On the generalized Stokes equations in Orlicz spaces

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Plan of the talk

- Existence of weak solutions to the generalized Stokes system
- (t, x)-dependent growth conditions of the stress tensor
- Implicit constitutive relations

Generalized Stokes system

Let $\Omega \subseteq \mathbb{R}^d$ be a bounded, open set and $Q = (0, T) \times \Omega$. The velocity field u and the pressure p describe the unsteady flow of the incompressible fluid

$$
\begin{aligned}
\partial_t u - \operatorname{div} S(t, x, Du) + \nabla p = f \quad & \text{in } (0, T) \times \Omega, \\
\operatorname{div} u = 0 & \\
u(0, x) = u_0 \quad & \text{in } \Omega, \\
u(t, x) = 0 \quad & \text{on } (0, T) \times \partial\Omega,
\end{aligned}
$$

where S is the stress tensor and f are given body forces.

Typical models

1. Navier-Stokes equations

$$S(Dv) = \nu Dv$$

2. Power-law fluids

$$S(Dv) \cdot Dv \geq c(1 + |Dv|)^q, \ |Dv| \leq c(1 + |Dv|)^{q-1}$$

More general setting

Let us assume that $S : \Omega \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ satisfies:

1. $S(x,\xi)$ is a Caratheodory function and $S(x,0) = 0$.
2. There exist a positive constant c, integrable function k and an N-function M satisfying for all ξ

$$S(x,\xi)\cdot\xi \geq c\{M(\xi) + M^*(S(x,\xi))\} - k(x)$$

3. S is monotone i.e. for all $\xi_1, \xi_2 \in \mathbb{R}^{d\times d}$ and a.a. $x \in \Omega$

$$[S(x,\xi_1) - S(x,\xi_2)]\cdot[\xi_1 - \xi_2] \geq 0.$$

$N-$ functions

Notation

M is an $N-$ function if M is convex, $M(x,0)=0$, has superlinear growth and $M(x,\xi)=M(x,-\xi)$
M^* is defined as $M^*(x,\eta)=\sup_{\xi\in\mathbb{R}^{d\times d}_{\mathrm{sym}}}(\eta\cdot\xi-M(x,\xi))$.

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Δ_2–condition

We say that M satisfies Δ_2–condition if for some constant $C>0$ and an integrable function m

$$M(x,2\xi)\le CM(x,\xi)+m(x)\quad \text{for all } \xi\in\mathbb{R}^{d\times d}_{\mathrm{sym}} \text{ and a.a. } x\in\Omega.$$

Motivation

Slow motion of a fluid of Prandtl–Eyring type

$$T = c_0 \frac{\operatorname{arcsinh}(\alpha|Du|)}{\alpha|Du|} Du$$

or Powell–Eyring

$$T = c_1 Du + c_0 \frac{\operatorname{arcsinh}(\alpha|Du|)}{\alpha|Du|} Du$$

For simplicity

$$M(\xi) = |\xi| \ln(1 + |\xi|)$$

Weak solutions

$$u \in L^\infty(0,T;L^2(\Omega)), \quad \int_Q M(x,Du)\,dx\,dt < \infty,$$

$$\int_Q M^*(x,S(Du))\,dx\,dt < \infty$$

$$\int_Q -u \cdot \partial_t \varphi + S(t,x,Du) \cdot D\varphi dxdt = \int_Q f \cdot \varphi dxdt - \int_\Omega u_0 \varphi(0) dx$$

for all $\varphi \in C_c^\infty((-\infty,T) \times \Omega)$, $\mathrm{div}\varphi = 0$.

Some properties of Orlicz spaces

The Orlicz class $\mathcal{L}_M(Q)$ is the set of all measurable functions $\xi : Q \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ such that

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By $L_M(Q)$ we mean the vector valued Orlicz space which is the set of all measurable functions $\xi : Q \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ which satisfy

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The generalized Orlicz space is a Banach space with respect to the Luxemburg norm

$$\|\xi\| = \inf\left\{\lambda > 0 : \quad \int_Q M\left(\frac{\xi}{\lambda}\right) dxdt \le 1\right\}.$$

Some properties of Orlicz spaces

The space E_M

By $E_M(Q)$ we denote the closure of $L^\infty(Q)$ in $L_M(Q)$

- $(E_M)^* = L_{M^*}$
- If M does not satisfy Δ_2–condition, then $E_M \subsetneq \mathcal{L}_M \subsetneq L_M$
- If M satisfies Δ_2–condition, L_M is separable and $L_M = E_M = \mathcal{L}_M$. If both M and M^* satisfy Δ_2–condition, then L_M is reflexive.

Definition

A sequence z^j converges modularly to z in $L_M(Q)$ if there exists $\lambda > 0$ such that $\int_Q M\left((z^j - z)/\lambda\right) dxdt \to 0$.

Properties

- Orlicz spaces are separable w.r.t. the modular convergence and smooth functions are dense

Integrating by parts

Classical case

H– Hilbert space, X– Banach space, $X \hookrightarrow H \simeq H^* \hookrightarrow X^*$,
$u \in L^p(0,T;X)$, $\frac{du}{dt} \in L^{p'}(0,T;X^*)$. Then for all $s_0, s_1 \in (0,T)$

$$\int_{s_0}^{s_1} \left\langle \frac{du(t)}{dt}, u(t) \right\rangle_X dt = \frac{1}{2}\|u(s_1)\|_H^2 - \frac{1}{2}\|u(s_0)\|_H^2.$$

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Problem:

$$L_M(Q) \neq L_M(0,T;L_M(\Omega))$$

Energy identity

- From the Galerkin method we have the identity $((S(t,x,Du) \stackrel{*}{\rightharpoonup} \chi$ in $L_{M_*}(Q))$

$$\int_Q u_t \varphi dxdt + \int_Q \chi \cdot D\varphi dxdt = \int_Q f \varphi dxdt$$

for each compactly supported, divergence-free and smooth function φ.

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- As a first step we will prove a function of the form

$$\varphi^j = \varrho^j * \varrho^j * u$$

  to be proper test function, where $\varrho \in C^\infty(\mathbb{R})$, ϱ has a compact support, $\int_{\mathbb{R}} \varrho(\tau) d\tau = 1$ and we define $\varrho^j(t) = j\varrho(jt)$

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  - If $\chi \in E_{M^*}$ we pass to the limit with weak star convergence
  - If $\chi \notin E_{M^*}$ we pass to the limit with modular convergence (here we need that modular and weak star limits coincide)

Integrating by parts

We observe that for $0 < s_0 < s < T$ it follows

$$\int_{s_0}^{s} \langle u_t, \varphi^j \rangle \, dt = \int_{s_0}^{s} \langle u_t, (\varrho^j * \varrho^j * u) \rangle \, dt = \int_{s_0}^{s} \langle (\varrho^j * u_t), (\varrho^j * u) \rangle \, dt$$
$$= \int_{s_0}^{s} \frac{1}{2} \frac{d}{dt} \|\varrho^j * u\|_2^2 \, dt = \frac{1}{2} \|\varrho^j * u(s)\|_2^2 - \frac{1}{2} \|\varrho^j * u(s_0)\|_2^2.$$

Next, we pass to the limit with $j \to \infty$ and obtain for a.a. s_0, s

$$\lim_{j \to \infty} \int_{s_0}^{s} \langle u_t, u^j \rangle \, dt = \frac{1}{2} \|u(s)\|_2^2 - \frac{1}{2} \|u(s_0)\|_2^2.$$

Anisotropic case $M : \mathbb{R}^{d\times d} \to \mathbb{R}_+$

1. the case of star-shaped domain and the anisotropic N-function with absolutely no restriction on the growth
2. the case of arbitrary Lipschitz domains. Here we define two functions $\underline{m}$, $\overline{m} : \mathbb{R}_+ \to \mathbb{R}_+$ as follows

$$\underline{m}(r) := \min_{\xi\in\mathbb{R}^{d\times d}_{\mathrm{sym}}, |\xi|=r} M(\xi),$$
$$\overline{m}(r) := \max_{\xi\in\mathbb{R}^{d\times d}_{\mathrm{sym}}, |\xi|=r} M(\xi)$$

and assume the control of anisotropy of N-function

$$\overline{m}(r) \leq c_m((\underline{m}(r))^{\frac{n}{n-1}} + |r|^2 + 1).$$

for all $r \in \mathbb{R}_+$, and $\underline{m}$ satisfies Δ_2-condition.

Tools: variant of the Sobolev-Korn inequality

$$\|u\|_{L^{\frac{n}{n-1}}(\Omega)} \leq \|Du\|_{L^1(\Omega)}$$

Generalization for Orlicz spaces

Let Ω be a bounded domain and
$u \in \{\varphi \in W_0^{1,1}(\Omega;\mathbb{R}^n);\ \int_\Omega M(|D\varphi|)dx < \infty\}$. Then

$$\|M(|u|)\|_{L^{\frac{n}{n-1}}(\Omega)} \leq C_n \|M(|Du|)\|_{L^1(\Omega)}.$$

Bogovski theorem

Let Ω be a bounded domain with a Lipschitz boundary. Let m be an N–function satisfying Δ_2-condition and such that m^γ is quasiconvex for some $\gamma \in (0,1)$. Then, for any $f \in L_m(\Omega)$ such that

$$\int_\Omega f\,dx = 0,$$

the problem of finding a vector field $v : \Omega \to \mathbb{R}^n$ such that

$$\begin{aligned} \operatorname{div} v = f \quad & \text{in } \Omega \\ v = 0 \quad & \text{on } \partial\Omega \end{aligned}$$

has at least one solution $v \in L_m(\Omega; \mathbb{R}^n)$ and $\nabla v \in L_m(\Omega; ; \mathbb{R}^{n\times n})$. Moreover, for some positive constant c

$$\int_\Omega m(|\nabla v|)dx \le c \int_\Omega m(|f|)dx.$$

$(t,x)-$dependent $N-$function

First question - density

Density of compactly supported smooth functions with respect to the modular topology

Second question - continuity

$$\int_Q M(t,x,(\rho^j * z)(t,x))\,dx\,dt \leq c \int_Q M(t,x,z(t,x))\,dx\,dt$$

for every $z \in L_M(Q)$.

Regularity w.r.t. (t,x)

There exists a constant $H > 0$ such that

$$\frac{M(t,x,\xi)}{M(s,y,\xi)} \leq |\xi|^{\frac{H}{\log\frac{1}{|t-s|+|x-y|}}}$$

Implicit constitutive relation

We look for u and S such that

$$\begin{aligned} u_t - \operatorname{div} S + \nabla p = f \quad & \text{in } Q, \\ (Du, S) \in \mathcal{A}(t,x) \quad & \text{in } Q \\ u(0,x) = u_0 \quad & \text{in } \Omega, \\ u(t,x) = 0 \quad & \text{on } (0,T) \times \partial\Omega. \end{aligned}$$

Introducing the graph $\mathcal{A} \subset \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^{d\times d}_{\mathrm{sym}}$ through the natural characterization

$$(D,S) \in \mathcal{A} \quad \Longleftrightarrow \quad G(D,S) = 0\,,$$

we can specify precisely the class of admissible responses G by formulating the assumptions on $\mathcal{A}$.

Graph

(A1) *$\mathcal{A}$ comes through the origin.*

(A2) *$\mathcal{A}$ is a monotone graph.*

$$(S_1 - S_2) \cdot (D_1 - D_2) \geq 0 \quad \text{for all } (D_1, S_1), (D_2, S_2) \in \mathcal{A}(t, x) \,.$$

(A3) *$\mathcal{A}$ is a maximal monotone graph.* Let $(D_2, S_2) \in \mathbb{R}^d \times \mathbb{R}^d$.

$$\text{If } (S_1 - S_2) \cdot (D_1 - D_2) \geq 0 \quad \text{for all } (D_1, S_1) \in \mathcal{A}(t, x)$$
$$\text{then } (D_2, S_2) \in \mathcal{A}(t, x).$$

(A4) *$\mathcal{A}$ is an $M-$ graph.* There are non-negative $k \in L^1(Q)$, $c_* > 0$ and N-function M such that for all $(D, S) \in \mathcal{A}(t, x)$

$$S \cdot D \geq -k(t, x) + c_*(M(t, x, D) + M^*(t, x, S))$$

(A5) *The existence of a measurable selection.*

Lemma

Let $\mathcal{A}$ be maximal monotone M-graph. Assume that there are sequences $\{S^n\}_{n=1}^{\infty}$ and $\{Du^n\}_{n=1}^{\infty}$ defined on Q such that the following conditions hold:

$$(Du^n, S^n) \in \mathcal{A} \qquad a.e.\ in\ Q,$$

$$Du^n \overset{*}{\rightharpoonup} Du \qquad weakly^*\, in\ L_M(Q),$$

$$S^n \overset{*}{\rightharpoonup} S \qquad weakly^*\, in\ L_{M^*}(Q),$$

$$\limsup_{n\to\infty} \int_Q S^n \cdot Du^n \, dz \le \int_Q S \cdot Du \, dz.$$

Then

$$(Du, S) \in \mathcal{A} \qquad a.e.\ in\ Q,$$