) The classification intensive vs extensive standardly relates to the independence or dependency of the properties upon the size or extent of the system. Typically temperature θ or chemical potential (here pressure p) are intensive properties of the system but e.g. energy, entropy, or here the heat content ϑ used in (3.9) are extensive. Also the ratio of two extensive properties is scale-invariant, and is therefore an intensive property. The damage variable α, meaning the ratio between damaged and undamaged material, is thus an intensive property. Also the other fractions used here, i.e. porosity ϕ and water content ζ, are intensive. If the medium is compressed or expands, intensive variables do not change, in contrast to extensive ones. Thus, when transported in a moving medium, the transport equations must be designed differently for both mentioned class of variables unless the medium would be incompressible, cf. (2.6b–e) vs (3.9). To this goal, as we neglect here mass density variation, the significant attribute is rather "volume-dependent" vs. "volume-invariant", reflecting how the particular variable reacts on multiplication of volume rather than addition. Here however all extensive (resp. intensive) variables are volume-dependent (resp. volume-invariant). Yet, e.g. concentration (with the standard physical dimension mol/m^3) would be an intensive variable but volume-dependent.

Remark 3.3 (Structural stresses.) The structural stresses $\mathfrak{s}_{\rm el}$ as chosen here in (2.4a) are motivated primarily to achieve the balance energy (3.1) and (3.10), which is essentially based on formula (3.3). Actually, such stresses are known in incompressible-fluid mechanics under the name Korteweg stresses [43]. If an extensive variable is transported in a compressible medium, this stress takes rather the form

$$K_{\rm ext}=\nabla z{\otimes}\nabla z-\Big(z\Delta z+\frac12|\nabla z|^2\Big)\mathbb{I}=\nabla z{\otimes}\nabla z-\frac12\big(\Delta z^2-|\nabla z|^2\big)\mathbb{I}. \tag{3.15}$$

Such stress balances the time derivative $\dot z+{\rm div}(\dot uz)=\Delta z$ when tested by $\dot z+{\rm div}(\dot uz)$, which gives, when again assuming isolated system on the boundary $\partial\Omega$, that

$$\begin{aligned}\int_\Omega\big|\dot z{+}{\rm div}(\dot uz)\big|^2+\frac12\frac{\partial}{\partial t}|\nabla z|^2\,{\rm d}x&=\int_\Omega\Delta z{\rm div}(\dot uz)\,{\rm d}x=\int_\Omega z\Delta z({\rm div}\dot u)+\Delta z(\dot u{\cdot}\nabla z)\,{\rm d}x\\&=\int_\Omega\Big(\frac12|\nabla z|^2+z\Delta z\Big){\rm div}\,\dot u-(\nabla z{\otimes}\nabla z){:}e(\dot u)\,{\rm d}x=\int_\Omega K_{\rm ext}{:}e(\dot u)\,{\rm d}x,\end{aligned} \tag{3.16}$$

where, for the last-but-one equality, we used (3.3). Then testing the momentum-equilibrium equation $\varrho\ddot u-{\rm div}(\sigma+K_{\rm ext})=f$ by $\dot u$, we enjoy cancellation of the terms $\int_\Omega K_{\rm ext}{:}e(\dot u)\,{\rm d}x$ and obtain the energy balance. It reveals that $|\dot z{+}{\rm div}(\dot uz)|^2=|\Delta z|^2$ is the specific dissipation rate due to diffusion/convection. Such a Korteweg stress (3.15) can be found relatively frequently in literature e.g. in [11, Sec.2], [46, Formulas (4.15) + (6.15)], or [64, Formula (2.20)], etc. On the other hand, when considering a transport of an intensive variable, we would have the structure $\frac{\rm D}{{\rm D}t}z=\Delta z$ to be tested by $\frac{\rm D}{{\rm D}t}z$. It modifies the above considerations only by taking

$$K_{\rm int}=\nabla z{\otimes}\nabla z-\frac12|\nabla z|^2\mathbb{I} \tag{3.17}$$

instead of (3.15), cf. [19]. Then, instead of (3.16), we would have

$$\begin{aligned}\int_\Omega\Big|\frac{{\rm D}z}{{\rm D}t}\Big|^2+\frac12\frac{\partial}{\partial t}|\nabla z|^2\,{\rm d}x&=-\int_\Omega\Delta z(\dot u{\cdot}\nabla z)\,{\rm d}x\\&=\int_\Omega(\nabla z{\otimes}\nabla z){:}e(\dot u)-\frac12|\nabla z|^2{\rm div}\,\dot u\,{\rm d}x=\int_\Omega K_{\rm int}{:}e(\dot u)\,{\rm d}x.\end{aligned} \tag{3.18}$$

Again it reveals that $|\frac{\rm D}{{\rm D}t}z|^2=|\Delta z|^2$ is the specific dissipation rate due to diffusion/convection. In particular, if the moving medium is incompressible, i.e. ${\rm tr}\,e(\dot u)={\rm div}\,\dot u=0$, both options of transport of extensive or intensive variable coincide with each other and, in particular, the term $\frac12|\nabla z|^2{\rm div}\,\dot u$ in (3.18) disappears and $K_{\rm int}$ in (3.17) reduces to $K_{\rm int}=\nabla z{\otimes}\nabla z$, which was used e.g. in [48, 26]. In geophysical models, such reduced Korteweg stress is used under the name structural stress in [51, 54] even in compressible situations but without deriving any global energy balance analogous to (3.1) or (3.10). Even more, if the diffusion/convection would be accompanied by a reaction term, then, e.g. in the case of an intensive variable, the equation in question would be $\frac{\rm D}{{\rm D}t}z=\Delta z+\psi'(z)$. Again it is to be tested by $\frac{\rm D}{{\rm D}t}z$, which leads to the structural stress

$$K_{\rm int}=\nabla z{\otimes}\nabla z-\Big(\frac12|\nabla z|^2+\psi(z)\Big)\mathbb{I} \tag{3.19}$$

instead of (3.17), cf. again [19]. The contribution $\psi(z)$ to the pressure in (3.19) arises by augmenting (3.18) by the calculation $\int_\Omega \psi'(z)(\dot u\cdot\nabla z)\,\mathrm{d}x = \int_\Omega \psi'(z)\dot u\cdot\nabla\psi(z)\,\mathrm{d}x = -\int_\Omega \psi(z)\mathrm{div}\,\dot u\,\mathrm{d}x$. For such a pressure term in the extensive-variable case, where such an additional pressure term takes the form $\psi(z)-z\psi'(z)$ instead of $\psi(z)$, we refer e.g. [20].

Remark 3.4 (The energy balance (3.10).) The last terms in the energy balances (3.1) and (3.10) are rather not desired and expected. Its meaning is not entirely clear, and it may be a result of a conceptual discrepancy of mixing the concept of small strains with large displacements (and using the additive decomposition which seems justified only up to higher-order terms, as claimed in Proposition 2.1)

or of the missing (and varying) mass density ϱ to be governed by the diffusion-less extensive-variable transport $\dot\varrho + \mathrm{div}(\dot u\varrho) = 0$. Yet, there is a certain experience that the influence of the structural stresses is anyhow very small in specific geophysical simulations [49]. In addition, in typical (rather stratified) geophysical simulations of moving lithospheric domains adjacent to straight faults as already mentioned in Proposition 2.1, the vectors $\dot u$ and $\nabla e_{ij}(u)$ are (mostly) orthogonal to each other so that our term $(\dot u\otimes\sigma_{\rm el})\vdots\nabla e_{ij}(u)$ in (3.1) and (3.10) is (nearly) zero. Likewise, the volume variations are (mostly) not substantial so that also the term $\psi_{\rm T}(\theta)\mathrm{div}\,\dot u$ might be nearly zero, too. This justifies in some sense this (to some extent still simplified) model from the viewpoint of geophysical applications. It should be also mentioned that considering $\mathfrak{s}_{\rm el}$ without any pressure-like term (as often done in literature, cf. Remark 3.3) would instead give rise many other terms, namely $\frac12(\kappa_1|\nabla\pi|^2+\kappa_2|\nabla\alpha|^2+\kappa_3|\nabla\phi|^2)\mathrm{div}\,\dot u - \tilde\psi'_\pi{:}(\dot u\cdot\nabla\pi) - \tilde\psi'_\phi(\dot u\cdot\nabla\phi) - \tilde\psi'_\alpha(\dot u\cdot\nabla\alpha) - \tilde\psi'_\zeta(\dot u\cdot\nabla\zeta)$ whose estimation like in Section 5 below would be questionable and which need not be small even in the stratified situations.

Remark 3.5 (Surface tension in water.) Notably, there is no contribution to the structural stresses in (2.4b) coming from $\nabla\zeta$, which is due to that we did not consider any gradient theory for the diffusion. Of course, if (2.2) were augmented also $\frac12\kappa_4|\nabla\zeta|^2$, the chemical potential would have been $p_{\rm tot} = \tilde\psi'_\zeta - \mathrm{div}\tilde\psi'_{\nabla\zeta}$ and (2.6d) would got a form of the Cahn-Hilliard [13] equation $\frac{\mathrm D}{\mathrm Dt}\zeta = \mathrm{div}(\mathrm m\nabla p_{\rm tot})$ and, if $\mathrm m$ is constant, the dissipation rate $\mathrm m|\nabla p|^2$ would been expanded for $\kappa_4|\mathrm m\Delta\zeta|^2$. Such additional term would describe a surface tension and there is wast literature addressing physics, analysis, and numerics possibly coupled with mechanical effect including also damage as here, cf. e.g. [23, 35, 73] and references therein.

4 Special cases towards geophysical applications

We now be still more specific about $\psi_{\rm M}$ in the ansatz (2.2), having in mind certain concrete geophysical applications. We introduce the notation for the two invariants of the elastic strain

$$I_1 = \mathrm{tr}\,\varepsilon_{\rm e} = \mathrm{tr}\,e(u) - \mathrm{tr}\,e_{\rm i} = \mathrm{div}\,u - 3\phi \qquad \text{and} \tag{4.1a}$$

$$I_2 = |\varepsilon_{\rm e}|^2 = |e(u)|^2 + 9\phi^2 + |\pi|^2 - 2\phi\,\mathrm{div}\,u - 2e(u){:}\pi, \tag{4.1b}$$

where we used already the ansatz (2.1), and further consider the following ansatz for the free energy:

$$\psi_{\rm M} = \psi_{\rm M}(\varepsilon_{\rm e},\alpha,\phi,\zeta,\theta) = \frac12\lambda(\alpha,\phi)I_1^2 + \mu(\alpha,\phi)I_2 - \gamma(\alpha,\phi)I_1\sqrt{I_2} + \frac12 M(\alpha,\phi)|\beta I_1 - \zeta + \phi|^2 + \chi\alpha. \tag{4.2}$$

For the Biot M-term see e.g. [53] while the non-Hookean γ-term was suggested in [55].

Note that, in view of (4.2) with $e = e(u)$, we have a simple expression

$$\sigma_{\rm el} = \Big(\lambda(\alpha,\phi) - \frac{\gamma(\alpha,\phi)}{\xi}\Big) I_1 \mathbb{I} + \beta M(\alpha,\phi)(\beta I_1 - \zeta + \phi)\big)\mathbb{I} \\ + \big(2\mu(\alpha,\phi) - \xi\gamma(\alpha,\phi)\big)\varepsilon_{\rm e} \quad \text{with the strain invariant ratio } \ \xi = \frac{I_1}{\sqrt{I_2}}, \tag{4.3a}$$

$$\sigma_{\rm i} = \big(2\mu(\alpha,\phi) - \gamma(\alpha,\phi)\big)\varepsilon_{\rm e}, \tag{4.3b}$$

$$\sigma_{\rm dam} = \frac{1}{2}\lambda'_\alpha(\alpha,\phi)I_1^2 + \mu'_\alpha(\alpha,\phi)I_2 - \gamma'_\alpha(\alpha,\phi)I_2\xi + \chi + \frac{1}{2}M'_\alpha(\alpha,\phi)|\beta I_1 - \zeta + \phi|^2, \tag{4.3c}$$

$$p_{\rm eff} = \frac{1}{2}\lambda'_\phi(\alpha,\phi)I_1^2 + \mu'_\phi(\alpha,\phi)I_2 - \gamma'_\phi(\alpha,\phi)I_2\xi + \frac{3}{2}(3\phi - {\rm div}\,u)\lambda(\alpha,\phi) + 2\mu(\alpha,\phi){\rm tr}\,\varepsilon_{\rm e} \\ - \gamma(\alpha,\phi)\xi'_\phi - p + \frac{1}{2}M'_\phi(\alpha,\phi)|\beta I_1 - \zeta + \phi|^2 + (1-3\beta)M(\alpha,\phi)\big((1-3\beta)\phi - \zeta + \beta{\rm div}\,u\big), \tag{4.3d}$$

$$p = M(\alpha,\phi)(\zeta - \beta I_1 - \phi), \tag{4.3e}$$

where $\mathbb{I}$ denotes the identity matrix, and while the heat capacity $c_{\rm v}(\theta) = -\theta\psi''_{\rm T}(\theta)$ is again as in (2.4g) which yield the form of the heat content:

$$\vartheta = C_{\rm v}(\theta) = \psi_{\rm T}(\theta) - \theta\psi'_{\rm T}(\theta). \tag{4.3f}$$

Here we already used that ${\rm tr}\,\pi = 0$. In (4.3a), we use the standard notation ξ for the so-called strain invariant ratio, ranging from $-\sqrt{3}$ for isotropic compaction to $\sqrt{3}$ for isotropic dilation.

Let us first specify the above (still general) model for a relatively simple particular case, ignoring the water flow and porosity, i.e. $M \equiv 0$ and $\phi \equiv 0 \equiv \zeta$, and (2.6c) simplifies while (2.6d) does not occur in the resulting system at all. Such sort of models has been devised in literature by setting

$$\lambda(\alpha) = \lambda_0, \qquad \mu(\alpha) = \mu_0 - \alpha\mu_{\rm r}, \qquad \gamma(\alpha) = \alpha\gamma_{\rm r}, \tag{4.4a}$$

cf. [30, 31] together with $\mathfrak{D}_2(\alpha;\cdot) = \partial_{\dot\alpha}\delta_2(\alpha;\cdot)$ given by the pseudopotential of dissipative forces for the damage evolution $\delta_2(\alpha,\cdot) : \mathbb{R} \to \mathbb{R}$ as

$$\delta_2(\alpha;\dot\alpha) = \begin{cases} \frac{1}{2c_0}\dot\alpha^2 & \text{if } \dot\alpha \ge 0, \\ \frac{1}{2c_1}{\rm e}^{-\alpha/c_2}\dot\alpha^2 & \text{if } \dot\alpha \le 0 \end{cases} \tag{4.4b}$$

with some c_0, c_1, and c_2 positive. Let us note that $\delta_2(\alpha;\cdot)$ is convex, degree-2 homogeneous, and non-differentiable at $\dot\alpha = 0$. This last property corresponds to the activation phenomena both for damage and for healing, while the degree-2 homogeneity yields that the damage dissipation rate is

$$\mathfrak{D}_2(\alpha;\dot\alpha)\dot\alpha = 2\delta_2(\alpha;\dot\alpha). \tag{4.5}$$

Then, denoting by $\delta_2^*(\alpha;\cdot)$ the conjugate functional to $\delta_2(\alpha;\cdot)$, by the convex-analysis calculus we obtain

$$[\mathfrak{D}_2(\alpha;\cdot)]^{-1}(z) = \partial_z\delta_2^*(\alpha;z) = \begin{cases} c_0 z & \text{if } z \ge 0, \\ c_1{\rm e}^{\alpha/c_2} z & \text{if } z \le 0, \end{cases} \tag{4.6}$$

which has been actually devised in [52, Formula (42)] and later used e.g. in [31, 50, 54] without identifying the underlying dissipation potential (4.4b), however. Here $\sigma_{\rm dam,tot}$ from (2.4f) now uses $\sigma_{\rm dam}$ from (4.3c) in the form

$$\sigma_{\rm dam,tot} = \gamma_{\rm r} I_1\sqrt{I_2} - \mu_{\rm r} I_2 = \gamma_{\rm r} I_2(\xi - \xi_0) \quad \text{with} \quad \xi_0 = -\frac{\mu_{\rm r}}{\gamma_{\rm r}} \tag{4.7}$$

with $\mu_{\rm r}$ and $\gamma_{\rm r}$ from (4.4) with ξ_0 having the meaning of a critical strain invariant ratio which is a decisive threshold between damaging and healing. The damage flow rule (2.6c), i.e. now $\partial_{\dot\alpha}\delta_2(\alpha;\frac{{\rm D}\alpha}{{\rm D}t})\ni\kappa_2\Delta\alpha-\sigma_{\rm dam}-d$ with the reaction force $d\in N_{[0,1]}(\alpha)$ = the normal cone to the interval $[0,1]$ where damage α is assumed to be valued, can be rewritten as $\frac{\rm D}{{\rm D}t}\alpha\in\partial_z\delta_2^*(\alpha;\kappa_2\Delta\alpha{-}\sigma_{\rm dam}{-}d)$. In view of (4.6), we arrive at

$$\frac{{\rm D}\alpha}{{\rm D}t}=\begin{cases}c_0\big(\gamma_{\rm r}I_2(\xi{-}\xi_0)+\kappa_2\Delta\alpha\big)&\text{if }\gamma_{\rm r}I_2(\xi{-}\xi_0)+\kappa_2\Delta\alpha\ge0,\\ c_1{\rm e}^{\alpha/c_2}\big(\gamma_{\rm r}I_2(\xi{-}\xi_0)+\kappa_2\Delta\alpha\big)&\text{otherwise.}\end{cases}\tag{4.8}$$

Nearly the same flow rule has been devised in [51, Formula (25)], except that the switching threshold in (4.8) has incorrectly ignored the κ_2-term in [51], however.

The flow rule (2.6b) can cover a conventional linearized plasticity model (either rate-dependent or rate-independent) with a yield stress $\sigma_{\rm y}=\sigma_{\rm y}(\alpha)$ for plastification possibly dependent on damage. Counting isotropic material, such a model is governed by the potential

$$\delta_1(\alpha;\dot\pi)=\sigma_{\rm y}(\alpha)|\dot\pi|.\tag{4.9}$$

Now, $\delta_1(\alpha;\cdot)$ is degree-1 homogeneous so that, in contrast to (4.5) where the factor "2" occurred, the dissipation rate is

$$\mathfrak{D}_1(\alpha;\dot\pi)\dot\pi=\delta_1(\alpha;\dot\pi).\tag{4.10}$$

A specific, even more general yield stress $\sigma_{\rm y}=\sigma_{\rm y}(\alpha,\xi)$ depending also on ξ has been devised in [50, Formula (11)].

Let us now briefly come to the model of poroelastic damageable rocks, and consider other internal variables: the porosity ϕ and the water content ζ. The nonlinearities (4.4) are then considered in [30, 53] modified as

$$\lambda(\alpha,\phi)=\lambda_0\big(1{-}\phi/\phi_{\rm cr}\big),\tag{4.11a}$$
$$\mu(\alpha,\phi)=\big(\mu_0-\alpha\mu_{\rm r}\big)\big(1{-}\phi/\phi_{\rm cr}\big),\tag{4.11b}$$
$$\gamma(\alpha,\phi)=\alpha\gamma_{\rm r}\big(1{-}\phi/\phi_{\rm cr}\big),\tag{4.11c}$$
$$M(\phi)=M_0\big(1{-}\phi/\phi_{\rm cr}\big)\tag{4.11d}$$

where $\phi_{\rm cr}$ denotes the porosity upper bound in which the material loses its stiffness. Moreover, the driving force for porosity evolution (4.3d) now simplifies as

$$p_{\rm eff}=-\frac{1}{3\phi_{\rm cr}}{\rm tr}\sigma_{\rm el}^0-p\tag{4.11e}$$

with $\sigma_{\rm el}^0$ from (4.3a) for $\phi=0$ and p again from (4.3e). The flow-rule for damage (4.8) may modify and a flow-rule for porosity added as

$$\frac{{\rm D}\alpha}{{\rm D}t}=\begin{cases}c_0\big(\gamma_{\rm r}I_2(\xi{-}\xi_0)+\kappa_2\Delta\alpha\big)&\text{if }\gamma_{\rm r}I_2(\xi{-}\xi_0)+\kappa_2\Delta\alpha\ge0,\\ c_1{\rm e}^{\alpha/c_2}{\rm e}^{b(\phi_0-\phi)}\big(\gamma_{\rm r}I_2(\xi{-}\xi_0)+\kappa_2\Delta\alpha\big)&\text{otherwise,}\end{cases}\tag{4.12a}$$
$$\frac{{\rm D}\phi}{{\rm D}t}=d(\phi)\big|p_{\rm eff}{+}\kappa_3\Delta\phi\big|^n\big(p_{\rm eff}{+}\kappa_3\Delta\phi\big).\tag{4.12b}$$

This would lead to a potential which is $(2,\frac{n+2}{n+1})$-homogeneous in terms of the rates $(\dot\alpha,\dot\phi)$:

$$\delta_2(\alpha,\phi;\dot\alpha,\dot\phi)=\tfrac{n+1}{n+2}d(\phi)^{-n-1}|\dot\phi|^{(n+2)/(n+1)}+\begin{cases}\frac{1}{2c_0}\dot\alpha^2&\text{if }\dot\alpha\ge0,\\ \frac{1}{2c_1}{\rm e}^{-\alpha/c_2}{\rm e}^{-b(\phi_0-\phi)}\dot\alpha^2&\text{if }\dot\alpha\le0.\end{cases}\tag{4.13}$$

Actually, a non-dissipative coupling between both flow rules in (4.12) has been considered in an antisymmetric way in [30, 31, 53]. Such dissipation does not have any potential and even does not control $\frac{\mathrm{D}\phi}{\mathrm{D}t}$ so that no standard existence theories for solutions is applicable, although sometimes [56] it is admitted that it is not necessary and a dissipative coupling makes sense, too (and in particular a symmetrical variant with a potential similarly as in [51] for a similar model with a granular-phase field instead of the porosity). In fact, even a non-monotone dependence of the right-hand side of (4.12b) has been devised in [30, 53] for $n \neq 0$, which is again not consistent with any standard existence theories for solutions.

Note that we count with dissipation of mechanical energy via diffusion $\mathfrak{m}|\nabla p|^2$ occurring in (3.1) through (2.6f), see also e.g. [30, Formula (11)].

It should be emphasized that no explicit energetics in a form like that one presented in Sect. 3 can be read from the cited (and many other) geophysical articles [49] and thus any attempt for rigorous analysis has ever not been made. Sometimes, even obviously misconceptual models are used, like negative entropy production due to negative friction (as already noted in [16], see also a critical survey with many such references in [72]) or strain acceleration contributing to the entropy production [54, Formulas (6) and (a10)]. Thus, using the concrete data in this section in the general model devised in Sections 2–3 yield a qualitative improvement of existing models in geophysical literature.

Remark 4.1 (Driving force for healing.) Let us note that the healing is possible due to two mechanisms, the possibly negative values of γ (which typically dominates in geophysical models [49]) and the χ-term in (4.2) which reflects the phenomenon that microcracks and microvoids related to damage bears a certain energy $\chi > 0$ and nature likes minimizing energy.

Remark 4.2 (Maxwell rheology.) Instead of (4.9), one could consider $\delta_1(\alpha;\dot{\pi}) = \frac{1}{2}\lambda_\mathrm{v}|\mathrm{tr}\dot{\pi}|^2 + \mu_\mathrm{v}\dot{\pi} : \dot{\pi}$ with some Lamé type coefficients $\mu_\mathrm{v} > 0$ and $\lambda_\mathrm{v} > 2\mu_\mathrm{v}/3$. This results to the Maxwell rheology. In geophysical models of the upper lithosphere, usually these "viscous moduli" takes very large values about $10^{22\pm2}\mathrm{Pa\,s}$ and this *creep* rheology applies under very slow load around µm/year which does not lead to earthquakes. A combination with activated plasticity (4.9) activated by large stress and leading to earthquakes is well possible by adding another strain into the splitting (2.1), cf. also e.g. [75]. Due to its hyperbolic character, the Maxwell rheology then exhibits a very low attenuation reflecting a relatively easy propagation of seismic waves on very long distances.

Remark 4.3 (Rate-and-state depedendent friction.) This model can be used also for the popular rate-and-state dependent friction [16] used for contact of elastic rocks on lithospheric faults. The damage α is then interpreted as the so-called aging variable, while the role of temperature may serve both to follow the velocity of mutual shift on the fault to reflect low-velocity laboratory experiments governed by the Dieterich-Ruina-type laws, cf. [72], or/and to reflect the so-called flash heating leading even to melting during high-velocity experiments or real earthquakes, cf. e.g. [7, 17, 68].

Remark 4.4 (Validity of the model in time and further transitions.) Damage models used in geophysics often play with an idea that their validity is terminated when the material is completely disintegrated. Then a so-called time-to-failure is considered as a vital outcome and sometimes estimates in 0-dimensional models do exist [49]. The model looses validity beyond this time. This is motivated by some laboratory experiments. Apart that this validity time is not estimated (and likely can be zero) in multi-dimensional cases, the physical time certainly goes on forever, in particular in the real geophysical applications. Models capturing global-in-time validity are certainly physically worth considering and should then either prevent complete disintegration (reflecting the phenomenon that rocks are staying relatively compact even when partly damaged in seismogenic zones many kilometers deep, as considered in this paper), or cope with possible complete disintegration of the material either by considering a complete damage (which is mathematically difficult and restricted so far on rather simpler models without healing, see [9, 61] or [60, Sect. 4.3.2.2]) or allow for a transition to a granular-material-type models (see e.g. [36, 37, 51], and also [38, 78] for a wide menagerie of granular-type models).

5 Appendix: mathematical analysis outlined

The mathematical analysis yields important theoretical justification of the system under consideration as far as mere existence of its (suitably defined) solution, and may yield a useful hint for a strategy concerning a numerical discretisation which would be stable and even convergent. Apart mere existence, a further and more difficult goal (not addressed in this paper) would be regularity (=smoothness) of (some or all) solutions which was, in fact, used in Propositions 3.1–3.3 in the position of an assumption.

Within the presented model, the rigorous energy conservation unfortunately seems not ensured and, related to this drawback, except Remark 5.1, we confine ourselves to a model either restricted to an isothermal situation or modified to small displacements with Kelvin-Voigt rheology. The later option is particularly relevant during ongoing earthquakes of medium magnitude when displacement is indeed small although the heat production is considerable. On top of it, we confine ourselves to rather special boundary conditions instead of (2.7a), cf. Remark 5.2, namely

$$u = 0 \quad \text{and} \quad \mathfrak{h}_{\rm el}{:}(\nu\otimes\nu) = 0 \quad \text{on } \Gamma. \tag{5.1}$$

The analysis is then similar to [76] except that the damage considered here is rate dependent and allows also for healing, and except that we consider 2nd-grade nonsimple material but do not consider any Kelvin-Voigt-type rheology so that the momentum equation (2.6a) has a hyperbolic instead of a parabolic character, even if another attenuation as in Remark 4.2 would be considered.

We only briefly outline main features and tricks. We use the standard notation for function spaces, namely L^p for the Lebesgue space of measurable functions whose p-power is integrable, $W^{k,p}$ for Sobolev spaces whose k-th derivatives are in L^p-spaces, and the abbreviation $H^k = W^{k,2}$. Also, $H^{-k} = (W_0^{k,2})^*$ denotes the dual space to H_0^k which is a subspace of H^k of functions with zero traces on the boundary. We further consider a fixed time interval $I = [0, T]$ and we denote by $L^p(I;X)$ the standard Bocher space of Bochner-measurable mappings $I \to X$ with X a Banach space. Also, $W^{k,p}(I;X)$ denotes the Banach space of mappings from $L^p(I;X)$ whose k-th distributional derivative in time is also in $L^p(I;X)$.

We consider an initial-boundary-value problem for the system (2.6) with the initial conditions at time $t = 0$:

$$u(0)=u_0,\quad \dot{u}(0)=v_0,\quad \pi(0)=\pi_0,\quad \zeta(0)=\zeta_0,\quad \phi(0)=\phi_0,\quad \alpha(0)=\alpha_0,\quad \vartheta(0)=\vartheta_0. \tag{5.2}$$

By a solution, we will understand a conventional weak solution to (2.6), cf. e.g. [71, Chap. 13] for the doubly-nonlinear inclusions involved in (2.6). Without going into any technical details, we assume that we have approximated somehow the system (2.4)–(2.6) in a constructive way, and denote its solution by $(u_h, \pi_h, \alpha_h, \phi_h, \zeta_h, \theta_h)$ as well as the corresponding pore pressure p_h and the rescaled temperature ϑ_h with $h > 0$ a discretisation parameter. One can imagine discretisation in space (i.e. the Galerkin approximation) based on a finite-element method with h denoting a mesh parameter, or a suitable semi-implicit decoupled discretisation in time like in [76, Sect. 5] with $h > 0$ denoting a time step.

Let us summarize the main assumptions considering the isothermal case $\psi_{\rm T} \equiv 0$, again without going into technical details and without optimizing:

$$\psi_{\rm M},\ \delta_1,\ \delta_2,\ \mathfrak{m},\ \mathfrak{k},\ \text{ continuous together with all their second derivatives in all their arguments,}$$
$$\text{considering } \alpha,\phi,\zeta \text{ ranging over in } [0,1]^3, \text{ and } \theta \text{ ranging over in } [0,\infty), \tag{5.3a}$$

$$\int_\Omega \tilde{\psi}(e(u),\pi,\alpha,\phi,\zeta,\nabla e,\nabla\pi,\nabla\alpha,\nabla\phi)\,{\rm d}x \ge \epsilon\|u\|^2_{H^2(\Omega;\mathbb{R}^3)} + \epsilon\|\nabla\pi\|^2_{L^2(\Omega;\mathbb{R}^{3\times3})} + \epsilon\|\phi\|^2_{H^1(\Omega)} + \epsilon\|\alpha\|^2_{H^1(\Omega)} - 1/\epsilon, \tag{5.3b}$$

$$\big|\psi_{\rm M}'(\varepsilon_{\rm e},\alpha,\phi,\zeta)\big| \le 1/\epsilon, \quad \epsilon \le [\psi_{\rm M}]''_{\zeta\zeta}(\varepsilon_{\rm e},\alpha,\phi,\zeta) \le 1/\epsilon, \tag{5.3c}$$

$$\big|[\psi_{\rm M}]''_{\zeta\varepsilon_{\rm e}}(\varepsilon_{\rm e},\alpha,\phi,\zeta)\big| \le 1/\epsilon,\ \ \big|[\psi_{\rm M}]''_{\zeta\alpha}(\varepsilon_{\rm e},\alpha,\phi,\zeta)\big| \le 1/\epsilon,\ \ \big|[\psi_{\rm M}]''_{\zeta\phi}(\varepsilon_{\rm e},\alpha,\phi,\zeta)\big| \le 1/\epsilon, \tag{5.3d}$$

$$\delta_1(\alpha,\phi,\theta;\cdot):\mathbb{R}^{3\times3}_{\rm dev}\to\mathbb{R}\ \text{convex, with quadratic growth and coercivity}, \tag{5.3e}$$
$$\delta_2(\alpha,\phi,\theta;\cdot):\mathbb{R}^{2}\to\mathbb{R}\quad \text{convex, with quadratic growth and coercivity}, \tag{5.3f}$$
$$\epsilon\le\mathfrak{m}(\alpha,\phi)\le 1/\epsilon,\ \ \epsilon\le\mathfrak{k}(\zeta,\theta)\le1/\epsilon, \tag{5.3g}$$
$$\varrho,\,\kappa_0,\,\kappa_1,\,\kappa_2,\,\kappa_3>0,\quad g\in L^2(0,T;L^1(\Omega;\mathbb{R}^3)),\quad f_{\rm ext}\in L^2(0,T;L^2(\Gamma)), \tag{5.3h}$$
$$u_0\!\in\! H^2(\Omega;\mathbb{R}^3),\ \ v_0\!\in\! L^2(\Omega;\mathbb{R}^3),\ \ \pi_0\!\in\! H^1(\Omega;\mathbb{R}^{3\times3}_{\rm dev}),$$
$$\zeta_0,\alpha_0,\phi_0\!\in\! H^1(\Omega),\ \ \text{and}\ \ \alpha_0(x),\,\phi_0(x),\,\zeta_0(x)\in[0,1]\ \ \text{for all}\ x\in\Omega, \tag{5.3i}$$

for some $\epsilon>0$, where $\psi_{\rm M}$ refers to (2.2) and $\tilde\psi$ to (2.3).

Lemma 5.1 (A-priori estimates) *Let the assumptions* (5.3) *hold. Then the (unspecified) approximate solution* $(u_h,\pi_h,\alpha_h,\phi_h,\zeta_h,\theta_h,p_h,\vartheta_h)$ *to the system* (2.4)–(2.6) *together with* (2.2) *and with the boundary conditions* (2.7b-d) *and* (5.1) *satisfies the following a-priori estimates:*

$$\big\|u_h\big\|_{L^\infty(0,T;H^2_0(\Omega;\mathbb{R}^3))\,\cap\, W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^3))\,\cap\, W^{2,1}(0,T;H^{-3}(\Omega;\mathbb{R}^3))}\le C, \tag{5.4a}$$
$$\big\|\pi_h\big\|_{L^\infty(0,T;H^1(\Omega;\mathbb{R}^{3\times3}_{\rm dev}))\,\cap\, H^1(0,T;L^1(\Omega;\mathbb{R}^{3\times3}_{\rm dev}))}\le C, \tag{5.4b}$$
$$\big\|\alpha_h\big\|_{L^\infty([0,T]\times\Omega)\,\cap\, L^\infty(0,T;H^1(\Omega))\,\cap\, H^1(0,T;L^1(\Omega))}\le C, \tag{5.4c}$$
$$\big\|\phi_h\big\|_{L^\infty([0,T]\times\Omega)\,\cap\, L^\infty(0,T;H^1(\Omega))\,\cap\, H^1(0,T;L^1(\Omega))}\le C, \tag{5.4d}$$
$$\big\|p_h\big\|_{L^\infty([0,T]\times\Omega)\,\cap\, L^2(0,T;H^1(\Omega))}\le C, \tag{5.4e}$$
$$\big\|\zeta_h\big\|_{L^\infty([0,T]\times\Omega)\,\cap\, L^2(0,T;H^1(\Omega))\,\cap\, H^1(0,T;H^1(\Omega)^*)}\le C \tag{5.4f}$$

with some $C<\infty$. In addition, if Ω is smooth, then for some C_Ω also

$$\big\|\pi_h\big\|_{L^2(0,T;H^2(\Omega;\mathbb{R}^{3\times3}_{\rm dev}))}\le C_\Omega,\ \ \big\|\phi_h\big\|_{L^2(0,T;H^2(\Omega))}\le C_\Omega,\ \ \big\|\alpha_h\big\|_{L^2(0,T;H^2(\Omega))}\le C_\Omega, \tag{5.4g}$$

and the estimates in (5.4a-d) improve by

$$\big\|\ddot u_h\big\|_{L^1(0,T;H^{-2}(\Omega;\mathbb{R}^3))}\le C_\Omega\quad\text{and}\quad\big\|(\dot\pi_h,\dot\alpha_h,\dot\phi_h)\big\|_{L^2(0,T;L^3(\Omega;\mathbb{R}^{3\times3}_{\rm dev}\times\mathbb{R}\times\mathbb{R}))}\le C_\Omega. \tag{5.4h}$$

Sketch of the proof. The estimates (5.4a-e) follow from the mechanical-energy balance (3.1) written for the approximate solution; for technicalities related to a fractional-step-type semi-implicit time-discretisation with a suitable regularization of the growth of the heat sources see [76]. The bulk right-hand side term in (3.1) can be estimated as

$$\Big|\int_\Omega(\dot u\otimes\sigma_{\rm el})\,\vdots\,\nabla e(u)\,\mathrm{d}x\Big|\le\big\|\sigma_{\rm el}\big\|_{L^\infty(\Omega;\mathbb{R}^{3\times3})}\big\|\dot u\big\|_{L^2(\Omega;\mathbb{R}^3)}\big\|\nabla e(u)\big\|_{L^2(\Omega;\mathbb{R}^{3\times3\times3})}$$
$$\le\frac{\sup|\tilde\psi_e'|}{4\epsilon}\big\|\dot u\big\|^2_{L^2(\Omega;\mathbb{R}^3)}+\epsilon\big\|\nabla e(u)\big\|^2_{L^2(\Omega;\mathbb{R}^{3\times3\times3})} \tag{5.5}$$

and then, for a sufficiently small $\epsilon>0$, use the coercivity (5.3b) and the Gronwall inequality, relying on the kinetic-energy term. Realizing that, due to (2.3), $\tilde\psi_\zeta'=[\psi_{\rm M}]_\zeta'(\varepsilon_{{\rm e},h},\alpha_h,\phi_h,\zeta_h)$ so that

$$\nabla p_h=\nabla(\tilde\psi_\zeta')=\tilde\psi_{\zeta\zeta}''\nabla\zeta_h+\tilde\psi_{\zeta\varepsilon_{\rm e}}''\nabla\varepsilon_{{\rm e},h}+\tilde\psi_{\zeta\alpha}''\nabla\alpha_h+\tilde\psi_{\zeta\phi}''\nabla\phi_h, \tag{5.6}$$

and using (5.3c) and (5.3d), we can perform also the estimation of the boundary term

$$\int_\Gamma p_hf_{\rm ext}\!\cdot\!\nu\,\mathrm{d}S\le\int_\Gamma\frac{1}{4\epsilon}f_{\rm ext}^2+\epsilon p_h^2\,\mathrm{d}S\le C_\epsilon\!\int_\Omega 1+\epsilon|\nabla\zeta_h|^2+\epsilon|\nabla\varepsilon_{{\rm e},h}|^2+\epsilon|\nabla\alpha_h|^2+\epsilon|\nabla\phi_h|^2\,\mathrm{d}x$$

and then, choosing $\epsilon > 0$ sufficiently small, to absorb the respective terms in the left-hand side by using the coercivity (5.3b). An estimate of π_h is little peculiar due to the absence of any hardening in the stored energy, but one can use the dissipation $\mathfrak{D}_1$ which controls $\frac{\mathrm{D}}{\mathrm{D}t}\pi_h \in L^2([0,T]\times\Omega;\mathbb{R}^{3\times3})$ due to (5.3e) and thus also $\dot{\pi}_h = \frac{\mathrm{D}}{\mathrm{D}t}\pi_h - \dot{u}_h\cdot\nabla\pi_h$ is controlled in $L^2(0,T;L^1(\Omega;\mathbb{R}^{3\times3}))$ because the velocity $\dot{u}_h$ is bounded in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$ thanks to the inertial term and because $\nabla\pi_h$ is bounded in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^{3\times3}))$. Therefore also π_h is controlled in $L^\infty(0,T;L^1(\Omega;\mathbb{R}^{3\times3}))$ and, due to the estimate of $\nabla\pi_h \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^{3\times3}))$ thanks to $\kappa_1 > 0$ assumed in (5.3h) we eventually have $\pi_h \in L^\infty(0,T;H^1(\Omega;\mathbb{R}^{3\times3}))$. Similar arguments show boundedness of $\dot{\alpha}_h = \frac{\mathrm{D}}{\mathrm{D}t}\alpha_h - \dot{u}_h\cdot\nabla\alpha_h$ and of $\dot{\phi}_h = \frac{\mathrm{D}}{\mathrm{D}t}\phi_h - \dot{u}_h\cdot\nabla\phi_h$ in $L^2(0,T;L^1(\Omega))$. Altogether, we thus have obtained the estimates (5.4a-e) except the $W^{2,1}$-estimate of u_h. This last estimate can be obtained by comparison $\ddot{u}_h = (\mathrm{div}\,\sigma_{\mathrm{el,tot},h} + g)/\varrho = (\mathrm{div}(\sigma_{\mathrm{el},h}{+}\mathfrak{s}_{\mathrm{el},h}) - \mathrm{div}^2\,\mathfrak{h}_{\mathrm{el},h} + g)/\varrho$ when realizing that, due to the already obtained estimates, $\sigma_{\mathrm{el},h} = \psi'_{\mathrm{M}}(e(u_h) - \pi_h - \phi_h\mathbb{I}, \alpha_h, \phi_h, \zeta_h) \in L^\infty([0,T]\times\Omega;\mathbb{R}^{3\times3}))$, $\mathfrak{s}_{\mathrm{el},h} \in L^\infty(0,T;L^1(\Omega;\mathbb{R}^{3\times3}))$, and $\mathfrak{h}_{\mathrm{el},h} = \kappa_0\nabla e(u_h) \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^{3\times3\times3}))$, cf. (2.4b).

Using again (5.6), also the L^2-estimate of $\nabla\zeta_h$ contained in (5.4f) then follows from

$$\nabla\zeta_h = \big[\tilde{\psi}''_{\zeta\zeta}\big]^{-1}\big(\nabla p_h - \tilde{\psi}''_{\zeta\varepsilon_{\mathrm{e}}}\nabla\varepsilon_{\mathrm{e},h} - \tilde{\psi}''_{\zeta\alpha}\nabla\alpha_h - \tilde{\psi}''_{\zeta\phi}\nabla\phi_h\big), \tag{5.7}$$

exploiting also the boundedness of $[\tilde{\psi}''_{\zeta\zeta}]^{-1}$ assumed in (5.3c) as well as of $\tilde{\psi}'_{\zeta\varepsilon_{\mathrm{e}}}$, $\tilde{\psi}''_{\zeta\alpha}$ and of $\tilde{\psi}''_{\zeta\phi}$, cf. (5.3d), and that the other gradients are already estimated. By comparison, we also obtain the estimate for $\dot{\zeta}_h$ contained in (5.4f), namely (written possibly only formally)

$$\sup_{\|v\|_{L^2(0,T;H^1(\Omega))}\le1}\int_0^T\!\!\int_\Omega \dot{\zeta}_h v\,\mathrm{d}x\mathrm{d}t = \sup_{\|v\|_{L^2(0,T;H^1(\Omega))}\le1}\int_0^T\!\!\int_\Omega \mathfrak{m}\nabla p_h{\cdot}\nabla v\,\mathrm{d}x\mathrm{d}t \le \|\mathfrak{m}\nabla p_h\|_{L^2(0,T;H^1(\Omega))}.$$

The regularity (5.4g) follows from the flow rules for π, α, and ϕ, from which we can see that $\kappa_1\Delta\pi_h$, $\kappa_2\Delta\alpha_h$, and $\kappa_3\Delta\phi_h$ are already estimated in $L^2(0,T;L^2(\Omega))$-norms. Using H^2-regularity theory on smooth domains, we obtain (5.4g). Actually, handling of the constraints formed by K is a bit technical and needs a smoothing technique, cf. [57]. Also let us emphasize that the concept of healing in damage and porosity is essential for these estimates. Then, we can also improve the bound of $\dot{u}_h\cdot\nabla\pi_h$ in $L^2(0,T;L^3(\Omega;\mathbb{R}^{3\times3}))$ because $\dot{u}_h$ is bounded in $L^\infty(0,T;L^3(\Omega;\mathbb{R}^3))$ and π_h in $L^2(0,T;L^6(\Omega;\mathbb{R}^{3\times3}))$ due to the Rellich embedding theorem $H^1(\Omega)\subset L^6(\Omega)$. Similar arguments hold for α_h and ϕ_h, so that we prove also (5.4h).

The first estimate in (5.4h) can be seen by comparison $\ddot{u}_h = (\mathrm{div}(\sigma_{\mathrm{el},h}{+}\mathfrak{s}_{\mathrm{el},h}) - \mathrm{div}^2\,\mathfrak{h}_{\mathrm{el},h} + g)/\varrho$ when realizing that now $\mathfrak{s}_{\mathrm{el},h} \in L^2(0,T;L^{3/2}(\Omega;\mathbb{R}^{3\times3}))$ which can be seen by interpolating $(\nabla\pi,\nabla\alpha,\nabla\phi)\in L^\infty(0,T;L^2(\Omega;\mathbb{R}^{3\times3+2}))\cap L^2(0,T;L^6(\Omega;\mathbb{R}^{3\times3+2}))\subset L^4(0,T;L^3(\Omega;\mathbb{R}^{3\times3+2}))$. $\square$

Proposition 5.1 (Convergence towards weak solutions in the isothermal case.) *Let the heat part of the problem is ignored (i.e. $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are independent of temperature so that the heat-transfer becomes separate and irrelevant for the mechanical part of the model) and let again (5.3) hold. Then, letting the abstract discretisation parameter h to converge to 0, there is a subsequence so that the (unspecified) approximate solutions $(u_h,\pi_h,\alpha_h,\phi_h,\zeta_h,p_h,)$ converge to some $(u,\pi,\alpha,\phi,\zeta,p)$ weakly* in the topologies indicated in (5.4a-f,i-k). Moreover, every such a limit is a solution (in a usual weak sense) to the initial-boundary-value problem for the system (2.6a-d) with the initial-boundary conditions (2.7b,c) and (5.1) and (5.2) and, in addition, the inclusions (2.6b-d) hold even almost everywhere.*

Sketch of the proof. First, the weakly* convergent subsequence in the topologies indicated in (5.4a-f,i) can be chosen by the Banach selection principle. The strong convergence of α, ϕ, and ζ needed for limit passage through the nonlinear terms is by the Aubin-Lions theorem based on the estimates of the gradients and of the time derivatives. In particular, note that the kinetic term $\varrho\ddot{u}_h$

in (2.6a) estimated in (5.4a) is needed to get strong convergence in $e(u_h)$ and thus also in $\varepsilon_{\mathrm{e},h}$. The highest-order gradient terms are linear so that the limit passage via the weak convergence works. $\square$

The energy conservation is a-priori not guaranteed just by basic energy estimates (5.4) which do not make the acceleration $\varrho\ddot{u}$ in duality with the velocity $\dot{u}$ so that possibly some energy may be lost on possible shock waves as, without considering the Kelvin-Voigt-type viscosity in the material, the momentum equation (2.6a) has a hyperbolic character. Even worse, it does not seem obvious that our "inviscid" model indeed allows for the expected energy conservation at all. More specifically, this can be seen from the analysis performed in [67, Prop. 2] where the energy conservation was essentially proved (without considering internal parameters) in the two-dimensional case if the elastic stress σ_{el}, cf. (2.4a), is twice differentiable as a nonlinear function of all involved variables and the initial conditions for u and $\dot{u}$ are more regular by employing differentiation of the momentum equation (2.6a) in time and testing it by $\ddot{u}$. For the physically relevant three-dimensional situation, the analysis in [67], if generalized for situations with internal variables, indicates that one would need to modify our model by using 3rd-grade nonsimple materials and 2nd-order gradients of the internal variables π, α, ϕ, and also the gradient theory for the diffusion leading to the Cahn-Hilliard model, cf. also Remark 3.5 above; to this goal, also a test of (2.6a) by $\Delta\dot{u}$ is employed.

A generalization of Proposition 5.1 for the full anisothermal model needs a limit passage in the heat source r from (2.6f) which needs (and is known to be essentially equivalent to) energy conservation which is however difficult, as discussed above. It is at least well possible in some sort of special cases (modifications) of the model for short times and thus small displacements where $\frac{\mathrm{D}}{\mathrm{D}t}$ is replaced by $\frac{\partial}{\partial t}$. Then the structural stresses are omitted, i.e. $\sigma_{\mathrm{el,tot}}$ from (2.4c) is considered with $\mathfrak{s}_{\mathrm{el}} = 0$. A need related to energy conservation is to get $\varrho\ddot{u}$ in duality to $\dot{u}$, which can be achieved by involving a Kelvin-Voigt type rheology in such a way that $\dot{u}$ would get estimated in $L^2(0,T;H^2_0(\Omega;\mathbb{R}^3))$. Altogether, (2.6) then modifies as:

$$\varrho\ddot{u} = \mathrm{div}\big(\sigma_{\mathrm{el}} + \lambda_{\mathrm{v}}\mathrm{tr}e(\dot{u}) + 2\mu_{\mathrm{v}}e(\dot{u}) - \mathrm{div}(\mathfrak{h}_{\mathrm{el}} + \kappa_{\mathrm{v}}\nabla e(\dot{u}))\big) + g, \tag{5.8a}$$

$$\mathfrak{D}_1\big(\alpha,\phi,\theta;\dot{\pi}\big) + \mathrm{dev}\,\sigma_{\mathrm{i,tot}} \ni 0, \tag{5.8b}$$

$$\mathfrak{D}_2\Big(\alpha,\phi,\theta;\begin{pmatrix}\dot{\alpha}\\ \dot{\phi}\end{pmatrix}\Big) + \begin{pmatrix}\sigma_{\mathrm{dam,tot}} + N_{[0,1]}(\alpha)\\ p_{\mathrm{eff,tot}} + N_{[0,1]}(\phi)\end{pmatrix} \ni 0, \tag{5.8c}$$

$$\dot{\zeta} \in \mathrm{div}\big(\mathfrak{m}(\alpha,\phi)\nabla p\big) - N_{[0,1]}(\zeta), \tag{5.8d}$$

$$\begin{aligned}c_{\mathrm{v}}(\theta)\dot{\theta} = {}&\mathrm{div}\big(\mathfrak{k}(\zeta,\theta)\nabla\theta\big) + \lambda_{\mathrm{v}}|\mathrm{div}\,\dot{u}|^2 + 2\mu_{\mathrm{v}}|e(\dot{u})|^2 + \kappa_{\mathrm{v}}|\nabla e(\dot{u})|^2\\ &+ \mathfrak{m}(\alpha,\phi)|\nabla p|^2 + \theta\eta\,\mathrm{div}\dot{u} + \mathfrak{D}_1\big(\alpha,\phi,\theta;\dot{\pi}\big){:}\dot{\pi} + \mathfrak{D}_2\Big(\alpha,\phi,\theta;\begin{pmatrix}\dot{\alpha}\\ \dot{\phi}\end{pmatrix}\Big)\cdot\begin{pmatrix}\dot{\alpha}\\ \dot{\phi}\end{pmatrix},\end{aligned} \tag{5.8e}$$

where $\lambda_{\mathrm{v}} \geq 0$, $\mu_{\mathrm{v}} > 0$, and $\kappa_{\mathrm{v}} > 0$ are coefficients related to the mentioned Kelvin-Voigt (linear) rheology.

Proposition 5.2 (Weak solutions in the small-displacement case.) *Let again* (5.3) *be valid together with*

$$\psi'_{\mathrm{T}}(\theta) \geq \epsilon(1+\theta)^{6/5+\epsilon} \quad \text{and} \quad |\psi_{\mathrm{T}}(\theta)| \leq (1+\theta)/\epsilon, \tag{5.9a}$$

$$j_{\mathrm{ext}} \in L^1(0,T;L^1(\Gamma)), \quad \vartheta_0 \in L^1(\Omega), \ \ \text{and} \ \ \vartheta_0(x) \geq 0 \ \text{for all}\ x \in \Omega. \tag{5.9b}$$

Then, for (again unspecified approximate solutions), the estimates (5.4a-f,h) hold completed now with $\|u_h\|_{H^1(0,T;H^2_0(\Omega;\mathbb{R}^3))} \leq C$ *due to the newly added Kelvin-Voigt viscosity, also the a-priori estimates*

$$\big\|\vartheta_h\big\|_{L^\infty(0,T;L^1(\Omega))} \leq C, \quad \big\|\vartheta_h\big\|_{L^r(0,T;W^{1,r}(\Omega))} \leq C_r, \quad \big\|\theta_h\big\|_{L^r(0,T;W^{1,r}(\Omega))} \leq C_r, \tag{5.10}$$

hold for any $1 \le r < 5/4$ *with some* C *and* $C_r < \infty$*, and any weakly* convergent subsequence has a limit which is a weak solution to the system* (5.8) *with the boundary conditions* (2.7b-d) *and* (5.1) *and with the initial condition* (5.2).

Sketch of the proof. The estimates (5.4) follow again by Hölder, Young, and Gronwall inequalities from (3.1) but simplified because the last bulk integral (related to large displacement model) is now omitted. Also the estimates of $\dot{u}_h$ is simplifies and the estimate (5.4h) of $\ddot{u}_h$ holds even without assuming Ω smooth because $\mathfrak{s}_{\rm el} \equiv 0$ here.

The first estimate in (5.10) then follows from (3.10) if one proves $\vartheta_h \ge 0$, which follows from the non-negativity of the initial conditions ζ_0 assumed in (5.3i) and of the heat sources (2.6f). Here (5.9) is used, and also a certain consistency of the discretisation scheme is needed but we omit technical details; for the spatial discretisation it is important that there are no adiabatic-cooling effects in the model while for the time-discretisation we refer to [76]. The resting two estimates in (5.10) can be obtained by a rather sophisticated test technique of the heat equation combined with Gagliardo-Nirenberg interpolation developed for nonlinear heat-transfer equation with L^1-data in [8]; cf. also e.g. [71].

The convergence in the semi-linear mechanical part toward (5.8-d) is quite standard by the weak convergence combined with Aubin-Lions compactness theorem. The limit passage towards (5.8e) needs the strong convergence in $\nabla e(\dot{u})$, $\dot{\pi}$, $\dot{\alpha}$, $\dot{\phi}$, and ∇p, which is however essentially equivalent to the energy conservation. For this, it is important that $\ddot{u}_h \in L^2(0,T;H^{-2}(\Omega;\mathbb{R}^3))$ is in duality with $\dot{u}_h \in L^2(0,T;H^2_0(\Omega;\mathbb{R}^3))$ due to the Kelvin-Voigt viscosity.

For various technicalities see also [75] for a model without porosity and water transport, or [73, 76] for a poroelastic model without the non-Hookean γ-term, and thus without the hyper-stresses. Moreover, if Galerkin's technique is used, two-step approximation has to be made because of a nonlinear test needed for (5.10), cf. e.g. [69] for details. $\square$

Remark 5.1 (An anisothermal model at large displacements.) The above outlined mathematical analysis works if one modifies (2.6) by replacing (2.6a) by (5.8a) and augmenting r in (2.6f) by $\lambda_{\rm v}|{\rm div}\,\dot{u}|^2 + 2\mu_{\rm v}|e(\dot{u})|^2$. Although conceptually a bit questionable, such system may still be used for large displacements during long periods of aseismic slips combined with Kelvin-Voigt attenuation of seismic waves emitted during short period of earthquakes under small displacements.

Remark 5.2 (More general boundary conditions.) If the Dirichlet condition for $u_{\rm D}{\cdot}\nu$ would be inhomogeneous, one should made the shift transformation $u \mapsto u + u_{\rm D}$ with $u_{\rm D}$ a suitable extension of the boundary data into the domain. This would lead to the homogeneous Dirichlet condition $u = 0$ on Γ but would give rise a lot of bulk terms in (2.6) not only in (2.6a), but e.g. $\dot{u}_{\rm D}{\cdot}\nabla\pi$ in (2.6b) or $\dot{u}_{\rm D}{\cdot}\nabla\alpha$ and $\dot{u}_{\rm D}{\cdot}\nabla\phi$ in (2.6c), etc. The above analysis could be performed but a lot of technicalities would arise. Coming back to the Robin-type boundary conditions considered in Section 3 would however bring a problem with estimation of the term $\int_\Gamma \psi_{\rm M}\dot{u}{\cdot}\nu\,{\rm d}S$ because the traces of $\dot{u}$ on Γ are not well defined and the by-part integration in time, which can otherwise be used for the term $\int_\Gamma h_{\rm ext}{\cdot}\dot{u}\,{\rm d}S$ in (3.1) if $h_{\rm ext} \in H^1(0,T;L^2(\Gamma;\mathbb{R}^3))$, does not seem to work. Again, a modification by a Kelvin-Voigt type viscosity like in [54] would be needed.

Remark 5.3 (Perfect plasticity.) For $\kappa_1 = 0$ and δ_1 degree-1 homogeneous in terms of $\dot{\pi}$, we would get the Prandtl-Reuss plasticity model which can exhibit shear bands and can thus model infinitesimally thin core of the lithospheric faults. It is important to see that no gradient of π is needed in (5.7). For a combination with damage without diffusion, porosity, and temperature, see also [77] where the concept of 2nd-grade nonsimple materials occurred again essential for analysis. This case however requires a special technique using the so-called bounded-deformation spaces and e.g. estimates (5.4a,b) are to be modified accordingly. Here it would still need to verify a so-called safe-load condition in case of gravity load, unless one confines on mere Dirichlet loading only. Yet, more important, $\nabla\pi$ is not controlled at all, and the term $\dot{u}{\cdot}\nabla\pi$ in the convective derivative could hardly have a meaning. This observation applies to geophysical models in literature using possibly visco-plastic models but without gradient plasticity, as e.g. [54].

Remark 5.4 (Geophysical models regularized.) Let us remark that a pointwise coercivity of $\psi_{\rm M}$ in all variables (5.3c) is not satisfied by (4.2) if γ is large, and without the gradient terms even the integral-type coercivity would not be satisfied. The interpretation of such model sometimes adopted in geophysical literature (cf. [52, Sect. 4.1]) is that its validity remains only until the non-Newtonian γ-term does not start dominating, believing that initial conditions and the loading regime keeps it valid at least for some short time, as discussed already in Remark 4.4. On the other hand, applicable in long time, the γ-term brings an essential driving force for healing of the damage [49]. It is therefore particularly important here that we involved the gradient terms and consider the integral-type coercivity (5.3b) which can be ensured by Korn's inequality in cooperation Dirichlet boundary conditions. Moreover, to satisfy some other assumptions in (5.3), the ansatz (4.2) must be regularized. In particular, to comply with the growth assumption (5.3b) which was casted to cope with the unpleasant growth of the last tri-linear term in (3.10), one can think about

$$\psi_{{\rm M},\epsilon}(\varepsilon_{\rm e},\alpha,\phi,\zeta)=\frac{1}{2}\frac{\lambda(\alpha,\phi)I_1^2+2\mu(\alpha,\phi)I_2-2\gamma(\alpha,\phi)I_1\sqrt{I_2}+M(\alpha,\phi)|\beta I_1{-}\zeta{+}\phi|^2}{\sqrt{1+\epsilon I_2}}+\chi\alpha \tag{5.11}$$

with $\epsilon>0$ presumably small. Obviously, for $\epsilon\to0$, $\psi_{{\rm M},\epsilon}$ from (5.11) approximates the original $\psi_{\rm M}$ from (4.2) although the rigorous proof of convergence of corresponding solutions is not clear. For fixed $\epsilon>0$, $\psi_{{\rm M},\epsilon}(\cdot,\alpha,\phi,\zeta):\mathbb{R}^{3\times3}\to\mathbb{R}$ has a linear growth and, in particular, the uniform boundedness of $\psi_{{\rm M},\epsilon}'(\varepsilon_{\rm e},\alpha,\phi,\zeta)$ assumed in (5.3c) and used in (5.5) is satisfied; sometimes, such nonlinear-elastic models are referred to as Hook's law with perfectly plastic domain [63, Sect. 3.5]. Here, choosing $\epsilon>0$ small, any visible deviation from the desired model $\epsilon=0$ can be made only for so big elastic stresses which anyhow no rock material can withstand so it does not represent any essential model modification while facilitates its rigorous mathematical support.

Acknowledgment

Many-year lasting discussions with Vladimir Lyakhovsky about the cited geophysical models and about their limited validity was an essential motivation of this paper. Also discussions with Josef Málek, František Maršík, and Christian Rohde about the structural-stress concept and with Alexander Mielke and Giuseppe Tomassetti about the small/large-strain concept are deeply appreciated. Thanks are also two anonymous referees for many comments that led to better readability of the paper and widening its context, and to Ondřej Souček for careful reading and commenting the final manuscript. This research was supported from the grants 13-18652S "Computational modeling of damage and transport processes in quasibrittle materials", 14-15264S "Experimentally justified multiscale modelling of shape memory alloys", and 16-03823S "Homogenization and multi-scale computational modelling of flow and nonlinear interactions in porous smart structures" of the Czech Science Foundation, together with the institutional support RVO: 61388998.

References

[1] E.C. AIFANTIS: On the microstructural origin of certain inelastic models. *ASME Jour. Eng. Mat. Tech.* **106** (1984), 326330.

[2] L. AMBROSIO AND V.M. TORTORELLI: On the approximation of free discontinuity problems, *Bollettino Unione Mat. Italiana* **7** (1992), 105123.

[3] J.M. BALL, R.D. JAMES: Proposed experimental tests of a theory of fine microstructure and the two-well problem. *Phil. Trans. Royal Soc. London A* **338** (1992), 389-450.

[4] Z.P. BAŽANT: *Scalling of Structural Strenght.* 2nd ed., Elsevier, Amsterdam, 2005.

[5] Z.P. BAŽANT AND J. PLANAS: *Fracture and Size Effect in Concrete and other quasibrittle Materials.* CRC Press, Boca Raton, 1998.

[6] M. A. BIOT: General theory of three-dimensional consolidation. *J. Appl. Phys.*, **12** (1941), 155-164.

[7] A. BIZZARRI, M. COCCO: Slip-weakening behavior during the propagation of dynamic ruptures obeying rate- and state-dependent friction laws. *J. Geophys. Res.* **108** (2003), 2373.

[8] L. BOCCARDO, T. GALLOUËT: Non-linear elliptic and parabolic equations involving measure data. *J. Funct. Anal.* **87** (1989), 149-169.

[9] G. BOUCHITTÉ, A. MIELKE, AND T. ROUBÍČEK: A complete damage problem at small strains. *Zeitschrift angew. Math. Phys.* **60** (2009), 205-236.

[10] B. BOURDIN, G.A. FRANCFORT, AND J.-J. MARIGO: The variational approach to fracture. *J. Elasticity* **91** (2008), 5-148.

[11] D. BRESCH, B. DESJARDINS, M. GISCLON, R. SART: Instability results related to compressible Korteweg system. *Ann Univ Ferrara*, DOI 10.1007/s11565-008-0043-3

[12] P.W. BRIDGMAN: *The Nature of Thermodynamics*. Harward Univ. Press, Cambridge (MA), 1943.

[13] J.W. CAHN, J.E. HILLIARD: Free energy of a uniform system I., Interfacial free energy, *J. Chem. Phys.*, **28** (1958), 258–267.

[14] M. CHIPOT: *Variational Inequalities and Flow in Porous Media.* Springer, Berlin, 1984.

[15] H. DARCY: *Les Fontaines Publiques de La Ville de Dijon*. Victor Dalmont, Paris, 1856.

[16] J.H. DIETERICH: Applications of rate- and state-dependent friction to models of fault slip and earthquake occurence. Chap. 4 in: *Earthquake Seismology* (Ed. H. Kanamori), Treatise on Geophys., Elsevier, 2007, pp. 107-129.

[17] G. DI TORO, R. HAN, T. HIROSE, N. DE PAOLA, S. NIELSEN, K. MIZOGUCHI, F. FERRI, M. COCCO, T. SHIMAMOTO: Fault lubrication during earthquakes. *Nature* **471** (2011), 494-498.

[18] O.W. DILLON AND J. KRATOCHVÍL: A strain gradient theory of plasticity. *Int. J. Solids Struct.* **6** (1970), 1513-1533.

[19] W. DREYER, J. GIESSELMANN, C. KRAUS: Modeling of compressible electrolytes with phase transition. Preprint: arXiv:1405.6625.

[20] W. DREYER, J. GIESSELMANN, C. KRAUS, C. ROHDE: Asymptotic analysis for Korteweg models. *Interfaces Free Bound.* **14** (2012), 105-143.

[21] P. DUHEM: *Traité dénergétique ou de thermodynamique générale*. Gauthier-Villars, Paris, 1911.

[22] G. DUVAUT AND J.L. LIONS: *Les Inéquations en Mécanique et en Physique.* Dunod, Paris, 1972 (Engl. transl. Springer, Berlin, 1976).

[23] H. GARCKE: On Cahn-Hilliard systems with elasticity. *Proc. Royal Soc. Edinburgh A* **133** (2003), 302–331.

[24] M. FRÉMOND: *Non-Smooth Thermomechanics*. Springer, Berlin, 2002.

[25] E. FEIREISL, H. PETZELTOV, AND E. ROCCA: Existence of solutions to a phase transition model with microscopic movements. *Math. Methods in the Appl. Sci.* **32** (2009), 1345–1369.

[26] E. FEIREISL, H. PETZELTOVÁ, E. ROCCA, G. SCHIMPERNA: Analysis of a phase-field model for two-phase compressible fluids. *Math. Models Meth. in the Appl. Sci.* **20** (2010), 1129–1160.

[27] E. FRIED, M. GURTIN: Tractions, balances, and boundary conditions for nonsimple materials with application to liquid flow at small-length scales. *Arch. Rational Mech. Anal.* **182** (2006), 513554.

[28] A. GREEN AND P. NAGHDI: A general theory of an elastic-plastic continuum. *Arch. Rational Mech. Anal.* **18** (1965), 251-281.

[29] B. HALPHEN, Q.S. NGUYEN: Sur les matériaux standards généralisés. *J. Mécanique* **14** (1975), 39–63.

[30] Y. HAMIEL, V. LYAKHOVSKY, AND A. AGNON: Coupled evolution of damage and porosity in poroelastic media: Theory and applications to deformation of porous rocks. *Geophys. J. Int.* **156** (2004), 701-713.

[31] Y. HAMIEL, V. LYAKHOVSKY, AND A. AGNON: Poroelastic damage rheology: dilation, compaction, and failure of rocks. *Geochem. Geophys. Geosyst.* (2005) No. 6, Q01008.

[32] H.M. HILBER, T.J.R. HUGHES, AND R.L. TAYLOR: Improved numerical dissipation for time integration algorithms in structural dynamics. *Earthquake Eng. Struct. Dyn.* **5** (1977), 283-292.

[33] I. Hlaváček, J. Haslinger, J. Nečas, J. Lovíšek: *Solution of Variational Inequalities in Mechanics*. Springer, New York, 1988.

[34] W. Han and B.D. Reddy: *Plasticity: Mathematical Theory and Numerical Analysis*. 2nd ed., Springer, New York, 2013.

[35] C. Heinemann and C. Kraus, Existence of weak solutions for Cahn-Hilliard systems coupled with elasticity and damage, *Adv. Math. Sci. Appl.* **21** (2011), 321359.

[36] K. Hutter: Geophysical granular and particle laden flows: review of the field. *Phil. Trans. R. Soc. London* A 363, 1497-1505 (2005)

[37] K. Hutter and K. Rajagopal: On flows of granular materials. *Continuum Mechanics and Thermodynamics* 6, 81-139 (1994)

[38] K. Hutter and K. Wilmanski (eds.): *Kinetic and Continuum Theories of Granular and Porous Media*. CISM Courses and Lectures No. 400, Springer, Vienna, 1999.

[39] M. Jirásek, J. Zeman: Localization study of a regularized variational damage model. *Intl. J. Solids and Structures* **69-70** (2015), 131–151.

[40] J. Kruis, T. Koudelka, and T. Krejčí: Multi-physics analyses of selected civil engineering concrete structures. *Comm. Comput. Phys.* **12** (2012), 885-918.

[41] L. M. Kachanov: Time of rupture process under Deep conditions. *Izv. Akad. Nauk SSSR*, **8** (1958), 26.

[42] L. M. Kachanov: *Introduction to Continuum Damage Mechanics* (1st ed. Springer, Dordrecht, 1986) 2nd ed., Kluwer, 1990.

[43] D.J. Korteweg: Sur la forme que prennent les équations du mouvement des fuides si lón tient compte des forces capillaires causées par des variations de densité considérables mais continues et sur la théorie de la capillarité dans l'hypothèse d'une variation continue de la densité. *Arch. Néerl. Sci. Exactes Nat.* **6** (1901), 1–24.

[44] E. Kröner: Allgemeine Kontinuumstheorie der Versetzungen und Eigenspannungen. *Archive Rational Mech. Anal.* **4** (1960), 273-334.

[45] M. Kružík, T. Roubíček: *Mathematical Methods in Continuum Mechanics of Solids*. (Interaction between Math. and Mech. Series) Springer, Cham, Heidelberg, 2018, to appear.

[46] A.G. Lamorgese, D. Molin, R. Mauri: Phase field approach to multiphase flow modeling. *Milan J. Math.* **79** (2011) 597-642.

[47] E. Lee and D. Liu: Finite-strain elastic-plastic theory with application to plain-wave analysis. *J. Applied Phys.*, **38** (1967), 19-27.

[48] F.-H. Lin and C. Liu: Nonparabolic dissipative systems modeling the flow of liquid crystals. *Comm. Pure Appl. Math.* **48** (1995), 501–537.

[49] V. Lyakhovsky: Personal communications. Praha & Jerusalem, 2013-2015.

[50] V. Lyakhovsky, Y. Ben-Zion: Scaling relations of earthquakes and aseismic deformation in a damage rheology model. *Geophys. J. Int.* **172** (2008), 651662

[51] V. Lyakhovsky, Y. Ben-Zion: A continuum damage-breakage faulting model and solid-granular transitions. *Pure Appl. Geophys.* **171** (2014), 3099-3123.

[52] V. Lyakhovsky, Y. Ben-Zion, A. Agnon: Distributed damage, faulting, and friction. *J. Geophys. Res.* **102** (1997), 27,635-27,649.

[53] V. Lyakhovsky, Y. Hamiel: Damage evolution and fluid flow in poroelastic rock. *Izvestiya, Physics of the Solid Earth* **43** (2007), 13-23.

[54] V. Lyakhovsky, Y. Hamiel, Y. Ben-Zion: A non-local visco-elastic damage model and dynamic fracturing. *J. Mech. Phys. Solids* **59** (2011), 1752-1776.

[55] V. Lyakhovsky, Z. Reches, R. Weiberger, T.E. Scott: Nonlinear elastic behaviour of damaged rocks. *Geophys. J. Int.* **130** (1997), 157–166.

[56] V. Lyakhovsky, W. Zhu, E. Shalev: Visco-poroelastic damage model for brittle-ductile failure of porous rocks. *J. Geophys. Res.: Solid Earth* **120** (2015), on line doi:10.1002/2014JB011805.

[57] F. LUTEROTTI, G. SCHMIPERNA, U. STEFANELLI: Global solution to a phase field model with irreversible and constrained phase evolution. *Quart. Appl. Math.* **60** (2002), 301-316.

[58] C. MATYSKA: *Mathematical Introduction to Geothermices and Geodymamics*. Lect. notes, Charles Uni., Prague.
http://geo.mff.cuni.cz/studium/Matyska-MathIntroToGeothermicsGeodynamics.pdf

[59] R.D. MINDLIN AND N.N. ESHEL: On first strain-gradient theories in linear elasticity. *Intl. J. Solid Structures* **4** (1968), 109-124.

[60] A. MIELKE, T. ROUBÍČEK: *Rate-Independent Systems - Theory and Application*. Springer, New York, 2015.

[61] A. MIELKE, T. ROUBÍČEK, AND J. ZEMAN: Complete damage in elastic and viscoelastic media and its energetics. *Computer Methods Appl. Mech. Engr.* **199** (2010), 1242–1253.

[62] A. MIELKE, U. STEFANELLI: Linearized plasticity is the evolutionary Γ-limit of finite plasticity. *J. Eur. Math. Soc.* **15** (2013), 923–948.

[63] J. NEČAS, HLAVÁČEK: *Mathematical Theory of Elastic and Elasto-Plastic Bodies*. Elsevier, Amsterdam, 1981.

[64] K. PIECHÓR: Non-local Korteweg stresses from kinetic theory point of view. *Arch. Mech.* **60** (2008), 23-58.

[65] P. PODIO-GUIDUGLI: Contact interactions, stress, and material symmetry, for nonsimple elastic materials *Theor. Appl. Mech.* **28–29** (2002), 261–276.

[66] P. PODIO-GUIDUGLI, M. VIANELLO: Hypertractions and hyperstresses convey the same mechanical information. *Cont. Mech. Thermodynam.* **22** (2010), 163–176.

[67] K.R. RAJAGOPAL, T. ROUBÍČEK: On the effect of dissipation in shape-memory alloys. *Nonlinear Anal., Real World Applications* **4** (2003), 581-597.

[68] J.R. RICE: Heating and weakening of faults during earthquake slip. *J.Geophys. Res.* **111** (2006), B05311.

[69] T.ROUBÍČEK: Thermo-visco-elasticity at small strains with L1-data. *Quarterly Appl. Math.* **67** (2009), 47-71.

[70] T.ROUBÍČEK: Approximation in multiscale modelling of microstructure evolution in shape-memory alloys. *Cont. Mech. Thermodynam.* **23** (2011), 491-507.

[71] T. ROUBÍČEK: *Nonlinear Partial Differential Equations with Applications*. 2nd ed., Birkhäuser, Basel, 2013.

[72] T. ROUBÍČEK: A note about the rate-and-state-dependent friction model in a thermodynamical framework of the Biot-type equation. *Geophysical J. Intl.* **199** (2014), 286-295.

[73] T. ROUBÍČEK: Energy-conserving time-discretisation scheme for poroelastic media with regularized fracture emitting waves and heat. (Preprint no. 2016-11, Nečas center, Prague) *Disc. Cont. Dynam. Syst. – S*, in print.

[74] T.ROUBÍČEK, K.-H.HOFFMANN: About the concept of measure-valued solutions to distributed parameter systems. *Math. Methods Appl. Sci.* **18** (1995), 671-685.

[75] T. ROUBÍČEK, O.SOUČEK, R.VODIČKA: A model of rupturing lithospheric faults with re-occurring earthquakes. *SIAM J. Appl. Math.* **73** (2013), 1460-1488.

[76] T. ROUBÍČEK, G.TOMASSETTI: Thermomechanics of demageable materials under diffusion: modeling and analysis. *Zeit. angew. Math. Phys.*, **66** (2015), 3535–3572.

[77] T. ROUBÍČEK, J. VALDMAN: Perfect plasticity with damage and healing at small strains, its modelling, analysis, and computer implementation. *SIAM J. Appl. Math.* **76** (2016), 314–340.

[78] O. SADOVSKAYA AND V. SADOVSKII: *Mathematical Modeling in Mechanics of Granular Materials*. Springer, Berlin, 2012.

[79] G. SCIARRA, S. VIDOLI: Asymptotic fracture modes in strain-gradient elasticity: size effects and characteristic lengths for isotropic materials. *J. Elast.* **113** (2013), 27–53.

[80] M. ŠILHAVÝ: Phase transitions in non-simple bodies. *Archive Rational Mech. Anal.* **88** (1985), 135–161.

[81] R.A. TOUPIN: Elastic materials with couple stresses. *Archive Rat. Mech. Anal.* **11** (1962), 385–414.

[82] N. TRIANTAFYLLIDIS AND E.C. AIFANTIS: A gradient approach to localization of deformation. I. Hyperelastic materials. *J. Elast.* **16** (1986), 225-237.