On power-law fluids with the power-law index proportional to the pressure

J. Málek*, K.R. Rajagopal†, J. Žabenský‡

July 5, 2016

Abstract

In this short note we study special unsteady flows of a fluid whose viscosity depends on both the pressure and the shear rate. Here we consider an interesting dependence of the viscosity on the pressure and the shear rate; a power-law of the shear rate wherein the exponent depends on the pressure. The problem is important from the perspective of fluid dynamics in that we obtain solutions to a technologically relevant problem, and also from the point of view of mathematics as the analysis of the problem rests on the theory of spaces with variable exponents. We use the theory to prove the existence of solutions to generalizations of Stokes' first and second problem.

Keywords

Power-law fluids, pressure-dependent viscosity, unsteady flows, existence theory, fluid dynamics

Acknowledgements

This work was supported by the ERC-CZ project LL1202 financed by the Ministry of Education, Youth and Sports of the Czech Republic. KRR thanks the Office of Naval Research for its support.

1 Introduction

It is well established that in practically all fluids, the viscosity of the fluid can depend on the pressure (provided the pressure range is particularly large, see Bridgman [3], Szeri [25]) and in a wide class of fluids, viscosity can also depend on the shear rate. The viscosity of certain fluids can vary by as much as 10^8 due to variations in the pressure[1] (see Bair and Koptke [1] and the various references in a survey by Málek and Rajagopal [16] or in the later paper [6]) and it can change by orders of magnitude with respect to the shear rate. While there has been considerable work concerning the flows of fluids with pressure-dependent viscosity as well as those with a shear-rate-dependent viscosity, there has been very little work concerning the response of fluids whose viscosity depends on both the pressure and the shear rate simultaneously. Additionally, in these few studies that are devoted to the viscosity depending on the shear rate and the pressure, dependence of the viscosity on the shear rate is of the power-law type with the power-law exponent being a fixed number and they address primarily mathematical questions concerning the existence and uniqueness of solutions (see Franta et al. [12], Málek et al. [15],

*Corresponding author; Charles University in Prague, Faculty of Mathematics and Physics, Mathematical Institute, Sokolovská 83, 186 75 Prague 8, Czech Republic, malek@karlin.mff.cuni.cz

†Department of Mechanical Engineering, Texas A&M University, College Station, TX 77843, USA, krajagopal@tamu.edu

‡University of Würzburg, Institute of Mathematics, Chair of Mathematics XI, Campus Hubland Nord, Emil-Fischer-Straße 40, 97074 Würzburg, Germany, josef.zabensky@mathematik.uni-wuerzburg.de

[1]To be more precise, the mean value of the stress; see the recent paper by Rajagopal [18] for a detailed discussion of the notion of pressure.

Bulíček et al. [7–9]) and are not concerned with the solution of specific initial-boundary value problems. Moreover, the dependence of the viscosity on both the pressure and the shear rate has to fulfill certain mathematical conditions which contradict the experimental observation that the viscosity tends to infinity with the pressure tending to infinity. There are a few numerical studies (see [10, 13, 14, 22]) that consider the dependence of the viscosity on the pressure and the shear rate. However, the effect of the pressure and the shear rate on the viscosity are in the form of a product of a term representing the effect of the pressure and another term, representing the effect of the shear rate. It is, however, possible that the two influences on the viscosity cannot be so decomposed. This is indeed the situation in most instances, where more than one quantity can influence the properties of the material, for instance the temperature and the shear rate. Electrorheological fluids represent a very interesting situation wherein the viscosity of the fluid depends on both the shear rate and the electric field, with the dependence being expressed by the shear rate raised to the power of the electrical field (see Růžička [21]). Such a situation takes into consideration not only a physical possibility but it also opens up an interesting area in mathematical analysis (see Diening et al. [11]).

In this short note, we consider fluids whose viscosity depends on the pressure as well as the shear rate, with the variation that is similar to that discussed in the papers on electrorheology cited above in that the viscosity depends on the shear rate that is raised to a power that depends on the pressure (see equation (3)) while in electrorheological fluids the shear rate is raised to the intensity of the applied electric field. Our study extends the seminal studies of two unsteady problems considered by Stokes for a Newtonian fluid that are popularly referred to as Stokes' first problem (see Stokes [24]) and Stokes' second problem (see Stokes [24], Rayleigh [20]), in which Stokes considers a fluid above a plane, flowing due to the oscillation of the plane, and the problem of the flow of a Newtonian fluid lying above a plane, due to the plane being accelerated suddenly. Stokes did not take into account the effect of gravity. Srinivasan and Rajagopal [23] extended Stokes' study to take into account the effect of gravity as well as the pressure-dependence of the viscosity. As the pressure changes with depth, due to gravity, the viscosity of the fluid changes with depth and this gives rise to interesting physical consequences in that the vorticity and the shear stresses at the wall differ markedly from what one expects in a Newtonian fluid.

Recently, Rajagopal, Saccomandi and Vergori [19] studied unidirectional unsteady flows of fluids with pressure-dependent viscosity, where the effects of gravity are taken into account. After discussing the qualitative properties of the governing equations and establishing uniqueness for such unidirectional flows, they found explicit exact solutions for generalizations of Stokes' first and second problem for a special case of pressure-dependence of the viscosity, namely an exponential dependence of the viscosity on the pressure that obeys what is popularly referred to as the Barus formula (see Barus [2]). Rajagopal, Saccomandi and Vergori [19] do not consider the possibility of the viscosity depending simultaneously on both the pressure and the shear rate, which is the subject matter of this note.

The governing partial differential equations (6)–(8) and the initial-boundary conditions (5) pose quite a challenging problem that requires us to appeal to results in the theory of Lebesgue and Sobolev spaces with variables exponents (see Diening et al. [11]) in order to establish the existence of a weak solution to the governing equations.

Let $\Omega \subset \mathbb{R}^3$ be a three-dimensional domain (i.e. an open, connected set). In Ω, we consider unsteady flows of an incompressible, homogeneous fluid with a constant (strictly positive) density ϱ. The velocity $\boldsymbol{v} = (u, v, w)$ and the mean normal stress (pressure) p satisfy the equations representing the balance of linear momentum and the constraint of incompressibility, i.e.

$$\left.\begin{aligned} \varrho\left(\partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v})\right) &= \operatorname{div} \boldsymbol{T} + \varrho \boldsymbol{f} \\ \operatorname{div} \boldsymbol{v} &= 0 \end{aligned}\right\} \text{ in } \Omega, \tag{1}$$

where $\boldsymbol{f}$ stands for the external forces. The Cauchy stress tensor $\boldsymbol{T}$ is supposed to be of the form

$$\boldsymbol{T} = -p\boldsymbol{I} + 2\mu(p, |\boldsymbol{D}\boldsymbol{v}|^2)\boldsymbol{D}\boldsymbol{v}. \tag{2}$$

Motivated by experimental works of Barus [2] and Bridgman [3–5], one is primarily interested in understanding flows of fluids with an exponential dependence of the viscosity μ on the pressure p. In this study, we investigate unsteady viscous incompressible flows where the viscosity is of the form

$$\mu(p,|\boldsymbol{D}\boldsymbol{v}|^2) = \mu_0 \exp\left(\frac{p-2}{2}\ln(1+|\boldsymbol{D}\boldsymbol{v}|^2)\right) = \mu_0(1+|\boldsymbol{D}\boldsymbol{v}|^2)^{(p-2)/2}, \qquad \mu_0 > 0. \tag{3}$$

Note that the pressure p in (2) appears as the power law exponent in (3). It is worth mentioning that experiments measuring the dependence of the viscosity on the pressure are indirect, based on a falling cylinder. This means that the material in these experiments is not static but flowing. To the best of our knowledge, however, the way in which μ varies with both the pressure and the shear rate have not been fully addressed in experimental studies. In addition, these experiments are performed under conditions which are not consistent with the assumptions that are made in the data reduction procedure. We refer here to study by Průša [17], where further details concerning the methodology as well as references to relevant studies are given.

At this moment, the mathematical tools are not in place to investigate the problem in its full generality. Therefore we solve a special problem, resorting to what is referred to as a semi-inverse method, which reduces the complexity of the problem, allowing us to obtain an *almost* explicit solution. Specifically, let $d > 0$ and $\Omega := \mathbb{R}^2 \times (0,d)$ be a layer of depth d and let $\varrho\boldsymbol{f} = (0,0,-\varrho g)^T$, with g being the gravitational constant. Denoting the Cartesian coordinates in $\mathbb{R}^3$ by x, y and z, we will further assume that

$$\boldsymbol{v} = \boldsymbol{v}(t,z). \tag{4}$$

We shall also assume that the boundary and initial conditions are of the form

$$\left.\begin{aligned} \boldsymbol{v}(0,z) &= (f(z),0,0),\\ \boldsymbol{v}(t,0) &= (g_0(t),0,0),\\ \boldsymbol{v}(t,d) &= (g_d(t),0,0),\\ p(t,d) &= p_0, \end{aligned}\right\} \tag{5}$$

for every $t > 0$, $z \in \mathbb{R}$, some smooth functions f, g_0, g_d and a constant reference pressure p_0.

A similar problem has been considered by Rajagopal, Saccomandi and Vergori [19], where the viscosity dependence (3) was (relatively) simplified as $\mu = \mu(p) = \mu_0\exp(\omega(p-p_0))$, for some $\omega > 0$. Our case can be seen as an interesting generalization of the power-law model, where the power-law index is actually given in terms of the pressure.

We will prove the existence of weak solutions to (1)–(5). Although quite a daring idea at first sight, the trick is that we are able to compute the pressure explicitly and then use the theory of Lebesgue and Sobolev spaces with variable exponents (see the monograph [11]) to show existence of a velocity field solving the problem in the weak sense.

Lemma 1 *Assume that there is a smooth solution satisfying* (1)–(5). *Then* $v = w = 0$ *and* $p = p_0 + \varrho g(d-z)$.

Proof. From $(1)_2$ and (4) it follows that $w = w(t)$ and $(5)_2$ then implies that $w = 0$. Hence $\operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v}) = \boldsymbol{0}$ and equation $(1)_1$ reads

$$\varrho\,\partial_t u = -\frac{\partial p}{\partial x} + 2\frac{\partial}{\partial z}\left(\mu(p,|\boldsymbol{D}\boldsymbol{v}|^2)\frac{\partial u}{\partial z}\right), \tag{6}$$

$$\varrho\,\partial_t v = -\frac{\partial p}{\partial y} + 2\frac{\partial}{\partial z}\left(\mu(p,|\boldsymbol{D}\boldsymbol{v}|^2)\frac{\partial v}{\partial z}\right), \tag{7}$$

$$0 = -\frac{\partial p}{\partial z} + 2\frac{\partial}{\partial x}\left(\mu(p,|\boldsymbol{D}\boldsymbol{v}|^2)\frac{\partial u}{\partial z}\right) + 2\frac{\partial}{\partial y}\left(\mu(p,|\boldsymbol{D}\boldsymbol{v}|^2)\frac{\partial v}{\partial z}\right) - \varrho g. \tag{8}$$

These three equations, combined with (4) and smoothness of the quantities involved, yields a wave equation for the pressure

$$\left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) p = 0, \tag{9}$$

where the spatial variable z plays the role of time in the wave equation. Eq. (9) can be solved easily: let $\widetilde{p}(t,x,y,z) := p(t,x,y,d-z)$ (time t is a fixed parameter). Then $\widetilde{p}$ satisfies (9) and $\widetilde{p}(t,x,y,0) = p_0$. From (4), $(5)_3$ and (8) we also infer

$$\frac{\partial \widetilde{p}}{\partial z}(t,x,y,z)\bigg|_{z=0} = -\frac{\partial p}{\partial z}(t,x,y,z)\bigg|_{z=d} = \varrho g.$$

Denoting $x' = (x,y)$ and $B_t(x',z) = \{(t,\bar{x},\bar{y},z); |x'-\bar{x}'| < z\}$, Poisson's formula therefore yields

$$\widetilde{p}(t,x,y,z) = \frac{1}{2\pi z^2}\int_{B_t(x',z)} \frac{p_0 z + \varrho g z^2}{(z^2 - |x'-\bar{x}'|^2)^{1/2}}\, d\bar{x}' = p_0 + \varrho g z,$$

so that $p = p_0 + \varrho g(d-z)$. Equation (7) thus finally implies

$$\frac{\varrho}{2}\frac{d}{dt}\|v(t,x,y)\|^2_{L^2(0,d)} = -2\int_0^d \mu(p,|\boldsymbol{D}\boldsymbol{v}|^2)\left|\frac{\partial v}{\partial z}\right|^2 dz \le 0,$$

leading to $v = 0$ (note that (5) implies $v(0) = 0$). □

We see that under our assumptions, system (1) simplifies to

$$\left.\begin{aligned} \varrho\,\partial_t u &= 2\frac{\partial}{\partial z}\left(\mu(p,|\partial_z u|^2)\partial_z u\right) && \text{in } (0,\infty)\times(0,d),\\ p &= p_0 + \varrho g(d-z) && \text{in } (0,\infty)\times(0,d),\\ u(0,z) &= f(z) && \text{in } (0,d),\\ u(t,0) &= g_0(t) && \text{in } (0,\infty),\\ u(t,d) &= g_d(t) && \text{in } (0,\infty). \end{aligned}\right\} \tag{10}$$

That is, a PDE for a scalar function of one spatial and one temporal variable. Let us further assume $p_0 > 1$ so that $\inf_z p > 1$.

Theorem 2 *There is a unique weak solution to equation* (10)*, i.e. a function u satisfying*

$$\begin{aligned} &u \in L^\infty(0,T;L^2(0,d)) \cap L^{p(\cdot)}(0,T;W^{1,p(\cdot)}(0,d)),\\ &\partial_t u \in \left(L^{p(\cdot)}(0,T;W^{1,p(\cdot)}(0,d))\right)^*,\\ &\lim_{t\to 0_+}\|u(t)-f\|_{L^2(0,d)} = 0 \end{aligned}$$

and solving $(10)_1$ *in the sense of distribution.*

Proof. Without loss of generality suppose that $g_0 = g_d = 0$, $\varrho = 2$ and $g = 1/2$. Let $\{w_i\}_i \subset C^\infty([0,d])$, $w_i(0) = w_i(d) = 0$ for every i, be an orthonormal basis of $L^2(0,d)$. In a standard way we could show existence of Faedo-Galerkin approximations

$$u^n(t,z) = \sum_{i=1}^n g_i^n(t) w_i(z),$$

existing on $[0, t_0)$ for some $t_0 > 0$ and satisfying, for every $i = 1, \dots, n$,

$$\begin{aligned} \frac{d}{dt} g_i^n &= -(\mu(p, |\partial_z u^n|^2)\partial_z u^n, w_i'), \\ g_i^n(0) &= (f, w_i). \end{aligned} \tag{11}$$

By $(\cdot, \cdot)$ we signify the scalar product in $L^2(0, d)$. Multiplying (11) by g_i^n and summing over i yields

$$\sup_{t\in[0,T]} \|u^n(t)\|_2^2 + 2\int_0^T \int_0^d \mu(p, |\partial_z u^n|^2)|\partial_z u^n(t, z)|^2 \, dz\, dt \le \|f\|_2^2, \tag{12}$$

which also gives the estimate

$$\sup_{t\in[0,T]} \|u^n(t)\|_2^2 + 2\int_0^T \int_0^d |\partial_z u^n(t, z)|^{p_0+(d-z)} \, dz\, dt \le \|f\|_2^2, \tag{13}$$

The variable exponent $p(z)$ is linear, $1 < p_0 \le p(\cdot) \le p_0 + d < \infty$, so that by the theory of spaces with variable exponents [11] we may select a subsequence[2] such that, for $n \to \infty$,

$$u^n \to u \qquad \text{weakly}^* \text{ in } L^\infty(0, T; L^2(0, d)), \tag{14}$$

$$u^n \to u \qquad \text{weakly in } L^{p(\cdot)}(0, T; W_0^{1,p(\cdot)}(0, d)), \tag{15}$$

$$\partial_t u^n \to \partial_t u \qquad \text{weakly in } L^{p'(\cdot)}(0, T; W^{-1,p'(\cdot)}(0, d)), \tag{16}$$

$$\mu(p, |\partial_z u^n|^2)\partial_z u^n \to \overline{\boldsymbol{S}} \qquad \text{weakly in } L^{p'(\cdot)}((0, T) \times (0, d)). \tag{17}$$

Since the nonlinearity $A(s) = \mu(p, s^2)s$ is monotone, we can use Minty's method for the identification $\overline{\boldsymbol{S}} = \mu(p, |\partial_z u|^2)\partial_z u$ and uniqueness of the weak solution follows likewise. Although this method is standard, we sketch it for the sake of completeness (it also yields an alternative and simpler proof than the one given in [19]).

First, from (11) we can easily show that $u(0) = f$ and then $u^n(T) \to u(T)$ weakly in $L^2(0, d)$ (note that $u \in C_{weak}([0, T]; L^2(0, d))$). Since (11) and (14)–(17) imply

$$\|u(T)\|_2^2 + \int_0^T \int_0^d \overline{\boldsymbol{S}} \cdot \partial_z u \, dz\, dt = \|f\|_2^2,$$

the energy inequality (12) and the weak lower-semicontinuity of the norm give that

$$\limsup_{n\to\infty} \int_0^T \int_0^d \mu(p, |\partial_z u^n|^2)\partial_z u^n \cdot \partial_z u^n \, dz\, dt \le \int_0^T \int_0^d \overline{\boldsymbol{S}} \cdot \partial_z u \, dz\, dt. \tag{18}$$

Now, let $\varphi \in L^{p(\cdot)}(0, T; W_0^{1,p(\cdot)}(0, d))$. By monotonicity of $A(s) = \mu(p, s^2)s$, for all n we have

$$0 \le \int_0^T \int_0^d \big(\mu(p, |\partial_z u^n|^2)\partial_z u^n - \mu(p, |\partial_z \varphi|^2)\partial_z \varphi\big) \cdot \big(\partial_z u^n - \partial_z \varphi\big) \, dz\, dt.$$

Due to (15), (17) and (18), we obtain

$$0 \le \int_0^T \int_0^d \big(\overline{\boldsymbol{S}} - \mu(p, |\partial_z \varphi|^2)\partial_z \varphi\big) \cdot \big(\partial_z u - \partial_z \varphi\big) \, dz\, dt.$$

Choosing $\varphi = u \pm \varepsilon w$ for arbitrary $\varepsilon > 0$ and $w \in L^{p(\cdot)}(0, T; W_0^{1,p(\cdot)}(0, d))$, we finally conclude $\overline{\boldsymbol{S}} = \mu(p, |\partial_z u|^2)\partial_z u$. □

[2] When the power law exponent $p(\cdot)$ satisfies the so-called log-Hölder condition, in particular when it is linear like here, then the resulting function spaces with a variable exponent behave much like their standard counterparts with respect to reflexivity, separability, density of smooth functions etc.

References

[1] S. Bair and P. Kottke. Pressure-viscosity relationships for elastohydrodynamics. *Tribology transactions*, 46(3):289–295, 2003.

[2] C. Barus. Isotherms, isopiestics and isometrics relative to viscosity. *Am. J. Sci.*, 45(3):87–96, 1893.

[3] P.W. Bridgman. The viscosity of liquids under pressure. *Proceedings of the National Academy of Sciences of the United States of America*, 11(10):603, 1925.

[4] P.W. Bridgman. The effect of pressure on the viscosity of forty-three pure liquids. *Proceedings of the American Academy of Arts and Sciences*, 61(3):57–99, 1926.

[5] P.W. Bridgman. The viscosity of mercury under pressure. *Proceedings of the American Academy of Arts and Sciences*, 62(7):187–206, 1927.

[6] M. Bulíček, M. Majdoub, and J. Málek. Unsteady flows of fluids with pressure dependent viscosity in unbounded domains. *Nonlinear Analysis: Real World Applications*, 11(5):3968–3983, 2010.

[7] M. Bulíček, M. Majdoub, and J. Málek. Unsteady flows of fluids with pressure dependent viscosity in unbounded domains. *Nonlinear Anal. Real World Appl.*, 11(5):3968–3983, 2010.

[8] M. Bulíček, J. Málek, and K.R. Rajagopal. Navier's slip and evolutionary Navier-Stokes-like systems with pressure and shear-rate dependent viscosity. *Indiana Univ. Math. J.*, 56(1):51–85, 2007.

[9] M. Bulíček, J. Málek, and K.R. Rajagopal. Mathematical analysis of unsteady flows of fluids with pressure, shear-rate, and temperature dependent material moduli that slip at solid boundaries. *SIAM J. Math. Anal.*, 41(2):665–707, 2009.

[10] A.R. Davies and X.K. Li. Numerical modeling of pressure and temperature effects in visoelastic flow between exccentrically rotating cylinders. *J. Non-Newt. Fluid Mech.*, 54:331–350, 1994.

[11] L. Diening, P. Harjulehto, P. Hästö, and M. Růžička. *Lebesgue and Sobolev spaces with variable exponents*, volume 2017 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2011.

[12] M. Franta, J. Málek, and K.R. Rajagopal. On steady flows of fluids with pressure- and shear-dependent viscosities. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 461(2055):651–670, 2005.

[13] D.Rh. Gwynllyw, A.R. Davies, and T.N. Phillips. On the effects of a piezoviscous lubricant on the dynamics of a journal bearing. *J. Rheol.*, 40(6):1239–1266, 1996.

[14] X.K. Li, D.Rh. Gwynllyw, A.R. Davies, and T.N. Phillips. On the influence of lubricant properties on the dynamics of two-dimensional journal bearings. *J. Non-Newtonian Fluid Mech*, 93:29–59, 2000.

[15] J. Málek, J. Nečas, and K.R. Rajagopal. Global analysis of the flows of fluids with pressure-dependent viscosities. *Arch. Ration. Mech. Anal.*, 165(3):243–269, 2002.

[16] J. Málek and K.R. Rajagopal. Mathematical Properties of the Solutions to the Equations Govering the Flow of Fluid with Pressure and Shear Rate Dependent Viscosities. In *Handbook of Mathematical Fluid Dynamics, Vol. IV*, Handb. Differ. Equ., pages 407–444. Elsevier/North-Holland, Amsterdam, 2007.

[17] V. Průša, S. Srinivasan, and K.R. Rajagopal. Role of pressure dependent viscosity in measurements with falling cylinder viscometer. *International Journal of Non-Linear Mechanics*, 47(7):743–750, 2012.

[18] K.R. Rajagopal. Remarks on the notion of "pressure". *International Journal of Non-Linear Mechanics*, 71:165–172, 2015.

[19] K.R. Rajagopal, G. Saccomandi, and L. Vergori. Unsteady flows of fluids with pressure dependent viscosity. *J. Math. Anal. Appl.*, 404(2):362–372, 2013.

[20] L. Rayleigh. On the motion of solid bodies through viscous liquid. *Philosophical Magazine Series 6*, 21(126):697–711, 1911.

[21] M. Růžička. *Electrorheological Fluids: Modeling and Mathematical Theory*, volume 1748 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2000.

[22] D.G. Schaeffer. Instability in the evolution equations describing incompressible granular flow. *J. Differential Equations*, 66(1):19–50, 1987.

[23] S. Srinivasan and K.R. Rajagopal. Study of a variant of stokes' first and second problems for fluids with pressure dependent viscosities. *International Journal of Engineering Science*, 47(11–12):1357–1366, 2009.

[24] G.G. Stokes. On the Effect of the Internal Friction of Fluids on the Motion of Pendulums. *Transactions of the Cambridge Philosophical Society*, 9:8–106, 1851.

[25] A.Z. Szeri. *Fluid Film Lubrication.* Cambridge University Press, 1998. Cambridge Books Online.