Localization of the $W^{-1,q}$ norm for local a posteriori efficiency

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Abstract

This paper gives a direct proof of localization of dual norms of bounded linear functionals on the Sobolev space $W_0^{1,p}(\Omega)$. The basic condition is that the functional in question vanishes over locally supported test functions from $W_0^{1,p}(\Omega)$ which form a partition of unity in Ω, apart from close to the boundary $\partial\Omega$. We also study how to weaken this condition. The results allow in particular to establish local efficiency and robustness of a posteriori estimates for nonlinear partial differential equations in divergence form, including the case of inexact solvers.

Key words: Sobolev space $W_0^{1,p}(\Omega)$, functional, dual norm, local structure, nonlinear partial differential equation, residual, finite element method, a posteriori error estimate

1 Introduction

The weak solution of the Dirichlet problem associated with the Laplace equation is a function u characterized by

$$u - u^{\mathrm{D}} \in W_0^{1,2}(\Omega), \tag{1.1a}$$

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in W_0^{1,2}(\Omega). \tag{1.1b}$$

Here $\Omega \subset \mathbb{R}^d$, $d \geq 1$, $f \in L^2(\Omega)$, and $u^{\mathrm{D}} \in W^{1,2}(\Omega)$. A typical numerical approximation of u gives u_h such that $u_h - u^{\mathrm{D}} \in V_h^0 \subset W_0^{1,2}(\Omega)$; we assume for simplicity that u^{D} lies in the same discrete space $V_h \subset W_0^{1,2}(\Omega)$ as u_h, so that there is no Dirichlet datum interpolation error.

The *intrinsic distance* of u_h to u is the *energy error* given by $\|\nabla(u - u_h)\|$. This distance is *localizable* in the sense that it is equal to a Hilbertian sum of the energy errors $\|\nabla(u - u_h)\|_K$ over elements K of a partition $\mathcal{T}_h$ of Ω, i.e.,

$$\|\nabla(u - u_h)\| = \left\{ \sum_{K \in \mathcal{T}_h} \|\nabla(u - u_h)\|_K^2 \right\}^{\frac{1}{2}}. \tag{1.2}$$

It is this problem-given intrinsic distance that is the most suitable for a posteriori error control. Under appropriate conditions, namely when the orthogonality $(f, \psi_{\mathbf{a}}) - (\nabla u_h, \nabla \psi_{\mathbf{a}}) = 0$ is fulfilled for the "hat" functions $\psi_{\mathbf{a}}$ associated with the interior vertices $\mathbf{a}$ of the partition $\mathcal{T}_h$, there exist a posteriori estimators $\eta_K(u_h)$, fully and locally computable from u_h, such that

$$\|\nabla(u - u_h)\| \leq \left\{ \sum_{K \in \mathcal{T}_h} \eta_K(u_h)^2 \right\}^{\frac{1}{2}} \tag{1.3}$$

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and such that

$$\eta_K(u_h) \le C \left\{ \sum_{K' \in \mathcal{T}_K} \|\nabla (u - u_h)\|_{K'}^2 \right\}^{\frac{1}{2}}, \tag{1.4}$$

where C is a generic constant and $\mathcal{T}_K$ is some local neighborhood of the element K, see Carstensen and Funken [9], Braess *et al.* [7], Veeser and Verfürth [22], or Ern and Vohralík [17] and the references therein. This last property is called *local efficiency* and is clearly only possible thanks to the local structure of the energy distance (1.2) itself. A different equivalence result where locality plays a central role is that of Veeser [21], who recently proved that the local and global best approximation errors in the energy norm are equivalent.

Many problems are nonlinear; the basic model we consider here is the Dirichlet problem associated with the p-Laplace equation, where, in place of (1.1), one looks for function u such that

$$\begin{aligned} u - u^{\mathrm{D}} &\in W_0^{1,p}(\Omega), \\ (\boldsymbol{\sigma}(\nabla u), \nabla v) &= (f, v) \qquad \forall v \in W_0^{1,p}(\Omega), \\ \boldsymbol{\sigma}(\mathbf{g}) &= |\mathbf{g}|^{p-2}\mathbf{g} \qquad \mathbf{g} \in \mathbb{R}^d \end{aligned}$$

for some $p \in (1, \infty)$, $u^{\mathrm{D}} \in W^{1,p}(\Omega)$, and $f \in L^q(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $u_h \in V_h \subset W^{1,p}(\Omega)$ such that $u_h - u^{\mathrm{D}} \in V_h^0 \subset W_0^{1,p}(\Omega)$ be a numerical approximation of the exact solution u and let $\mathcal{R}(u_h)$ be the *residual* of u_h given by

$$\langle \mathcal{R}(u_h), v \rangle_{(W_0^{1,p}(\Omega))', W_0^{1,p}(\Omega)} := (f, v) - (\boldsymbol{\sigma}(\nabla u_h), \nabla v) \qquad v \in W_0^{1,p}(\Omega); \tag{1.6}$$

$\mathcal{R}(u_h)$ belongs to $\left(W_0^{1,p}(\Omega)\right)'$, the set of bounded linear functionals on $W_0^{1,p}(\Omega)$, see Example 3.2 below for more details. The intrinsic distance of u_h to u is here given by the *dual norm* of the residual $\mathcal{R}(u_h)$

$$\|\mathcal{R}(u_h)\|_{(W_0^{1,p}(\Omega))'} := \sup_{v \in W_0^{1,p}(\Omega);\, \|\nabla v\|_p = 1} \langle \mathcal{R}(u_h), v \rangle_{(W_0^{1,p}(\Omega))', W_0^{1,p}(\Omega)}; \tag{1.7}$$

of course $\|\mathcal{R}(u_h)\|_{(W_0^{1,2}(\Omega))'} = \|\nabla(u - u_h)\|$ when $p = 2$ and $\boldsymbol{\sigma}(\mathbf{g}) = \mathbf{g}$. The analog of (1.3) can then be obtained: there are a posteriori estimators $\eta_K(u_h)$, fully and locally computable from u_h, such that

$$\|\mathcal{R}(u_h)\|_{(W_0^{1,p}(\Omega))'} \le \left\{ \sum_{K \in \mathcal{T}_h} \eta_K(u_h)^q \right\}^{\frac{1}{q}}, \tag{1.8}$$

see, e.g., Verfürth [24, 25], Veeser and Verfürth [22], El Alaoui *et al.* [15], Ern and Vohralík [16], or Kreuzer and Süli [20]. This can typically be proved under the *orthogonality condition*

$$\langle \mathcal{R}, \psi_{\mathbf{a}} \rangle_{(W_0^{1,p}(\Omega))', W_0^{1,p}(\Omega)} = 0 \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}, \tag{1.9}$$

where $\mathcal{V}_h^{\mathrm{int}}$ stands for interior vertices of $\mathcal{T}_h$ and $\psi_{\mathbf{a}}$ are test functions forming a partition of unity over all vertices $\mathbf{a} \in \mathcal{V}_h$. However, the analog of the local efficiency (1.4) does not seem to be obvious. The foremost reason is that the intrinsic dual error measure (1.7) does not seem to be localizable in the sense that

$$\|\mathcal{R}(u_h)\|_{(W_0^{1,p}(\Omega))'} \ne \left\{ \sum_{K \in \mathcal{T}_h} \|\mathcal{R}(u_h)\|_{(W_0^{1,p}(K))'}^q \right\}^{\frac{1}{q}},$$

in contrast to (1.2).

For the estimators from (1.8) with piecewise polynomial u_h, global efficiency in the form

$$\left\{ \sum_{K \in \mathcal{T}_h} \eta_K(u_h)^q \right\}^{\frac{1}{q}} \le C \|\mathcal{R}\|_{(W_0^{1,p}(\Omega))'}$$

has been shown previously, cf. [25, 15, 16, 20] and the references therein. Building on the results of [24, 25], it, however, actually follows that

$$\eta_K(u_h) \leq C \left\{ \sum_{\mathbf{a}\in\mathcal{V}_K} \|\mathcal{R}\|^q_{\left(W_0^{1,p}(\omega_{\mathbf{a}})\right)'} \right\}^{\frac{1}{q}}, \tag{1.10}$$

where $\mathcal{V}_K$ stands for the vertices of the element $K \in \mathcal{T}_h$ and $\omega_{\mathbf{a}}$ is a patch of mesh elements around the vertex $\mathbf{a}$, see for example [16, Theorem 5.3], [15, proof of Lemma 4.3], [20, proof of Theorem 21], or [14, equation (3.10)] for the Hilbertian setting $p = 2$. Consequently:

1. Inequality (1.8) together with (1.10) imply

$$\|\mathcal{R}\|_{\left(W_0^{1,p}(\Omega)\right)'} \leq C_1 \left\{ \sum_{\mathbf{a}\in\mathcal{V}_h} \|\mathcal{R}\|^q_{\left(W_0^{1,p}(\omega_{\mathbf{a}})\right)'} \right\}^{\frac{1}{q}}; \tag{1.11a}$$

   for the Hilbertian setting $p = 2$, this has probably first been shown in [2, Theorem 2.1.1].

2. It can also be shown that

$$\left\{ \sum_{\mathbf{a}\in\mathcal{V}_h} \|\mathcal{R}\|^q_{\left(W_0^{1,p}(\omega_{\mathbf{a}})\right)'} \right\}^{\frac{1}{q}} \leq C_2 \|\mathcal{R}\|_{\left(W_0^{1,p}(\Omega)\right)'}; \tag{1.11b}$$

   see in particular [2, Theorem 2.1.1], [14, equation (3.23)], and [13, Theorem 5.1] for the Hilbertian setting $p = 2$.

3. Thus, for the error measure $\|\mathcal{R}\|_{\left(W_0^{1,p}(\Omega)\right)'}$, the a posteriori estimators $\eta_K(u_h)$ lead to an a posteriori analysis framework where one has localization of the error measure (1.11), global reliability (1.8), and local efficiency (1.10). This is thus a fully consistent and analogous situation to (1.2), (1.3), and (1.4) of the $W_0^{1,2}(\Omega)$ setting.

The main purpose of the present paper is to give a minimalist and direct proof of the two inequalities (1.11) in a general context, without considering any particular partial differential equation or a posteriori error estimators. In particular, Theorem 3.5 shows that *dual norms of all functionals in* $\left(W_0^{1,p}(\Omega)\right)'$ are *localizable* in the sense that (1.11) holds. The orthogonality condition (1.9) is necessary for (1.11a) to hold but not for (1.11b). Moreover, we show that the constants C_1 and C_2 *only depend* on the *regularity* of the partition $\mathcal{T}_h$; they are in particular independent of the exponent p. The result of Theorem 3.5 applies to, but is not limited to, the dual norms of residuals of (nonlinear) partial differential equations of the form (1.6). We discuss similar considerations for the localization of the energy error in Remark 3.3 and for the localization of the global lifting of $\mathcal{R}$ into $W_0^{1,p}(\Omega)$ in Remark 3.4 below. Theorem 3.7 further shows that (1.11b) can be strengthened to hold patch by patch $\omega_{\mathbf{a}}$, with a global lifting of $\mathcal{R}$ on the right-hand side. All these results are presented in Section 3, after we set up the notation and gather the preliminaries in Section 2.

We are also interested in the situations where the orthogonality condition (1.9) is not satisfied. In practical applications, this is typically connected with inexact algebraic/nonlinear solvers. Our Theorem 4.1 gives two-sided bounds on $\|\mathcal{R}\|_{\left(W_0^{1,p}(\Omega)\right)'}$ in this setting and Corollary 4.2 proves therefrom that the localization result of Theorem 3.5 can be recovered provided that the loss of orthogonality is small with respect to the leading term. Theorem 4.3 finally presents an extension to vectorial setting, with typical practical applications in fluid dynamics (see [20, 6]) or elasticity. We collect these results in Section 4. Finally, Section 5 illustrates our theoretical findings on several numerical experiments.

2 Setting

We describe the setting and notation in this section. We then state cut-off estimates based on Poincaré–Friedrichs inequalities that we shall need later.

2.1 Notation and assumptions

We suppose that $\Omega \subset \mathbb{R}^d$, $d \geq 1$, is a domain (open, bounded, and connected set) with a Lipschitz-continuous boundary and let $1 < p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. We will work with standard Sobolev spaces $W^{1,p}(\Omega)$ of functions with $L^p(\Omega)$-integrable weak derivatives, see, e.g., Evans [18], Brenner and Scott [8], and the references therein. The space $W^{1,p}_0(\Omega)$ then stands for functions that are zero in the sense of traces on $\partial\Omega$. Similarly notation is used on subdomains of Ω.

For measurable $\omega \subset \Omega$ and $u \in L^q(\omega)$, $v \in L^p(\omega)$, $(u,v)_\omega$ stands for $\int_\omega u(\mathbf{x}) v(\mathbf{x})\,\mathrm{d}\mathbf{x}$ and similarly $(\mathbf{u},\mathbf{v})_\omega = \int_\omega \mathbf{u}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\,\mathrm{d}\mathbf{x}$ for $\mathbf{u} \in [L^q(\omega)]^d$ and $\mathbf{v} \in [L^p(\omega)]^d$; we simply write (u,v) instead of $(u,v)_\Omega$ when $\omega = \Omega$ and similarly in the vectorial case. We follow the convention $\|v\|_{p,\omega} := \left(\int_\omega |v(\mathbf{x})|^p\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{p}}$ and $\|\mathbf{v}\|_{p,\omega} := \left(\int_\omega |\mathbf{v}(\mathbf{x})|^p\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{p}}$, where $|\mathbf{v}| = \left(\sum_{i=1}^d |\mathbf{v}_i|^2\right)^{\frac{1}{2}}$ is the Euclidean norm in $\mathbb{R}^d$. Note that, when $p \neq 2$, $\|\nabla v\|_{p,\omega}$ is different from (but equivalent to) $|v|_{1,p,\omega} := \left(\sum_{i=1}^d \|\partial_{\mathbf{x}_i} v\|_{p,\omega}^p\right)^{\frac{1}{p}}$ for $v \in W^{1,p}(\omega)$; we will often use below the equivalence of l^p norms in $\mathbb{R}^d$:

$$\|\mathbf{x}\|_p \leq \|\mathbf{x}\|_q \leq d^{\frac{1}{q}-\frac{1}{p}} \|\mathbf{x}\|_p \qquad \forall \mathbf{x} \in \mathbb{R}^d,\ 1 \leq q \leq p \in \mathbb{R}. \tag{2.1}$$

We shall note by $\nabla\mathbf{v}$ for $\mathbf{v} \in [W^{1,p}(\omega)]^d$ a matrix with lines given by $\nabla \mathbf{v}_i$, $1 \leq i \leq d$; in accordance with above, $\|\nabla\mathbf{v}\|_{p,\omega} := \left(\int_\omega \left(\sum_{i=1}^d \sum_{j=1}^d |\partial_{\mathbf{x}_j} \mathbf{v}_i(\mathbf{x})|^2\right)^{\frac{p}{2}} \mathrm{d}\mathbf{x}\right)^{\frac{1}{p}}$. For vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$, $\mathbf{u} \otimes \mathbf{v}$ defines a tensor $\mathbb{t} \in \mathbb{R}^{d\times d}$ such that $\mathbb{t}_{i,j} := \mathbf{u}_i \mathbf{v}_j$.

We will work with partitions of the domain Ω into subdomains (*patches*) $\omega_\mathbf{a}$ with a nonzero d-dimensional measure and a Lipschitz-continuous boundary which satisfy $\cup_{\mathbf{a}\in\mathcal{V}_h} \omega_\mathbf{a} = \Omega$. Each member $\mathbf{a}$ of the finite index set $\mathcal{V}_h$ is called a *vertex* and it indeed denotes a point inside $\overline{\omega_\mathbf{a}}$. The set of vertices $\mathcal{V}_h$ is further decomposed to interior vertices $\mathcal{V}_h^{\mathrm{int}}$ lying inside the domain Ω and to vertices $\mathcal{V}_h^{\mathrm{ext}}$ located on the boundary $\partial\Omega$. In the latter case, $\partial\omega_\mathbf{a} \cap \partial\Omega$ is supposed to have a nonzero $(d-1)$-dimensional measure. Necessarily, the partition by $\omega_\mathbf{a}$ is overlapping, i.e., the intersection of several different patches has a nonzero d-dimensional measure. We collect the minimal intersections into a nonoverlapping partition $\mathcal{T}_h$ of Ω with *elements* denoted by K. We suppose that each point in Ω lies in at most N patches. Equivalently, each $K \in \mathcal{T}_h$ is the intersection of at most N patches, and we collect their vertices in the set $\mathcal{V}_K$. Vice-versa, the elements $K \in \mathcal{T}_\mathbf{a}$ form the patch $\omega_\mathbf{a}$. There in particular holds

$$\sum_{\mathbf{a}\in\mathcal{V}_h} \|v\|_{p,\omega_\mathbf{a}}^p \leq N \|v\|_p^p \qquad \forall v \in L^p(\Omega), \tag{2.2}$$

and similarly in the vectorial case.

We shall frequently use below the patchwise Sobolev spaces given by

$$\begin{aligned} W^{1,p}_*(\omega_\mathbf{a}) &:= \{v \in W^{1,p}(\omega_\mathbf{a});\ (v,1)_{\omega_\mathbf{a}} = 0\}, && \mathbf{a} \in \mathcal{V}_h^{\mathrm{int}},\\ W^{1,p}_*(\omega_\mathbf{a}) &:= \{v \in W^{1,p}(\omega_\mathbf{a});\ v = 0 \text{ on } \partial\omega_\mathbf{a} \cap \partial\Omega\}, && \mathbf{a} \in \mathcal{V}_h^{\mathrm{ext}}, \end{aligned}$$

with mean value over $\omega_\mathbf{a}$ zero in the first case and the trace on the boundary of Ω equal to zero in the second case. The Poincaré–Friedrichs inequality then states that

$$\|v\|_{p,\omega_\mathbf{a}} \leq C_{\mathrm{PF},p,\omega_\mathbf{a}} h_{\omega_\mathbf{a}} \|\nabla v\|_{p,\omega_\mathbf{a}} \qquad \forall v \in W^{1,p}_*(\omega_\mathbf{a}). \tag{2.3}$$

Here $h_{\omega_\mathbf{a}}$ stands for the diameter of the patch $\omega_\mathbf{a}$; similarly we let h_Ω denote the diameter of Ω. In particular, for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$ and $\omega_\mathbf{a}$ convex, $C_{\mathrm{PF},p,\omega_\mathbf{a}} = 2\left(\frac{p}{2}\right)^{\frac{1}{p}}$, see [12]. This implies $1 < C_{\mathrm{PF},p,\omega_\mathbf{a}} \leq C_{\mathrm{PF},2\mathrm{e},\omega_\mathbf{a}} = 2\,\mathrm{e}^{\frac{1}{2\mathrm{e}}} \approx 2.404$ for all $1 < p < \infty$ in this case. The values for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$ and nonconvex patches $\omega_\mathbf{a}$ are identified in, e.g., Veeser and Verfürth [23, Theorems 3.1 and 3.2]. Finally, $C_{\mathrm{PF},p,\omega_\mathbf{a}} = 1$ for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{ext}}$ when $\partial\omega_\mathbf{a} \cap \partial\Omega$ can be reached in a constant direction from any point inside $\omega_\mathbf{a}$; bounds in the general case can be obtained for instance as in [22, Lemma 5.1].

We denote by $\psi_\mathbf{a} \in W^{1,\infty}(\omega_\mathbf{a})$ a function which takes values between 1 and 0 on $\omega_\mathbf{a}$; $\psi_\mathbf{a}$ is zero in the sense of traces on the whole boundary $\partial\omega_\mathbf{a}$ for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$ and on $\partial\omega_\mathbf{a} \setminus \partial\Omega$ for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{ext}}$. We extend

$\psi_{\mathbf{a}} \in W^{1,\infty}(\omega_{\mathbf{a}})$ to $\psi_{\mathbf{a}} \in W^{1,\infty}(\Omega)$ by zero outside of $\omega_{\mathbf{a}}$ and assume that these functions form a partition of unity in the sense that

$$\sum_{\mathbf{a}\in\mathcal{V}_h} \psi_{\mathbf{a}} = 1 \quad \text{ a.e. in } \Omega. \tag{2.4}$$

We describe the *regularity* of the partition by the number

$$C_{\text{cont,PF}} \coloneqq \max_{\mathbf{a}\in\mathcal{V}_h}\{1 + C_{\text{PF},p,\omega_{\mathbf{a}}} h_{\omega_{\mathbf{a}}} \|\nabla\psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}}\}, \tag{2.5}$$

which we suppose to be uniformly bounded on families of partitions indexed by the parameter h.

Remark 2.1 (Simplicial or quadrilateral/hexahedral meshes)**.** *A prototypal example we have in mind is the case where Ω is a polytope, $\cup_{K\in\mathcal{T}_h}\overline{K} = \overline{\Omega}$, each element K is a d-dimensional simplex or a d-dimensional parallelepiped, and the intersection of closures of two different elements K is either empty or their d'-dimensional common face, $0 \le d' \le d-1$. Then $N = d+1$ for simplices and $N = 2^d$ for parallelepipeds, $\omega_{\mathbf{a}}$ is the patch of all elements sharing the given vertex $\mathbf{a} \in \mathcal{V}_h$, and (2.2) takes form of equality. In particular, for the seminorm on $W^{1,p}(\Omega)$,*

$$\sum_{\mathbf{a}\in\mathcal{V}_h} \|\nabla v\|^p_{p,\omega_{\mathbf{a}}} = \sum_{K\in\mathcal{T}_h}\sum_{\mathbf{a}\in\mathcal{V}_K} \|\nabla v\|^p_{p,K} = N \sum_{K\in\mathcal{T}_h} \|\nabla v\|^p_{p,K} = N\|\nabla v\|^p_p \qquad \forall v \in W^{1,p}(\Omega). \tag{2.6}$$

We then take $\psi_{\mathbf{a}}$ as the continuous, piecewise (d-)affine "hat" function of finite element analysis, taking value 1 at the vertex $\mathbf{a} \in \mathcal{V}_h$ and 0 in all other vertices from $\mathcal{V}_h$. Denoting by $\kappa_{\mathcal{T}_h}$ the maximal ratio of the diameter of K to the diameter of the largest ball inscribed into K over all $K \in \mathcal{T}_h$, it follows from Veeser and Verfürth [23, Theorems 3.1 and 3.2] and Carstensen and Funken [9] or Braess et al. *[7] that both $C_{\text{PF},p,\omega_{\mathbf{a}}}$ of (2.3) and $C_{\text{cont,PF}}$ of (2.5) only depend on $\kappa_{\mathcal{T}_h}$.*

2.2 Poincaré–Friedrichs cut-off estimates

The forthcoming result, following the lines of Carstensen and Funken [9, Theorem 3.1] or Braess *et al.* [7, Section 3], with $W^{1,p}(\omega_{\mathbf{a}})$-Poincaré–Friedrichs inequalities of Chua and Wheeden [12] and Veeser and Verfürth [23], will form the basic building block for our considerations:

Lemma 2.2 (Cut-off estimate)**.** *For the constant $C_{\text{cont,PF}}$ from (2.5), there holds, for all $\mathbf{a} \in \mathcal{V}_h$,*

$$\|\nabla(\psi_{\mathbf{a}} v)\|_{p,\omega_{\mathbf{a}}} \le C_{\text{cont,PF}} \|\nabla v\|_{p,\omega_{\mathbf{a}}} \qquad \forall v \in W^{1,p}_*(\omega_{\mathbf{a}}).$$

Proof. Let $\mathbf{a} \in \mathcal{V}_h$. We have, employing the triangle inequality, $\|\psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}} = 1$, and (2.3),

$$\begin{aligned}\|\nabla(\psi_{\mathbf{a}} v)\|_{p,\omega_{\mathbf{a}}} &= \|\nabla\psi_{\mathbf{a}} v + \psi_{\mathbf{a}}\nabla v\|_{p,\omega_{\mathbf{a}}}\\ &\le \|\nabla\psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}}\|v\|_{p,\omega_{\mathbf{a}}} + \|\psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}}\|\nabla v\|_{p,\omega_{\mathbf{a}}}\\ &\le (1 + C_{\text{PF},p,\omega_{\mathbf{a}}} h_{\omega_{\mathbf{a}}}\|\nabla\psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}})\|\nabla v\|_{p,\omega_{\mathbf{a}}},\end{aligned}$$

and the assertion follows from the definition (2.5). □

The vectorial variant is:

Lemma 2.3 (Cut-off estimate in vectorial setting)**.** *There exists a constant $C_{\text{cont,PF},d} > 0$, only depending on the space dimension d and on the constant $C_{\text{cont,PF}}$ from (2.5), such that for all $\mathbf{a} \in \mathcal{V}_h$, there holds*

$$\|\nabla(\psi_{\mathbf{a}}\mathbf{v})\|_{p,\omega_{\mathbf{a}}} \le C_{\text{cont,PF},d}\|\nabla\mathbf{v}\|_{p,\omega_{\mathbf{a}}} \qquad \forall \mathbf{v} \in [W^{1,p}_*(\omega_{\mathbf{a}})]^d.$$

Proof. Using the scalar Poincaré–Friedrichs inequality (2.3) and the norm equivalence (2.1),

$$\begin{aligned}\|\mathbf{v}\|_{p,\omega_{\mathbf{a}}} &= \left(\int_{\omega_{\mathbf{a}}}\left(\sum_{i=1}^d |\mathbf{v}_i(\mathbf{x})|^2\right)^{\frac{p}{2}} \mathrm{d}\mathbf{x}\right)^{\frac{1}{p}} \le \underline{C}_{p,d}\left(\sum_{i=1}^d \int_{\omega_{\mathbf{a}}} |\mathbf{v}_i(\mathbf{x})|^p \,\mathrm{d}\mathbf{x}\right)^{\frac{1}{p}}\\ &\le \underline{C}_{p,d} C_{\text{PF},p,\omega_{\mathbf{a}}} h_{\omega_{\mathbf{a}}}\left(\sum_{i=1}^d \|\nabla\mathbf{v}_i\|^p_{p,\omega_{\mathbf{a}}}\right)^{\frac{1}{p}} \le \underline{C}_{p,d}\overline{C}_{p,d} C_{\text{PF},p,\omega_{\mathbf{a}}} h_{\omega_{\mathbf{a}}}\|\nabla\mathbf{v}\|_{p,\omega_{\mathbf{a}}} \qquad \forall \mathbf{v} \in [W^{1,p}_*(\omega_{\mathbf{a}})]^d,\end{aligned}$$

where

$$\underline{C}_{p,d} := \begin{cases} 1 & \text{if } p \le 2, \\ d^{\frac{1}{2}-\frac{1}{p}} & \text{if } p \ge 2 \end{cases}$$

and

$$\overline{C}_{p,d} := \begin{cases} d^{\frac{1}{p}-\frac{1}{2}} & \text{if } p \le 2, \\ 1 & \text{if } p \ge 2. \end{cases}$$

Denote $C_{p,d} := \underline{C}_{p,d}\overline{C}_{p,d} = d^{\left|\frac{1}{2}-\frac{1}{p}\right|}$ and notice that $1 \le C_{p,d} < \sqrt{d}$. Then, we readily arrive at

$$\begin{aligned} \|\nabla(\psi_{\mathbf{a}}\mathbf{v})\|_{p,\omega_{\mathbf{a}}} &= \|\mathbf{v} \otimes \nabla\psi_{\mathbf{a}} + \psi_{\mathbf{a}}\nabla\mathbf{v}\|_{p,\omega_{\mathbf{a}}} \\ &\le \|\nabla\psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}}\|\mathbf{v}\|_{p,\omega_{\mathbf{a}}} + \|\psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}}\|\nabla\mathbf{v}\|_{p,\omega_{\mathbf{a}}} \\ &\le \left(1 + C_{p,d}C_{\mathrm{PF},p,\omega_{\mathbf{a}}}h_{\omega_{\mathbf{a}}}\|\nabla\psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}}\right)\|\nabla\mathbf{v}\|_{p,\omega_{\mathbf{a}}}, \end{aligned}$$

and the assertion follows with $C_{\mathrm{cont,PF},d} := \max_{\mathbf{a}\in\mathcal{V}_h}\{1 + C_{p,d}C_{\mathrm{PF},p,\omega_{\mathbf{a}}}h_{\omega_{\mathbf{a}}}\|\nabla\psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}}\}$. □

3 Localization of dual functional norms

This section states and proves our main localization result and discusses some of its consequences. We shall, however, first fix some notation and present the context in more details.

3.1 Context

Denote

$$V := W_0^{1,p}(\Omega)$$

and consider a bounded linear functional $\mathcal{R} \in V'$. We denote the action of $\mathcal{R}$ on $v \in V$ by $\langle \mathcal{R}, v \rangle_{V',V}$ and define

$$\|\mathcal{R}\|_{V'} := \sup_{v\in V;\, \|\nabla v\|_p=1} \langle \mathcal{R}, v \rangle_{V',V}. \tag{3.1}$$

Similarly, let, for $\mathbf{a} \in \mathcal{V}_h$,

$$V^{\mathbf{a}} := W_0^{1,p}(\omega_{\mathbf{a}})$$

and define the restriction of the functional $\mathcal{R}$ to $V^{\mathbf{a}}$, still denoted by $\mathcal{R}$, via

$$\langle \mathcal{R}, v \rangle_{(V^{\mathbf{a}})',V^{\mathbf{a}}} := \langle \mathcal{R}, v \rangle_{V',V} \qquad v \in V^{\mathbf{a}}, \tag{3.2}$$

where $v \in V^{\mathbf{a}}$ is extended by zero outside of the patch $\omega_{\mathbf{a}}$ to $v \in V$. Let

$$\|\mathcal{R}\|_{(V^{\mathbf{a}})'} := \sup_{v\in V^{\mathbf{a}};\, \|\nabla v\|_{p,\omega_{\mathbf{a}}}=1} \langle \mathcal{R}, v \rangle_{(V^{\mathbf{a}})',V^{\mathbf{a}}}. \tag{3.3}$$

To fix ideas, we give two examples.

Example 3.1 ($\mathcal{R}$ being divergence of an integrable function)**.** *Let $\boldsymbol{\xi} \in [L^q(\Omega)]^d$. A simple example of $\mathcal{R} \in V'$ is*

$$\langle \mathcal{R}, v \rangle_{V',V} := (\boldsymbol{\xi}, \nabla v) \qquad v \in V.$$

In this case, immediately,

$$\langle \mathcal{R}, v \rangle_{(V^{\mathbf{a}})',V^{\mathbf{a}}} = (\boldsymbol{\xi}, \nabla v)_{\omega_{\mathbf{a}}} \qquad v \in V^{\mathbf{a}}.$$

Example 3.2 ($\mathcal{R}$ given by a residual of a partial differential equation)**.** *Let $u^{\mathrm{D}} \in W^{1,p}(\Omega)$, $f \in L^q(\Omega)$, and let u be the solution of the nonlinear elliptic partial differential equation in divergence form*

$$\begin{aligned} -\operatorname{div}\boldsymbol{\sigma}(u, \nabla u) &= f \qquad \text{in } \Omega, \\ u &= u^{\mathrm{D}} \qquad \text{on } \partial\Omega. \end{aligned}$$

Here $\boldsymbol{\sigma}$ is a nonlinear function of u and/or of ∇u such that $\boldsymbol{\sigma}(v, \nabla v) \in [L^q(\Omega)]^d$ for any $v \in W^{1,p}(\Omega)$. Prototypical example is the p-Laplacian where $\boldsymbol{\sigma}(u, \nabla u) = |\nabla u|^{p-2}\nabla u$. We suppose that the weak formulation: find a function u such that

$$u - u^{\mathrm{D}} \in V, \tag{3.4a}$$
$$(\boldsymbol{\sigma}(u, \nabla u), \nabla v) = (f, v) \qquad \forall v \in V \tag{3.4b}$$

admits one and only one solution.[1] *This gives rise to the notion of the residual $\mathcal{R} \in V'$ of an arbitrary function $u_h \in W^{1,p}(\Omega)$ defined via*

$$\langle \mathcal{R}, v\rangle_{V',V} \coloneqq (f, v) - (\boldsymbol{\sigma}(u_h, \nabla u_h), \nabla v) \qquad v \in V. \tag{3.5}$$

Note that the Hölder inequality and (2.3), used in the entire domain Ω on the space V, imply that

$$|\langle \mathcal{R}, v\rangle| \le (\|f\|_q C_{\mathrm{PF},p,\Omega} h_\Omega + \|\boldsymbol{\sigma}(u_h, \nabla u_h)\|_q)\, \|\nabla v\|_p.$$

Consequently, $\mathcal{R} \in V'$. Note also that $\mathcal{R} = 0$ if and only if $u_h = u$, using the assumption of well-posedness of (3.4). Then $\|\mathcal{R}\|_{V'}$ is the intrinsic distance of u_h to u, *the* dual norm of the residual. *Note that this problem can also be cast in the form of Example 3.1: it is enough to set $\boldsymbol{\xi} \coloneqq \boldsymbol{\sigma}(u, \nabla u) - \boldsymbol{\sigma}(u_h, \nabla u_h)$.*

We would now like to see whether $\|\mathcal{R}\|_{V'}$, a priori just a number defined for any $\mathcal{R} \in V'$, expressing its size over the entire computational domain Ω, can be bounded from above and from below by the sizes $\|\mathcal{R}\|_{(V^{\mathbf{a}})'}$ of $\mathcal{R}$ localized over the patches $\omega_{\mathbf{a}}$. We still add two motivating remarks:

Remark 3.3 (Energy error localization)**.** *Remark that similarly to (1.2), there always holds*

$$\|\nabla v\|_p = \left\{ \sum_{K \in \mathcal{T}_h} \|\nabla v\|_{p,K}^p \right\}^{\frac{1}{p}}, \qquad v \in V,$$

and, in particular,

$$\|\nabla(u - u_h)\|_p = \left\{ \sum_{K \in \mathcal{T}_h} \|\nabla(u - u_h)\|_{p,K}^p \right\}^{\frac{1}{p}} \tag{3.6}$$

in the context of Example 3.2, on meshes from Remark 2.1. The energy norm $\|\nabla \cdot\|_p$ is always localizable, but it seems difficult/unoptimal to derive a posteriori error estimates of the form (1.8) for $\|\nabla(u - u_h)\|_p$ in place of $\|\mathcal{R}\|_{V'}$, see, e.g., the discussions in Belenki et al. *[3] and [16].*

Remark 3.4 (Localization of the p-Laplacian lifting of $\mathcal{R}$)**.** *Let $\mathfrak{z} \in V$ be the analogue of the Riesz representation of the functional $\mathcal{R}$ by the p-Laplacian solve on Ω, i.e., $\mathfrak{z} \in V$ is such that*

$$\left(|\nabla \mathfrak{z}|^{p-2}\nabla \mathfrak{z}, \nabla v\right) = \langle \mathcal{R}, v\rangle_{V',V} \qquad \forall v \in V. \tag{3.7}$$

Then, we readily obtain

$$\|\nabla \mathfrak{z}\|_p^p = \left(|\nabla \mathfrak{z}|^{p-2}\nabla \mathfrak{z}, \nabla \mathfrak{z}\right) = \langle \mathcal{R}, \mathfrak{z}\rangle_{V',V} = \|\mathcal{R}\|_{V'}^q. \tag{3.8}$$

Consequently, on meshes from Remark 2.1,

$$\|\nabla \mathfrak{z}\|_p = \left\{ \sum_{K \in \mathcal{T}_h} \|\nabla \mathfrak{z}\|_{p,K}^p \right\}^{\frac{1}{p}} \tag{3.9}$$

suggests itself as a way to measure the error with localization and a posteriori estimate of the form (1.8). Also an equivalent of (1.10),

$$\eta_K(u_h) \le C \left\{ \sum_{K' \in \mathcal{T}_K} \|\nabla \mathfrak{z}\|_{K'}^p \right\}^{\frac{1}{q}},$$

would hold, but the trouble here is that (3.7) is a nonlocal *problem, obtained itself by a global solve.*

[1]This holds, for example, if $\boldsymbol{\sigma}(u, \nabla u) = \tilde{\boldsymbol{\sigma}}(\nabla u)$, $(\tilde{\boldsymbol{\sigma}}(\mathbf{g}_1) - \tilde{\boldsymbol{\sigma}}(\mathbf{g}_2)) \cdot (\mathbf{g}_1 - \mathbf{g}_2) > 0$ for all $\mathbf{g}_1, \mathbf{g}_2 \in \mathbb{R}^d$, $\mathbf{g}_1 \ne \mathbf{g}_2$, $C_1|\mathbf{g}|^p \le \tilde{\boldsymbol{\sigma}}(\mathbf{g}) \cdot \mathbf{g}$, and $|\tilde{\boldsymbol{\sigma}}(\mathbf{g})| \le C_2(1 + |\mathbf{g}|)^{p-1}$ for all $\mathbf{g} \in \mathbb{R}^d$.

3.2 Main result

Recall that $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $V = W_0^{1,p}(\Omega)$, the partition $\cup_{\mathbf{a} \in \mathcal{V}_h} \omega_{\mathbf{a}}$ covers the domain Ω with the overlap N, the patches $\omega_{\mathbf{a}}$ are indexed by the vertices $\mathbf{a}$ where $\mathbf{a} \in \mathcal{V}_h^{\text{int}}$ lies inside Ω and $\mathbf{a} \in \mathcal{V}_h^{\text{ext}}$ on the boundary of Ω, and $C_{\text{cont,PF}}$ is the constant from (2.5). Our localization result is:

Theorem 3.5 (Localization of dual norms of functionals with $\psi_{\mathbf{a}}$-orthogonality). *Let $\mathcal{R} \in V'$. If*

$$\langle \mathcal{R}, \psi_{\mathbf{a}} \rangle_{V',V} = 0 \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\text{int}}, \tag{3.10}$$

then

$$\|\mathcal{R}\|_{V'} \leq N C_{\text{cont,PF}} \left\{ \frac{1}{N} \sum_{\mathbf{a} \in \mathcal{V}_h} \|\mathcal{R}\|^q_{(V^{\mathbf{a}})'} \right\}^{\frac{1}{q}}. \tag{3.11}$$

Conversely, there always holds

$$\left\{ \frac{1}{N} \sum_{\mathbf{a} \in \mathcal{V}_h} \|\mathcal{R}\|^q_{(V^{\mathbf{a}})'} \right\}^{\frac{1}{q}} \leq \|\mathcal{R}\|_{V'}. \tag{3.12}$$

Proof. We first show (3.11). Let $v \in V$ with $\|\nabla v\|_p = 1$ be fixed. The partition of unity (2.4), the linearity of $\mathcal{R}$, definition (3.2), and the orthogonality with respect to $\psi_{\mathbf{a}}$ (3.10) give

$$\begin{aligned} \langle \mathcal{R}, v \rangle_{V',V} &= \sum_{\mathbf{a} \in \mathcal{V}_h} \langle \mathcal{R}, \psi_{\mathbf{a}} v \rangle_{V',V} = \sum_{\mathbf{a} \in \mathcal{V}_h} \langle \mathcal{R}, \psi_{\mathbf{a}} v \rangle_{(V^{\mathbf{a}})',V^{\mathbf{a}}} \\ &= \sum_{\mathbf{a} \in \mathcal{V}_h^{\text{int}}} \langle \mathcal{R}, \psi_{\mathbf{a}} (v - \Pi_{0,\omega_{\mathbf{a}}} v) \rangle_{(V^{\mathbf{a}})',V^{\mathbf{a}}} + \sum_{\mathbf{a} \in \mathcal{V}_h^{\text{ext}}} \langle \mathcal{R}, \psi_{\mathbf{a}} v \rangle_{(V^{\mathbf{a}})',V^{\mathbf{a}}}, \end{aligned} \tag{3.13}$$

where $\Pi_{0,\omega_{\mathbf{a}}} v$ is the mean value of the test function v on the patch $\omega_{\mathbf{a}}$. There holds $(v - \Pi_{0,\omega_{\mathbf{a}}} v)|_{\omega_{\mathbf{a}}} \in W_*^{1,p}(\omega_{\mathbf{a}})$ and $(\psi_{\mathbf{a}} (v - \Pi_{0,\omega_{\mathbf{a}}} v))|_{\omega_{\mathbf{a}}} \in V^{\mathbf{a}}$ for $\mathbf{a} \in \mathcal{V}_h^{\text{int}}$. Similarly, $v|_{\omega_{\mathbf{a}}} \in W_*^{1,p}(\omega_{\mathbf{a}})$ and $(\psi_{\mathbf{a}} v)|_{\omega_{\mathbf{a}}} \in V^{\mathbf{a}}$ for $\mathbf{a} \in \mathcal{V}_h^{\text{ext}}$. Thus, using (3.3) and Lemma 2.2 yields, for $\mathbf{a} \in \mathcal{V}_h^{\text{int}}$,

$$\begin{aligned} \langle \mathcal{R}, \psi_{\mathbf{a}} (v - \Pi_{0,\omega_{\mathbf{a}}} v) \rangle_{(V^{\mathbf{a}})',V^{\mathbf{a}}} &\leq \|\mathcal{R}\|_{(V^{\mathbf{a}})'} \|\nabla (\psi_{\mathbf{a}} (v - \Pi_{0,\omega_{\mathbf{a}}} v))\|_{p,\omega_{\mathbf{a}}} \\ &\leq C_{\text{cont,PF}} \|\mathcal{R}\|_{(V^{\mathbf{a}})'} \|\nabla (v - \Pi_{0,\omega_{\mathbf{a}}} v)\|_{p,\omega_{\mathbf{a}}} \\ &= C_{\text{cont,PF}} \|\mathcal{R}\|_{(V^{\mathbf{a}})'} \|\nabla v\|_{p,\omega_{\mathbf{a}}}. \end{aligned}$$

A similar estimate holds for $\mathbf{a} \in \mathcal{V}_h^{\text{ext}}$. Thus, the Hölder inequality gives

$$\langle \mathcal{R}, v \rangle_{V',V} \leq N^{\frac{1}{q}} C_{\text{cont,PF}} \left\{ \frac{1}{N} \sum_{\mathbf{a} \in \mathcal{V}_h} \|\mathcal{R}\|^q_{(V^{\mathbf{a}})'} \right\}^{\frac{1}{q}} \left\{ \sum_{\mathbf{a} \in \mathcal{V}_h} \|\nabla v\|^p_{p,\omega_{\mathbf{a}}} \right\}^{\frac{1}{p}}.$$

Combining (2.2) used for ∇v with (3.1) now implies (3.11).

We now pass to (3.12). Let $\mathbf{a} \in \mathcal{V}_h$ and let $\mathscr{r}^{\mathbf{a}} \in V^{\mathbf{a}}$ be defined by the lifting via the p-Laplacian solve on the patch $\omega_{\mathbf{a}}$:

$$\left(|\nabla \mathscr{r}^{\mathbf{a}}|^{p-2} \nabla \mathscr{r}^{\mathbf{a}}, \nabla v \right)_{\omega_{\mathbf{a}}} = \langle \mathcal{R}, v \rangle_{(V^{\mathbf{a}})',V^{\mathbf{a}}} \qquad \forall v \in V^{\mathbf{a}}. \tag{3.14}$$

Then, as in (3.8),

$$\|\nabla \mathscr{r}^{\mathbf{a}}\|^p_{p,\omega_{\mathbf{a}}} = \left(|\nabla \mathscr{r}^{\mathbf{a}}|^{p-2} \nabla \mathscr{r}^{\mathbf{a}}, \nabla \mathscr{r}^{\mathbf{a}} \right)_{\omega_{\mathbf{a}}} = \langle \mathcal{R}, \mathscr{r}^{\mathbf{a}} \rangle_{(V^{\mathbf{a}})',V^{\mathbf{a}}} = \|\mathcal{R}\|^q_{(V^{\mathbf{a}})'}. \tag{3.15}$$

Consequently, exploiting that the local supremum $\|\mathcal{R}\|^q_{(V^{\mathbf{a}})'}$ writes as the action $\langle \mathcal{R}, \mathscr{r}^{\mathbf{a}} \rangle_{(V^{\mathbf{a}})',V^{\mathbf{a}}}$ and setting, extending $\mathscr{r}^{\mathbf{a}}$ by zero outside of the patch $\omega_{\mathbf{a}}$, $\underline{\mathscr{r}} := \sum_{\mathbf{a} \in \mathcal{V}_h} \mathscr{r}^{\mathbf{a}} \in V$,

$$\sum_{\mathbf{a} \in \mathcal{V}_h} \|\mathcal{R}\|^q_{(V^{\mathbf{a}})'} = \sum_{\mathbf{a} \in \mathcal{V}_h} \langle \mathcal{R}, \mathscr{r}^{\mathbf{a}} \rangle_{(V^{\mathbf{a}})',V^{\mathbf{a}}} = \langle \mathcal{R}, \underline{\mathscr{r}} \rangle_{V',V} \leq \|\mathcal{R}\|_{V'} \|\nabla \underline{\mathscr{r}}\|_p.$$

In now remains to notice that, since $\mathfrak{r}|_K = \sum_{\mathbf{a}\in\mathcal{V}_K} \mathfrak{r}^{\mathbf{a}}|_K$ and using the triangle and Hölder inequalities,

$$\begin{aligned}
\|\nabla \mathfrak{r}\|_p^p &= \sum_{K\in\mathcal{T}_h} \left\| \sum_{\mathbf{a}\in\mathcal{V}_K} (\nabla \mathfrak{r}^{\mathbf{a}})|_K \right\|_{p,K}^p = \sum_{K\in\mathcal{T}_h} \int_K \left| \sum_{\mathbf{a}\in\mathcal{V}_K} \nabla \mathfrak{r}^{\mathbf{a}}(\mathbf{x}) \right|^p \mathrm{d}\mathbf{x} \\
&\le \sum_{K\in\mathcal{T}_h} \int_K \left(\sum_{\mathbf{a}\in\mathcal{V}_K} |\nabla \mathfrak{r}^{\mathbf{a}}(\mathbf{x})| \right)^p \mathrm{d}\mathbf{x} \le N^{\frac{p}{q}} \sum_{K\in\mathcal{T}_h} \int_K \sum_{\mathbf{a}\in\mathcal{V}_K} |\nabla \mathfrak{r}^{\mathbf{a}}(\mathbf{x})|^p \mathrm{d}\mathbf{x} \\
&= N^{p-1} \sum_{\mathbf{a}\in\mathcal{V}_h} \sum_{K\in\mathcal{T}_{\mathbf{a}}} \int_K |\nabla \mathfrak{r}^{\mathbf{a}}(\mathbf{x})|^p \mathrm{d}\mathbf{x} = N^{p-1} \sum_{\mathbf{a}\in\mathcal{V}_h} \|\nabla \mathfrak{r}^{\mathbf{a}}\|_{p,\omega_{\mathbf{a}}}^p,
\end{aligned}$$

so that

$$\sum_{\mathbf{a}\in\mathcal{V}_h} \|\mathcal{R}\|_{(V^{\mathbf{a}})'}^q \le \|\mathcal{R}\|_{V'} N^{\frac{p-1}{p}} \left\{ \sum_{\mathbf{a}\in\mathcal{V}_h} \|\mathcal{R}\|_{(V^{\mathbf{a}})'}^q \right\}^{\frac{1}{p}}$$

and (3.12) follows. □

3.3 Remarks

We collect here a couple of remarks linked to Theorem 3.5.

Remark 3.6 (Localization based on weighted Poincaré–Friedrichs inequalities)**.** *Poincaré–Friedrichs inequalities can be derived for the weighted $L^p(\Omega)$-norm of v on $\omega_{\mathbf{a}}$, $\|\psi_{\mathbf{a}}^{\frac{1}{p}} v\|_{p,\omega_{\mathbf{a}}}$ in place of $\|v\|_{p,\omega_{\mathbf{a}}}$ in (2.3), see Chua and Wheeden [12] and Veeser and Verfürth [23]. Then, in the spirit of Carstensen and Funken [9] and Veeser and Verfürth [22], weighted equivalents of Lemma 2.2 and Theorem 3.5 could be given. This might reduce the size of the constants in (3.11)–(3.12), originating from overlapping of the supports of the test functions $\psi_{\mathbf{a}}$, at the price of most likely making the formulas a little more complicated.*

Next we show that inequality (3.12) can be split into local contributions when passing from dual norms of the functional $\mathcal{R}$ to its liftings.

Theorem 3.7 (Splitting (3.12) into local contributions using lifted norms)**.** *Let $\mathcal{R} \in V'$ and $\mathbf{a} \in \mathcal{V}_h$ be given. Define the* global lifting *$\mathfrak{r} \in V$ of the functional $\mathcal{R}$ by (3.7) and the* local lifting *$\mathfrak{r}^{\mathbf{a}} \in V^{\mathbf{a}}$ by (3.14). Then it holds*

$$\|\mathcal{R}\|_{(V^{\mathbf{a}})'} = \|\nabla \mathfrak{r}^{\mathbf{a}}\|_{p,\omega_{\mathbf{a}}}^{p-1} \le \|\nabla \mathfrak{r}\|_{p,\omega_{\mathbf{a}}}^{p-1}. \tag{3.16}$$

Proof. The equality has been shown in equation (3.15). The inequality follows using definition (3.3), definition of the global lifting (3.7), and the Hölder inequality

$$\|\mathcal{R}\|_{(V^{\mathbf{a}})'} = \sup_{v\in V^{\mathbf{a}};\,\|\nabla v\|_{p,\omega_{\mathbf{a}}}=1} \langle \mathcal{R}, v\rangle_{V',V} = \sup_{v\in V^{\mathbf{a}};\,\|\nabla v\|_{p,\omega_{\mathbf{a}}}=1} \left(|\nabla \mathfrak{r}|^{p-2}\nabla \mathfrak{r}, \nabla v\right)_{\omega_{\mathbf{a}}} \le \|\nabla \mathfrak{r}\|_{p,\omega_{\mathbf{a}}}^{p-1}.$$

□

Note that, indeed, summing (3.16) in q-th power over all vertices $\mathcal{V}_h$ and using (2.2) and (3.8) one gets (3.12) as a trivial consequence.

4 Extensions

This section collects various extensions of the main result of Theorem 3.5.

4.1 Localization without any orthogonality

We begin by the generalization of Theorem 3.5 to the case without the orthogonality to the hat functions $\psi_{\mathbf{a}}$. Let us introduce the notation

$$r_{\mathrm{rem}}^{\mathbf{a}} := \frac{h_\Omega C_{\mathrm{PF},p,\Omega}}{|\omega_{\mathbf{a}}|^{\frac{1}{p}}} |\langle \mathcal{R}, \psi_{\mathbf{a}}\rangle_{(V_{\mathbf{a}})',V_{\mathbf{a}}}|.$$

Theorem 4.1 (Localization of dual norms of functionals without $\psi_{\mathbf{a}}$-orthogonality)**.** *For each $\mathcal{R} \in V'$, there holds*

$$\|\mathcal{R}\|_{V'} \le N C_{\mathrm{cont,PF}} \left\{ \frac{1}{N} \sum_{\mathbf{a}\in\mathcal{V}_h} \|\mathcal{R}\|^q_{(V^{\mathbf{a}})'} \right\}^{\frac{1}{q}} + N \left\{ \frac{1}{N} \sum_{\mathbf{a}\in\mathcal{V}_h^{\mathrm{int}}} (r^{\mathbf{a}}_{\mathrm{rem}})^q \right\}^{\frac{1}{q}}, \tag{4.1a}$$

$$\left\{ \frac{1}{N} \sum_{\mathbf{a}\in\mathcal{V}_h} \|\mathcal{R}\|^q_{(V^{\mathbf{a}})'} \right\}^{\frac{1}{q}} \le \|\mathcal{R}\|_{V'}. \tag{4.1b}$$

Proof. Estimate (4.1b) has been proven in Theorem 3.5. Estimate (4.1a) is proven along the lines of Theorem 3.5, counting for the additional nonzero term

$$\sum_{\mathbf{a}\in\mathcal{V}_h^{\mathrm{int}}} (\Pi_{0,\omega_{\mathbf{a}}} v) \, \langle \mathcal{R}, \psi_{\mathbf{a}} \rangle_{(V^{\mathbf{a}})', V^{\mathbf{a}}}$$

in (3.13). For each $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$, the Hölder inequality gives

$$|\omega_{\mathbf{a}}|^{\frac{1}{p}} \, (\Pi_{0,\omega_{\mathbf{a}}} v) = |\omega_{\mathbf{a}}|^{\frac{1}{p}} \, (v,1)_{\omega_{\mathbf{a}}} |\omega_{\mathbf{a}}|^{-1} \le |\omega_{\mathbf{a}}|^{\frac{1}{p}} \|v\|_{p,\omega_{\mathbf{a}}} |\omega_{\mathbf{a}}|^{\frac{1}{q}} |\omega_{\mathbf{a}}|^{-1} = \|v\|_{p,\omega_{\mathbf{a}}}.$$

Thus, the Hölder inequality, the Poincaré–Friedrichs inequality (2.3) used in the entire domain Ω on the space V, and (2.2) lead to

$$\sum_{\mathbf{a}\in\mathcal{V}_h^{\mathrm{int}}} (\Pi_{0,\omega_{\mathbf{a}}} v) \, \langle \mathcal{R}, \psi_{\mathbf{a}} \rangle_{(V^{\mathbf{a}})', V^{\mathbf{a}}} = \sum_{\mathbf{a}\in\mathcal{V}_h^{\mathrm{int}}} |\omega_{\mathbf{a}}|^{-\frac{1}{p}} \langle \mathcal{R}, \psi_{\mathbf{a}} \rangle_{(V^{\mathbf{a}})', V^{\mathbf{a}}} |\omega_{\mathbf{a}}|^{\frac{1}{p}} \, (\Pi_{0,\omega_{\mathbf{a}}} v)$$
$$\le \left\{ \sum_{\mathbf{a}\in\mathcal{V}_h^{\mathrm{int}}} \left(|\omega_{\mathbf{a}}|^{-\frac{1}{p}} |\langle \mathcal{R}, \psi_{\mathbf{a}} \rangle_{(V^{\mathbf{a}})', V^{\mathbf{a}}}| \right)^q \right\}^{\frac{1}{q}} \left\{ \sum_{\mathbf{a}\in\mathcal{V}_h^{\mathrm{int}}} \|v\|^p_{p,\omega_{\mathbf{a}}} \right\}^{\frac{1}{p}} \le \left\{ \sum_{\mathbf{a}\in\mathcal{V}_h^{\mathrm{int}}} (r^{\mathbf{a}}_{\mathrm{rem}})^q \right\}^{\frac{1}{q}} N^{\frac{1}{p}} \|\nabla v\|_p,$$

and (3.1) gives the assertion. □

Introduce the notation $r_{\mathrm{rem}} := \left\{ \frac{1}{N} \sum_{\mathbf{a}\in\mathcal{V}_h^{\mathrm{int}}} (r^{\mathbf{a}}_{\mathrm{rem}})^q \right\}^{\frac{1}{q}}$. This additional *remainder* term in inequality (4.1a) is known in the a posteriori estimation context, see, e.g., [19, equation (7.1)] or [16, equation (3.5a)]. One typically requires to control adaptively its size with respect to the principal contribution $\left\{ \frac{1}{N} \sum_{\mathbf{a}\in\mathcal{V}_h} \|\mathcal{R}\|^q_{(V^{\mathbf{a}})'} \right\}^{\frac{1}{q}}$, see [16, equation (3.10)]. The following corollary shows that localization can be restored in this way.

Corollary 4.2 (Localization of dual norms of functionals with controlled loss of orthogonality)**.** *Let $\mathcal{R} \in V'$ be arbitrary and assume that*

$$r_{\mathrm{rem}} \le \gamma_{\mathrm{rem}} C_{\mathrm{cont,PF}} \left\{ \frac{1}{N} \sum_{\mathbf{a}\in\mathcal{V}_h} \|\mathcal{R}\|^q_{(V^{\mathbf{a}})'} \right\}^{\frac{1}{q}}$$

for some parameter $\gamma_{\mathrm{rem}} \ge 0$. Then there holds

$$\|\mathcal{R}\|_{V'} \le (1 + \gamma_{\mathrm{rem}}) \, N C_{\mathrm{cont,PF}} \left\{ \frac{1}{N} \sum_{\mathbf{a}\in\mathcal{V}_h} \|\mathcal{R}\|^q_{(V^{\mathbf{a}})'} \right\}^{\frac{1}{q}},$$

$$\left\{ \frac{1}{N} \sum_{\mathbf{a}\in\mathcal{V}_h} \|\mathcal{R}\|^q_{(V^{\mathbf{a}})'} \right\}^{\frac{1}{q}} \le \|\mathcal{R}\|_{V'}.$$

4.2 Localization in vectorial setting

Now we present a vectorial variant of Theorem 3.5. We only make a concise presentation, as the extension from the scalar case is rather straightforward. We denote

$$\begin{aligned}\mathbf{V} &\coloneqq [W_0^{1,p}(\Omega)]^d,\\ \mathcal{R} &\in \mathbf{V}',\\ \|\mathcal{R}\|_{\mathbf{V}'} &\coloneqq \sup_{\mathbf{v}\in\mathbf{V};\, \|\nabla\mathbf{v}\|_p=1} \langle \mathcal{R},\mathbf{v}\rangle_{\mathbf{V}',\mathbf{V}}.\end{aligned}$$

For a vertex $\mathbf{a} \in \mathcal{V}_h$, the local setting is

$$\begin{aligned}\mathbf{V}^{\mathbf{a}} &\coloneqq [W_0^{1,p}(\omega_{\mathbf{a}})]^d,\\ \langle \mathcal{R},\mathbf{v}\rangle_{(\mathbf{V}^{\mathbf{a}})',\mathbf{V}^{\mathbf{a}}} &\coloneqq \langle \mathcal{R},\mathbf{v}\rangle_{\mathbf{V}',\mathbf{V}} \qquad \mathbf{v}\in\mathbf{V}^{\mathbf{a}},\\ \|\mathcal{R}\|_{(\mathbf{V}^{\mathbf{a}})'} &\coloneqq \sup_{\mathbf{v}\in\mathbf{V}^{\mathbf{a}};\, \|\nabla\mathbf{v}\|_{p,\omega_{\mathbf{a}}}=1} \langle \mathcal{R},\mathbf{v}\rangle_{(\mathbf{V}^{\mathbf{a}})',\mathbf{V}^{\mathbf{a}}}.\end{aligned}$$

Define $\boldsymbol{\psi}_{\mathbf{a},m}$, $1 \le m \le d$, as the vectorial variant of the partition of unity functions $\psi_{\mathbf{a}}$ such that $(\boldsymbol{\psi}_{\mathbf{a},m})_m = \psi_{\mathbf{a}}$ and $(\boldsymbol{\psi}_{\mathbf{a},m})_n = 0$ for $1 \le n \le d$, $n \ne m$. The following is a generalization of Theorem 3.5 to vectorial setting:

Theorem 4.3 (Localization of dual norms of functionals in vectorial case)**.** *Let* $\mathcal{R} \in \mathbf{V}'$*. If*

$$\langle \mathcal{R}, \boldsymbol{\psi}_{\mathbf{a},m}\rangle_{\mathbf{V}',\mathbf{V}} = 0 \qquad \forall 1 \le m \le d,\ \forall \mathbf{a} \in \mathcal{V}_h^{\mathrm{int}},$$

then

$$\|\mathcal{R}\|_{\mathbf{V}'} \le N C_{\mathrm{cont,PF},d} \left\{ \frac{1}{N} \sum_{\mathbf{a}\in\mathcal{V}_h} \|\mathcal{R}\|_{(\mathbf{V}^{\mathbf{a}})'}^q \right\}^{\frac{1}{q}}.$$

Conversely, there always holds

$$\left\{ \frac{1}{N} \sum_{\mathbf{a}\in\mathcal{V}_h} \|\mathcal{R}\|_{(\mathbf{V}^{\mathbf{a}})'}^q \right\}^{\frac{1}{q}} \le \|\mathcal{R}\|_{\mathbf{V}'}.$$

Proof. Along the lines of proof of Theorem 3.5, using Lemma 2.3 instead of Lemma 2.2. □

Extension of Theorems 3.7, 4.1 and of Corollary 4.2 to vectorial case is straightforward.

5 Numerical illustration

We now numerically demonstrate the validity of Theorem 3.5. The experiments were implemented using `dolfin-tape` [4] package built on top of the FEniCS Project [1]. The complete supporting code for reproducing the experiments can be obtained at [5].

Let $V_h \coloneqq \mathbb{P}_1(\mathcal{T}_h) \cap W^{1,p}(\Omega)$ be the space of continuous, piecewise first-order polynomials with respect to a matching triangular mesh $\mathcal{T}_h$ of the domain $\Omega \subset \mathbb{R}^2$, see Remark 2.1. Let $V_h^0 \coloneqq V_h \cap W_0^{1,p}(\Omega)$ be its zero-trace subspace and let u_h be the finite element approximation to the p-Laplace problem of Example 3.2, i.e.,

$$u_h - u_h^{\mathrm{D}} \in V_h^0, \tag{5.1a}$$

$$\left(|\nabla u_h|^{p-2}\nabla u_h, \nabla v_h\right) = (f_h, v_h) \qquad \forall v_h \in V_h^0, \tag{5.1b}$$

where $u_h^{\mathrm{D}} \in V_h$ is a $\mathbb{P}_1$-nodal interpolant of $u^{\mathrm{D}} \in W^{1,p}(\Omega) \cap C^0\left(\overline{\Omega}\right)$ (approximation error of u^{D} by u_h^{D} is neglected) and $(f_h,\cdot)$ approximates $(f,\cdot)$ by a Gauss quadrature (of degree 4). We consider $\mathcal{R} \in V'$, the residual of u_h with respect to equation (3.4b) (with $\boldsymbol{\sigma}(u,\nabla u) = |\nabla u|^{p-2}\nabla u$) given by (3.5). Taking $v_h = \psi_{\mathbf{a}}$ in (5.1b) immediately gives the orthogonality property (3.10) for all interior vertices $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$. Computationally, regularization and linearization of the degenerate p-Laplace operator is employed to approximately solve (5.1). The arising errors are secured to be small by error-distinguishing a posteriori estimation techniques of [16], thus ensuring sufficiently approximate fulfilment of the Galerkin orthogonality (3.10).

The evaluation of the norms $\|\mathcal{R}\|_{V'}$ and $\|\mathcal{R}\|_{(V^{\mathbf{a}})'}$ in (3.11)–(3.12) is equivalent to solving respectively for the *global lifting* $\mathfrak{r}$ on Ω defined by (3.7) and for the *local liftings* $\mathfrak{r}^{\mathbf{a}}$ on every patch $\omega_{\mathbf{a}}$ defined by (3.14). Again, only approximations $\mathfrak{r}_h \in V$ and $\mathfrak{r}_h^{\mathbf{a}} \in V^{\mathbf{a}}$ are available, where the evaluation error $\mathcal{E}_h \in V'$ is given by

$$\langle \mathcal{E}_h, v\rangle_{V',V} \coloneqq \left(|\nabla \mathfrak{r}_h|^{p-2}\nabla \mathfrak{r}_h, \nabla v\right) - \langle \mathcal{R}, v\rangle_{V',V} \qquad v \in V.$$

Since, simultaneously,

$$\|\mathcal{R}\|_{V'} \le \|\mathcal{E}_h\|_{V'} + \|\nabla \mathfrak{r}_h\|_p^{p-1},$$
$$\|\nabla \mathfrak{r}_h\|_p^{p-1} \le \|\mathcal{R}\|_{V'} + \|\mathcal{E}_h\|_{V'},$$

we obtain

$$\frac{\left|\|\nabla \mathfrak{r}_h\|_p^{p-1} - \|\mathcal{R}\|_{V'}\right|}{\|\nabla \mathfrak{r}_h\|_p^{p-1}} \le \frac{\|\mathcal{E}_h\|_{V'}}{\|\nabla \mathfrak{r}_h\|_p^{p-1}}.$$

Consequently, using a posteriori techniques from [16], the approximation

$$\|\mathcal{R}\|_{V'} \approx \|\nabla \mathfrak{r}_h\|_p^{p-1}$$

is guaranteed to hold with a given relative accuracy that we set to 10^{-2}. Similarly, we secure the relative accuracy of the approximation

$$\|\mathcal{R}\|_{(V^{\mathbf{a}})'} \approx \|\nabla \mathfrak{r}_h^{\mathbf{a}}\|_{p,\omega_{\mathbf{a}}}^{p-1}$$

to 10^{-2}. For clarity of notation, we drop the subscript h in what follows.

In order to plot local distributions, we find natural to define two non-negative functions from $\mathbb{P}_1(\mathcal{T}_h)$

$$\epsilon_{\mathrm{glob}}^q \coloneqq \sum_{\mathbf{a}\in\mathcal{V}_h} \|\nabla \mathfrak{r}\|_{p,\omega_{\mathbf{a}}}^p \frac{\psi_{\mathbf{a}}}{|\omega_{\mathbf{a}}|}, \tag{5.2a}$$

$$\epsilon_{\mathrm{loc}}^q \coloneqq \sum_{\mathbf{a}\in\mathcal{V}_h} \|\nabla \mathfrak{r}^{\mathbf{a}}\|_{p,\omega_{\mathbf{a}}}^p \frac{\psi_{\mathbf{a}}}{|\omega_{\mathbf{a}}|}. \tag{5.2b}$$

The employed normalization gives on simplicial meshes $|\omega_{\mathbf{a}}|^{-1}(\psi_{\mathbf{a}}, 1)_{\omega_{\mathbf{a}}} = N^{-1}$ (with $N = d + 1$) and together with (2.6) ensures that

$$\|\epsilon_{\mathrm{glob}}\|_q^q = \|\nabla \mathfrak{r}\|_p^p \overset{(3.8)}{=} \|\mathcal{R}\|_{V'}^q,$$
$$\|\epsilon_{\mathrm{loc}}\|_q^q = \frac{1}{N}\sum_{\mathbf{a}\in\mathcal{V}_h} \|\nabla \mathfrak{r}^{\mathbf{a}}\|_{p,\omega_{\mathbf{a}}}^p \overset{(3.15)}{=} \frac{1}{N}\sum_{\mathbf{a}\in\mathcal{V}_h} \|\mathcal{R}\|_{(V^{\mathbf{a}})'}^q.$$

Consequently, Theorem 3.5 can be rephrased as

$$\|\epsilon_{\mathrm{glob}}\|_q \le N C_{\mathrm{cont,PF}} \|\epsilon_{\mathrm{loc}}\|_q,$$
$$\|\epsilon_{\mathrm{loc}}\|_q \le \|\epsilon_{\mathrm{glob}}\|_q.$$

Moreover the second inequality above can be split into local contributions using Theorem 3.7, so that

$$\epsilon_{\mathrm{loc}} \le \epsilon_{\mathrm{glob}}. \tag{5.3}$$

Let us also introduce the effectivity index of an inequality (`ineq`)

$$\mathrm{Eff}_{(\mathtt{ineq})} \coloneqq \frac{\text{rhs of } (\mathtt{ineq})}{\text{lhs of } (\mathtt{ineq})} \ge 1.$$

For testing, we choose

- *Chaillou–Suri* [11, 16], $\Omega = (0,1)^2$, $p \in \{1.5, 10\}$, $u^{\mathrm{D}}(\mathbf{x}) = q^{-1}\left(0.5^q - |\mathbf{x} - (0.5, 0.5)|^q\right)$, $f = -\Delta_p u^{\mathrm{D}} = 2$,

Case	#cells	$C_{\text{cont,PF}}$	$\|\epsilon_{\text{glob}}\|_q$	$\|\epsilon_{\text{loc}}\|_q$	$\text{Eff}_{(3.11)}$	$\text{Eff}_{(5.5)}$	$\text{Eff}_{(3.12)}$
Chaillou–Suri $p = 1.5$, $N = 3$	100	5.670	0.0502	0.0431	14.6	13.8	1.17
	400	5.670	0.0259	0.0220	14.4	14.1	1.18
	900	5.670	0.0174	0.0147	14.4	14.2	1.18
	1600	5.670	0.0131	0.0111	14.4	14.2	1.18
Chaillou–Suri $p = 10.0$, $N = 3$	100	7.645	0.0604	0.0484	18.4	16.6	1.25
	400	7.645	0.0312	0.0255	18.8	17.8	1.22
	900	7.645	0.0214	0.0175	18.8	18.1	1.22
	1600	7.645	0.0161	0.0132	18.8	18.4	1.22
Carstensen–Klose $p = 4.0$, $N = 3$	40	9.706	0.1611	0.1236	22.3	16.3	1.30
	189	13.844	0.0930	0.0753	33.6	19.0	1.23
	428	12.981	0.0635	0.0518	31.8	19.4	1.23
	739	12.801	0.0471	0.0383	31.2	19.9	1.23

Table 1: Computed quantities of localization inequalities (3.11), (5.5), and (3.12) for the chosen model problems

- *Carstensen–Klose* [10, Example 3], $\Omega = (-1,1)^2 \setminus [0,1] \times [-1,0]$, $p = 4$, $u^{\mathrm{D}}(r,\theta) = r^{\frac{7}{8}} \sin\left(\frac{7}{8}\theta\right)$, $f = -\Delta_p u^{\mathrm{D}}$.

As we have the exact solution $u = u^{\mathrm{D}}$ in our hands, we can also check the distribution of the energy error (3.6). Therefore, as above, we define the non-negative function from $\mathbb{P}_1(\mathcal{T}_h)$

$$\epsilon_{\text{en}}^p := \sum_{\mathbf{a}\in\mathcal{V}_h} \|\nabla(u-u_h)\|_{p,\omega_{\mathbf{a}}}^p \frac{\psi_{\mathbf{a}}}{|\omega_{\mathbf{a}}|} \tag{5.4}$$

having the property

$$\|\epsilon_{\text{en}}\|_p = \|\nabla(u-u_h)\|_p.$$

The results of numerical experiments are shown in Table 1 and Figures 1–5. Effectivity indices in Table 1 show that the bound (3.12) is quite tight but the reverse bound (3.11) suffers with a larger, though still reasonable and predictable, overestimation. This overestimation decreases a little when improving (3.11) to

$$\|\mathcal{R}\|_{V'} \le N \left\{ \frac{1}{N} \sum_{\mathbf{a}\in\mathcal{V}_h} \left(C_{\text{cont,PF},\omega_{\mathbf{a}}} \|\mathcal{R}\|_{(V^{\mathbf{a}})'} \right)^q \right\}^{\frac{1}{q}}, \tag{5.5}$$

where $C_{\text{cont,PF},\omega_{\mathbf{a}}} := 1 + C_{\text{PF},p,\omega_{\mathbf{a}}} h_{\omega_{\mathbf{a}}} \|\nabla\psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}}$ is the continuity constant of each patch. Figures 1, 2, and 3 nicely demonstrate the local inequalities (3.16) as expressed by (5.3). Finally, Figures 4, and 5 show that there is no hope of locally comparing the energy error $\|\nabla(u-u_h)\|_p^p$ expressed here by ϵ_{en}^p of (5.4) and the lifted residual error $\|\nabla \mathfrak{r}\|_p^p$ expressed here by ϵ_{glob}^q of (5.2a).

In conclusion, the main results, Theorems 3.5 and 3.7, are well supported by the performed numerical experiments.

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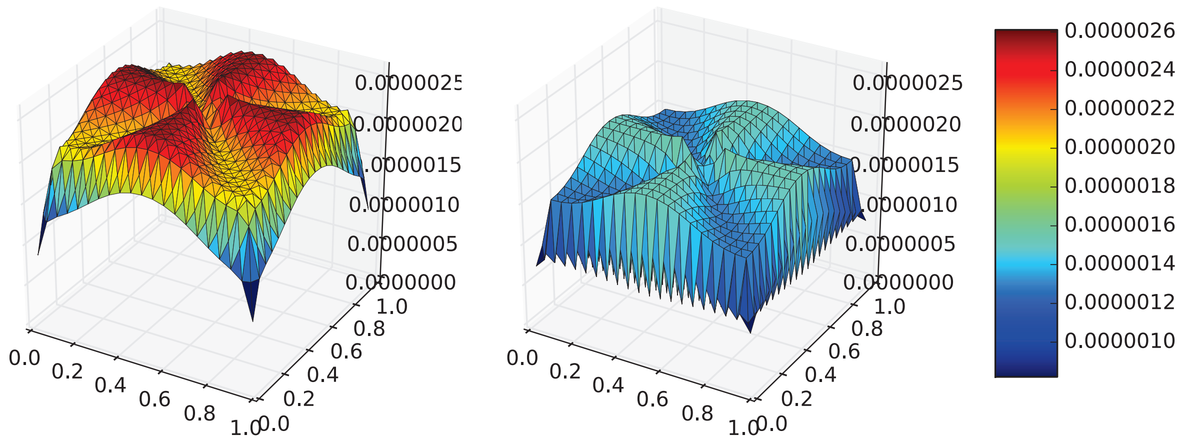

Figure 1: Functions ϵ^q_{glob} (left) and ϵ^q_{loc} (right) for the case *Chaillou–Suri*, $p = 1.5$, #cells=1600

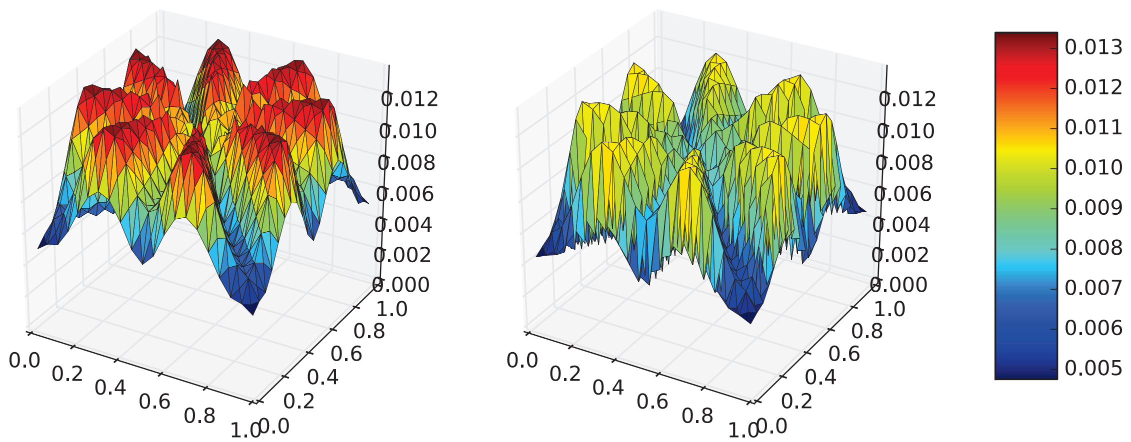

Figure 2: Functions ϵ^q_{glob} (left) and ϵ^q_{loc} (right) for the case *Chaillou–Suri*, $p = 10$, #cells=1600

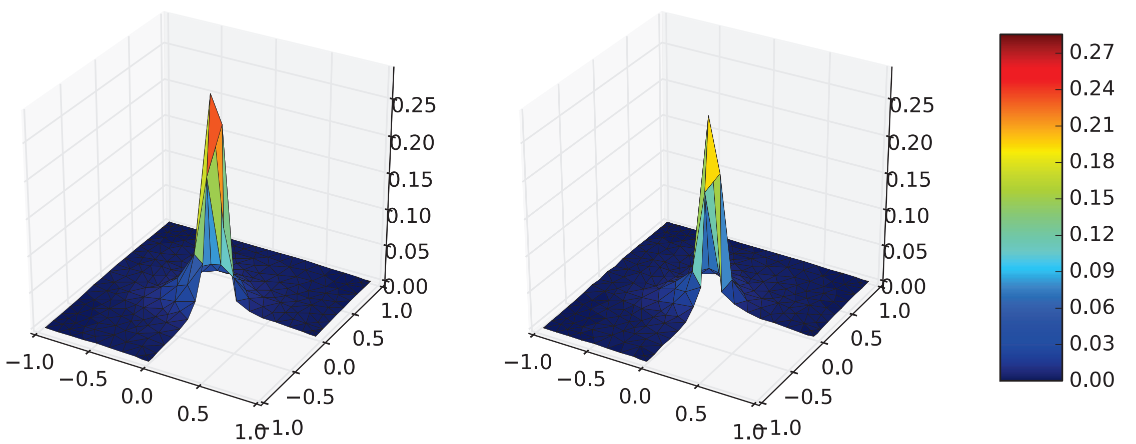

Figure 3: Functions ϵ^q_{glob} (left) and ϵ^q_{loc} (right) for the case *Carstensen–Klose*, $p = 4$, #cells=428

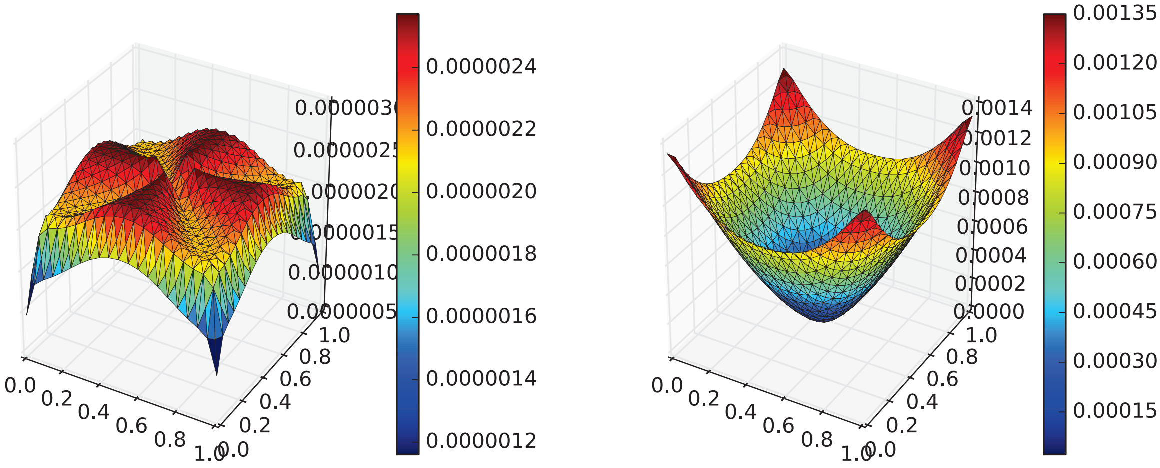

Figure 4: Functions $\epsilon^q_{\mathrm{glob}}$ (left) and ϵ^p_{en} (right) for the case *Chaillou–Suri*, $p = 1.5$, #cells=1600

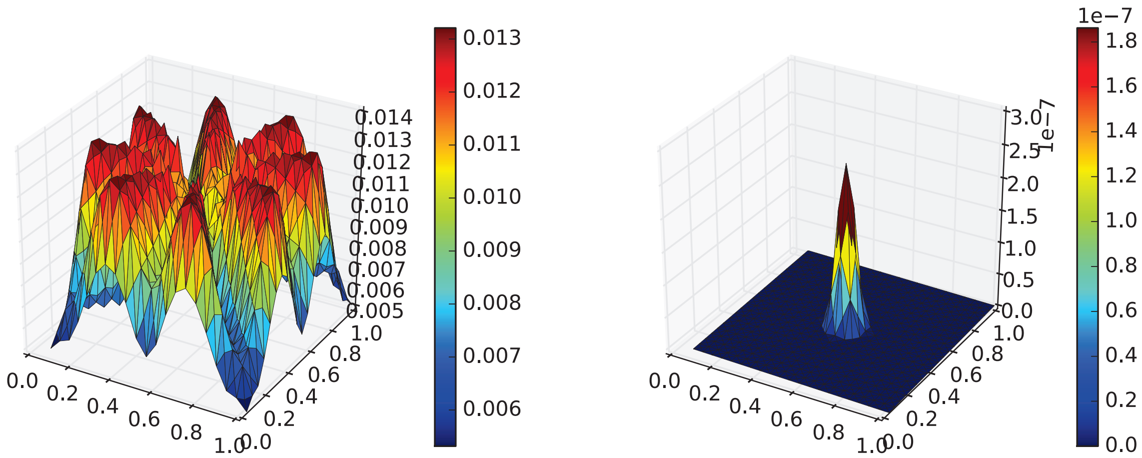

Figure 5: Functions $\epsilon^q_{\mathrm{glob}}$ (left) and ϵ^p_{en} (right) for the case *Chaillou–Suri*, $p = 10$, #cells=1600

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