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EXISTENCE OF GLOBAL WEAK SOLUTIONS TO IMPLICITLY CONSTITUTED KINETIC MODELS OF INCOMPRESSIBLE HOMOGENEOUS DILUTE POLYMERS

MIROSLAV BULÍČEK, JOSEF MÁLEK, AND ENDRE SÜLI

Abstract. We show the existence of global weak solutions to a general class of kinetic models of homogeneous incompressible dilute polymers. The main new feature of the model is the presence of a general implicit constitutive equation relating the viscous part $\mathbf{S}_v$ of the Cauchy stress and the symmetric part $\mathbf{D}$ of the velocity gradient. We consider implicit relations that generate maximal monotone (possibly multivalued) graphs, and the corresponding rate of dissipation is characterized by the sum of a Young function and its conjugate depending on $\mathbf{D}$ and $\mathbf{S}_v$, respectively. Such a framework is very general and includes, among others, classical power-law fluids, stress power-law fluids, fluids with activation criteria of Bingham or Herschel–Bulkley type, and shear-rate dependent fluids with discontinuous viscosities as special cases. The appearance of $\mathbf{S}_v$ and $\mathbf{D}$ in all the assumptions characterizing the implicit relationship $\mathbf{G}(\mathbf{S}_v, \mathbf{D}) = \mathbf{0}$ is fully symmetric. The elastic properties of the flow, characterizing the response of polymer macromolecules in the viscous solvent, are modelled by the elastic part $\mathbf{S}_e$ of the Cauchy stress tensor, whose divergence appears on the right-hand side of the momentum equation, and which is defined by the Kramers expression involving the probability density function, associated with the random motion of the polymer molecules in the solvent. The probability density function satisfies a Fokker–Planck equation, which is nonlinearly coupled to the momentum equation. We establish long-time and large-data existence of weak solutions to such a system, completed by an initial condition and either a no-slip or Navier's slip boundary condition, by using properties of maximal monotone operators and Lipschitz approximations of Sobolev-space-valued Bochner functions via a weak compactness arguments based on the Div-Curl Lemma and Chacon's Biting Lemma. A key ingredient in the proof is the strong compactness in L^1 of the sequence of Galerkin approximations to the probability density function and of the associated sequence of approximations to the elastic part $\mathbf{S}_e$ of the Cauchy stress tensor.

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1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded open Lipschitz domain, let T denote the length of the time interval of interest and let $Q := \Omega \times (0, T)$ signify the associated space-time domain. We consider the following system of nonlinear partial differential equations, modelling the motion of an incompressible homogeneous fluid:

$$\rho\,(\boldsymbol{v}_{,t} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v})) - \operatorname{div} \mathbf{T} = \rho \boldsymbol{f} \qquad \text{in } Q, \tag{1.1}$$

$$\operatorname{div} \boldsymbol{v} = 0 \qquad \text{in } Q, \tag{1.2}$$

subject to the initial condition

$$\boldsymbol{v}(\cdot, 0) = \boldsymbol{v}_0(\cdot) \qquad \text{in } \Omega, \tag{1.3}$$

and the boundary conditions

$$\boldsymbol{v} \cdot \boldsymbol{n} = 0 \qquad \text{on } \partial\Omega \times (0, T), \tag{1.4}$$

$$\lambda(\mathbf{T}\boldsymbol{n})_{\boldsymbol{\tau}} + (1 - \lambda)\gamma_* \boldsymbol{v}_{\boldsymbol{\tau}} = 0 \qquad \text{on } \partial\Omega \times (0, T). \tag{1.5}$$

In the equations (1.1)–(1.2), $\boldsymbol{v} : Q \to \mathbb{R}^d$ is the velocity of the fluid, $\mathbf{T} : Q \to \mathbb{R}^{d\times d}$ denotes the Cauchy stress, $\boldsymbol{f} : Q \to \mathbb{R}^d$ is the density of external body forces, and ρ is the density of the fluid, which we assume here to be constant. We note in connection with the boundary conditions (1.4)–(1.5) that we shall only consider internal flows here, i.e., the equation (1.4) will be assumed to hold, where $\boldsymbol{n} = \boldsymbol{n}(x)$ denotes the outer unit normal vector at a point $x \in \partial\Omega$. For the tangential part $\boldsymbol{v}_{\boldsymbol{\tau}}$ of the velocity vector $\boldsymbol{v}$ (where the symbol $\boldsymbol{w}_{\boldsymbol{\tau}}(x)$ denotes the projection of $\boldsymbol{w}(x)$ on the tangent plane to the boundary at $x \in \partial\Omega$, i.e., $\boldsymbol{w}_{\boldsymbol{\tau}} := \boldsymbol{w} - (\boldsymbol{w} \cdot \boldsymbol{n})\boldsymbol{n}$), we assume that the fluid either slips along $\partial\Omega$ (when $\lambda = 1$ in (1.5)) or partially slips (when $\lambda \in (0, 1)$, and then the condition (1.5) is called the Navier slip boundary condition), or it adheres to the boundary (when $\lambda = 0$), in which case (1.5) represents the standard no-slip boundary condition $\boldsymbol{v} = \mathbf{0}$ on $\partial\Omega \times (0, T)$. The positive constant γ_* denotes a friction coefficient whose actual value is of no significance for the discussion that will follow, and we therefore set it to 1; we follow the same convention for the fluid density ρ, which we also set to 1 in the sequel.

To complete the system (1.1)–(1.5), we need to state the constitutive equation for the Cauchy stress $\mathbf{T}$. In what follows we assume that the Cauchy stress is decomposed as

$$\mathbf{T} = -p\,\mathbf{I} + \mathbf{S}_v + \mathbf{S}_e, \tag{1.6}$$

where $\mathbf{S}_v : Q \to \mathbb{R}^{d\times d}_{sym}$ represents the viscous part of the stress, $\mathbf{S}_e : Q \to \mathbb{R}^{d\times d}_{sym}$ corresponds to the elastic part of the stress and $p : Q \to \mathbb{R}$ is the pressure. The viscous part $\mathbf{S}_v$ of the Cauchy stress $\mathbf{T}$ and the symmetric part of the velocity gradient $\mathbf{D}(\boldsymbol{v}) := \frac{1}{2}(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^{\mathrm{T}})$ will be assumed to be related through a maximal monotone graph (cf. below for details) described by an implicit relation of the form

$$\mathbf{G}(\mathbf{S}_v, \mathbf{D}(\boldsymbol{v})) = \mathbf{0}, \tag{1.7}$$

where $\mathbf{G} : \mathbb{R}^{d\times d}_{sym} \times \mathbb{R}^{d\times d}_{sym} \to \mathbb{R}^{d\times d}_{sym}$ is a continuous mapping. The class of fluids described by (1.7) is very general and includes not only Newtonian (Navier–Stokes) fluids ($\mathbf{S} = 2\mu_* \mathbf{D}(\boldsymbol{v})$

with μ_* being a positive constant), but also standard power-law fluid models, where $\mathbf{S}_v = 2\mu_*|\mathbf{D}(\boldsymbol{v})|^{r-2}\,\mathbf{D}(\boldsymbol{v})$, $1 \le r < \infty$, and their generalizations ($\mathbf{S}_v = 2\tilde{\mu}(|\mathbf{D}(\boldsymbol{v})|^2)\,\mathbf{D}(\boldsymbol{v})$), stress power-law fluid flow models and their generalizations of the form $\mathbf{D}(\boldsymbol{v}) = \alpha(|\mathbf{S}_v|^2)\,\mathbf{S}_v$, fluids with the viscosity depending on the shear rate and the shear stress

$$\mathbf{S}_v = 2\hat{\mu}(|\mathbf{D}(\boldsymbol{v})|^2, |\mathbf{S}_v|^2)\,\mathbf{D}(\boldsymbol{v}),$$

as well as activated fluids, such as Bingham and Herschel–Bulkley fluids, characterized via the equation[1]

$$2\nu(|\mathbf{D}|^2)\left(\tau_* + (|\mathbf{S}| - \tau_*)^+\right)\mathbf{D} = (|\mathbf{S}| - \tau_*)^+\,\mathbf{S},$$

where $\tau_* > 0$ and $x^+ = \max\{x, 0\}$. For further details concerning the physical background of the implicit constitutive theory we refer the reader to the papers by Rajagopal [35, 36] and Rajagopal & Srinivasa [37], and the introductory parts of Bulíček, Gwiazda, Málek & Świerczewska-Gwiazda [11, 15] and Bulíček, Gwiazda, Málek, Rajagopal & Świerczewska-Gwiazda [14].

Dilute polymers can be viewed as mixtures of some base liquid (the solvent) and polymer macromolecules flowing in it. This is reflected here in the form of the equation for the Cauchy stress, see (1.6), which is, similarly as in a Kelvin–Voigt model, the sum of two parts of the stress: one corresponding to the elastic response and the other corresponding to the fluid response (plus a spherical stress, reflecting the fact that the fluid is incompressible). Usually the liquid is considered to be a Newtonian (Navier–Stokes) fluid. There are however many liquids that shear-thin or shear-thicken, or they may change their properties dramatically once a certain critical value of the stress (or the shear rate) is reached. Such a behavior can be conveniently described within the class of implicitly constituted fluids characterized by the constitutive equation (1.7).

Finally, we state the constitutive relation for the elastic part $\mathbf{S}_e$ of the Cauchy stress tensor $\mathbf{T}$. In a *bead-spring chain model* for dilute polymers, consisting of $K+1$ beads coupled with K elastic springs to represent a polymer chain, the elastic extra-stress tensor $\mathbf{S}_e$ is defined by the *Kramers expression* as a weighted average of ψ, the probability density function of the (random) conformation $\boldsymbol{q} := (\boldsymbol{q}^1, \ldots, \boldsymbol{q}^K) \in \mathbb{R}^{d\times K}$ of the chain (see eq. (1.11) below), with the (column) vector $\boldsymbol{q}^j := (q^j_1, \ldots, q^j_d)^{\mathrm{T}}$ representing the d-component conformation/orientation vector of the jth spring in the bead-spring chain. For ease of exposition, superscripts throughout the paper are related to the number of springs (i.e., the number of configuration space dimensions) and can attain the values $1, \ldots, K$, while subscripts are generally related to the number of physical space dimensions d, so they can attain the values $1, \ldots, d$.

The Kolmogorov equation satisfied by ψ is a second-order parabolic equation, the Fokker–Planck equation (see eq. (1.13) below), whose transport coefficients depend on the velocity

[1] A usual description of fluids of Bingham or Herschel–Bulkley type reads as follows (see, for example, the monograph of Duvaut & Lions [21]):

$$|\mathbf{S}| \le \tau_* \Leftrightarrow \mathbf{D} = \mathbf{0} \quad \text{and} \quad |\mathbf{S}| > \tau_* \Leftrightarrow \mathbf{S} = \frac{\tau_*\mathbf{D}}{|\mathbf{D}|} + 2\nu(|\mathbf{D}|^2)\,\mathbf{D}.$$

field $\boldsymbol{v}$. The domain D of admissible conformation vectors $D \subset \mathbb{R}^{d\times K} \cong \mathbb{R}^{Kd}$ is a K-fold Cartesian product $D^1 \times \cdots \times D^K$ of balanced convex open sets $D^j \subset \mathbb{R}^d$, $j = 1,\dots,K$; the term *balanced* means that $\boldsymbol{q}^j \in D^j$ if, and only if, $-\boldsymbol{q}^j \in D^j$. Hence, in particular, $\mathbf{0} \in D^j$, $j = 1,\dots,K$. Typically D^j is the whole of $\mathbb{R}^d$ or a bounded open d-dimensional ball centred at the origin $\mathbf{0} \in \mathbb{R}^d$ for each $j = 1,\dots,K$. When $K = 1$, the model is referred to as the *dumbbell model.*

Let $I^j \subset [0,\infty)$ denote the image of D^j under the mapping $\boldsymbol{q}^j \in D^j \mapsto \frac{1}{2}|\boldsymbol{q}^j|^2$, and consider the *spring potential* $U^j \in \mathcal{C}^{0,1}_{loc}(I^j;\mathbb{R}_{\geq 0})$, $j = 1,\dots,K$. Clearly, $0 \in I^j$. Typically, $U^j(0) = 0$ and U^j is monotonic increasing and unbounded on I^j for each $j = 1,\dots,K$; however, we do not explicitly require these properties in our analysis. The elastic spring-force $\mathsf{F}^j : D^j \subseteq \mathbb{R}^d \to \mathbb{R}^d$ of the jth spring in the chain is defined by

$$\mathsf{F}^j(\boldsymbol{q}^j) = (U^j)'(\tfrac{1}{2}|\boldsymbol{q}^j|^2)\,\boldsymbol{q}^j, \qquad j = 1,\dots,K. \tag{1.8}$$

Example 1.1. In the Hookean dumbbell model $K = 1$, and the spring force is defined by $\mathsf{F}(\boldsymbol{q}) = \boldsymbol{q}$, with $\boldsymbol{q} \in D = \mathbb{R}^d$, corresponding to $U(s) = s$, $s \in I = [0,\infty)$. This model is physically unrealistic as it admits an arbitrarily large extension of the spring modelling the polymer molecule. $\diamond$

We shall therefore assume in what follows that D is a Cartesian product of K *bounded* open balls $D^j \subset \mathbb{R}^d$, each centred at the origin $\mathbf{0} \in \mathbb{R}^d$, $j = 1,\dots,K$, with $K \geq 1$. We define the (normalized) Maxwellian M^j with respect to the variable $\boldsymbol{q}^j$ by

$$M^j(\boldsymbol{q}^j) = \frac{1}{\mathcal{Z}^j}\,\mathrm{e}^{-U^j(\frac{1}{2}|\boldsymbol{q}^j|^2)}, \qquad \mathcal{Z}^j := \int_{D^j} \mathrm{e}^{-U^j(\frac{1}{2}|\boldsymbol{q}^j|^2)}\,\mathrm{d}\boldsymbol{q}^j,$$

where $\mathrm{d}\boldsymbol{q}^j := \mathrm{d}q^j_1 \cdots \mathrm{d}q^j_d$, $j = 1,\dots,K$. The (full) Maxwellian in the model is then defined by

$$M(\boldsymbol{q}) := \prod_{j=1}^K M^j(\boldsymbol{q}^j) \qquad \forall\, \boldsymbol{q} := (\boldsymbol{q}^1,\dots,\boldsymbol{q}^K) \in D := \bigtimes_{j=1}^K D^j. \tag{1.9}$$

Observe that, for $\boldsymbol{q} \in D$ and $j = 1,\dots,K$,

$$M(\boldsymbol{q})\,\nabla_{\boldsymbol{q}^j}[M(\boldsymbol{q})]^{-1} = -[M(\boldsymbol{q})]^{-1}\,\nabla_{\boldsymbol{q}^j} M(\boldsymbol{q}) = \nabla_{\boldsymbol{q}^j} U^j(\tfrac{1}{2}|\boldsymbol{q}^j|^2) = (U^j)'(\tfrac{1}{2}|\boldsymbol{q}^j|^2)\,\boldsymbol{q}^j. \tag{1.10}$$

Here $\nabla_{\boldsymbol{q}^j} := (\partial/\partial q^j_1,\dots,\partial/\partial q^j_d)^{\mathrm{T}}$, for $j = 1,\dots,K$. We define $\mathrm{div}_{q^j} := \nabla_{\boldsymbol{q}^j}\cdot = (\nabla_{\boldsymbol{q}^j})^{\mathrm{T}}$, and for a mapping $\boldsymbol{q} \in D \to B(\boldsymbol{q}) \in \mathbb{R}^{d\times K}$ we let $\mathrm{div}_{\boldsymbol{q}} B := \mathrm{div}_{\boldsymbol{q}^1} B^1 + \cdots + \mathrm{div}_{\boldsymbol{q}^K} B^K$, where B^j, $j = 1,\dots,K$, denote the (d-component) column vectors of the matrix $B = B(\boldsymbol{q})$. Finally, we define the $(d \times K)$-component differential operator $\nabla_{\boldsymbol{q}} := (\nabla_{\boldsymbol{q}^1},\dots,\nabla_{\boldsymbol{q}^K})$.

We use the standard notation for differential operators when we differentiate with respect to the spatial variable x or the time variable t; in particular, differential operators, such as div, ∇, Δ, bearing no subscript, signify differential operators with respect to the variable x. Differential operators with respect to the configuration space variable $\boldsymbol{q} = (\boldsymbol{q}^1,\dots,\boldsymbol{q}^K) \in D$ will be subscripted according to the notational conventions introduced in the previous paragraph. We shall denote by $\nabla_{x,\boldsymbol{q}}$ the gradient operator with respect to the variable $(x,\boldsymbol{q})$, with analogous definitions of the differential operators $\nabla_{t,x,\boldsymbol{q}}$ and $\mathrm{div}_{t,x,\boldsymbol{q}}$.

Example 1.2. In the FENE (finitely extensible nonlinear elastic) dumbbell model $K = 1$ and the spring force is given by $\mathsf{F}(\boldsymbol{q}) = (1 - |\boldsymbol{q}|^2/b)^{-1}\,\boldsymbol{q}$, $\boldsymbol{q} \in D = B(\mathbf{0}, b^{\frac{1}{2}})$, corresponding to $U(s) = -\frac{b}{2}\ln\left(1 - \frac{2s}{b}\right)$, $s \in I = [0, \frac{b}{2})$. Here $B(\mathbf{0}, b^{\frac{1}{2}})$ is a bounded open ball in $\mathbb{R}^d$ centred at the origin $\mathbf{0} \in \mathbb{R}^d$ and of fixed radius $b^{\frac{1}{2}}$, with $b > 0$. The usual assumption in the case of the FENE model is that $b > 2$ (cf. [7] and [8]). For the purposes of our large-data global existence result herein, the weaker assumption $b > 0$ will suffice. $\diamond$

The governing equations of the general FENE-type bead-spring chain model with centre-of-mass diffusion are (1.1)–(1.7), where the extra-stress tensor S_e is defined by the *Kramers expression*:

$$\mathsf{S}_e(x,t) := k\left(\sum_{j=1}^{K}\int_D \psi(x,\boldsymbol{q},t)\,\boldsymbol{q}^j\,{\boldsymbol{q}^j}^{\mathrm{T}}\,(U^j)'\left(\tfrac{1}{2}|\boldsymbol{q}^j|^2\right)\mathrm{d}\boldsymbol{q} - K\varrho(x,t)\,\mathsf{I}\right), \tag{1.11}$$

with I denoting the $d\times d$ unit matrix, $\mathrm{d}\boldsymbol{q} := \mathrm{d}\boldsymbol{q}^1\cdots\mathrm{d}\boldsymbol{q}^K$, and the density of polymer chains (not to be confused with the constant density ρ of the fluid, which we set to 1,) located at x at time t defined by

$$\varrho(x,t) := \int_D \psi(x,\boldsymbol{q},t)\,\mathrm{d}\boldsymbol{q}. \tag{1.12}$$

The probability density function ψ is a solution of the Fokker–Planck equation

$$\begin{aligned}&\psi_{,t} + \mathrm{div}\,(\psi\boldsymbol{v}) + \sum_{j=1}^{K}\mathrm{div}_{\boldsymbol{q}^j}\left((\nabla\boldsymbol{v})\boldsymbol{q}^j\,\psi\right)\\ &= \varepsilon\,\Delta\,\psi + \frac{1}{4\,\lambda}\sum_{i=1}^{K}\sum_{j=1}^{K}A_{ij}\,\mathrm{div}_{\boldsymbol{q}^i}\left(M\,\nabla_{\boldsymbol{q}^j}\left(\frac{\psi}{M}\right)\right) \quad \text{in } \Omega\times D\times(0,T].\end{aligned} \tag{1.13}$$

The dimensionless constant $k > 0$ featuring in (1.11) is a constant multiple of the product of the Boltzmann constant k_B and the absolute temperature $\mathtt{T}$. In (1.13), $\varepsilon > 0$ is the centre-of-mass diffusion coefficient defined as $\varepsilon := (\ell_0/L_0)^2/(4(K+1)\lambda)$ with $\ell_0 := \sqrt{k_B\mathtt{T}/\mathtt{H}}$ signifying the characteristic microscopic length-scale and $\lambda := (\zeta/4\mathtt{H})(U_0/L_0)$, where $\zeta > 0$ is a friction coefficient and $\mathtt{H} > 0$ is a spring-constant. The dimensionless positive parameter λ characterizes the elastic relaxation property of the fluid. In the subsequent discussion we shall simply take $\varepsilon = 1$ and $\lambda = 1/4$, since none of our results depend on the specific values of these positive parameters.

Further, $A = (A_{ij})_{i,j=1}^{K} \in \mathbb{R}^{K\times K}_{\mathrm{sym}}$ is a constant symmetric positive definite matrix, referred to as the *Rouse matrix*. We associate with A the linear mapping $\mathbb{A} : \mathbb{R}^{d\times K} \to \mathbb{R}^{d\times K}$ defined, for any $B = (B_i^j)_{i=1,\dots,d}^{j=1,\dots,K} \in \mathbb{R}^{d\times K}$, by $(\mathbb{A}(B))_i^j := \sum_{k=1}^{K} B_i^k A_{kj}$, and let $\mathbb{A}^j : \mathbb{R}^{d\times K} \to \mathbb{R}^d$ be the linear mapping defined by $(\mathbb{A}^j(B))_i := (\mathbb{A}(B))_i^j$, for $i = 1,\dots,d$ and $j = 1,\dots,K$. We then deduce from the assumed positive definiteness of the Rouse matrix $A \in \mathbb{R}^{K\times K}_{\mathrm{sym}}$ the existence of positive constants C_1 and C_2 such that

$$C_1|B|^2 \le \mathbb{A}(B) : B \le C_2|B|^2 \qquad \forall\, B \in \mathbb{R}^{d\times K}. \tag{1.14}$$

Let us define

$$\widehat{\psi} := \frac{\psi}{M}.$$

With this notation, we have from (1.11) and (1.12) that the elastic part of the Cauchy stress defined by the Kramers expression can be rewritten as

$$\mathbf{S}_e(x,t) := k \sum_{j=1}^{K} \int_D M(\boldsymbol{q}) \nabla_{\boldsymbol{q}^j} \widehat{\psi}(x, \boldsymbol{q}, t) \otimes \boldsymbol{q}^j \, \mathrm{d}\boldsymbol{q}, \tag{1.15}$$

and the Fokker–Planck equation (1.13) becomes

$$(M\widehat{\psi})_{,t} + \operatorname{div}\left(M\widehat{\psi}\boldsymbol{v}\right) + \operatorname{div}_{\boldsymbol{q}}\left(M\widehat{\psi}(\nabla\boldsymbol{v})\boldsymbol{q}\right) - \triangle(M\widehat{\psi}) - \operatorname{div}_{\boldsymbol{q}} \mathbb{A}(M\nabla_{\boldsymbol{q}}\widehat{\psi}) = 0 \tag{1.16}$$

in $\mathcal{O} \times (0,T)$, with $\mathcal{O} := \Omega \times D$. The Fokker–Planck equation (1.16) will be supplemented by the following boundary conditions:

$$M\nabla\widehat{\psi} \cdot \boldsymbol{n} = 0 \qquad \text{on } \partial\Omega \times D \times (0,T), \tag{1.17}$$

$$\left(M\widehat{\psi}(\nabla\boldsymbol{v})\boldsymbol{q}^j - \mathbb{A}^j(M\nabla_{\boldsymbol{q}}\widehat{\psi})\right) \cdot \boldsymbol{n}^j = 0 \qquad \text{on } \Omega \times \partial\bar{D}^j \times (0,T), \tag{1.18}$$

for all $j = 1, \dots, K$, and the initial condition

$$\widehat{\psi}(x, \boldsymbol{q}, 0) = \widehat{\psi}_0(x, \boldsymbol{q}) \qquad \text{in } \mathcal{O}. \tag{1.19}$$

In (1.15)–(1.18), we used the following notations and abbreviations that will be also used in what follows. Concerning the notation related to the boundary terms, $\partial\bar{D}^j$ signifies

$$\partial\bar{D}^j := D^1 \times \cdots \times D^{j-1} \times \partial D^j \times D^{j+1} \times \cdots \times D^K$$

and $\boldsymbol{n}^j = (n^j_1, \dots, n^j_d)^{\mathrm{T}}$ is the unit outward normal vector to ∂D^j, $j = 1, \dots, K$.

We continue with a brief literature survey that will be followed by an overview of the main contributions of the paper. Unless otherwise stated, in the survey, the centre-of-mass diffusion term is absent from the model considered in the cited reference (i.e., ε is set to 0), the viscous part, $\mathbf{S}_v$, of the Cauchy stress, $\mathbf{T}$, is assumed to be a linear function of the symmetric part of the velocity gradient, and $K = 1$, i.e., a simple dumbbell model is considered rather than a bead-spring chain model.

An early contribution to the existence and uniqueness of local-in-time solutions to a family of dumbbell type polymeric flow models is due to Renardy [39]. While the class of potentials considered by Renardy [39] (cf. hypotheses (F) and (F′) on pp. 314–315) does include the case of Hookean dumbbells, it excludes the practically relevant case of the FENE dumbbell model (see Example 1.2 above). E, Li & Zhang [22] and Li, Zhang & Zhang [29] revisited the question of local existence of solutions for dumbbell models. A further development in this direction is the work of Zhang & Zhang [42], where the local existence of regular solutions to FENE-type dumbbell models was shown. All of these papers require high regularity of the initial data. Constantin [16] considered the Navier–Stokes equations coupled to nonlinear Fokker–Planck equations describing the evolution of the probability distribution of the particles interacting with the fluid. Subsequently, in [17], Constantin & Seregin explored the question of global regularity of solutions to a coupled system involving the incompressible Navier–Stokes equations and a nonlinear Fokker–Planck equation. Otto & Tzavaras [34] investigated the Doi model (which is similar to a Hookean model (cf. Example 1.1 above), except that $D = S^2$) for suspensions of rod-like molecules in the dilute regime. Jourdain, Lelièvre & Le Bris [27] studied the existence of solutions to the FENE dumbbell model in the case of a simple Couette flow. By using tools from the theory of stochastic differential equations, they showed the existence of a unique local-in-time solution to the FENE dumbbell model for $d = 2$ when the velocity field $\boldsymbol{v}$ is unidirectional and of the particular form $\boldsymbol{v}(x_1, x_2) = (v_1(x_2), 0)^{\mathrm{T}}$.

In the case of Hookean dumbbells ($K = 1$), and assuming $\varepsilon = 0$ and constant solvent density ρ, the coupled microscopic-macroscopic model described above yields (with a linear relationship between the S_v and $\mathsf{D}(\boldsymbol{v})$ in the momentum equation), formally, taking the second moment of $\boldsymbol{q} \mapsto \psi(x, \boldsymbol{q}, t)$, the fully macroscopic, Oldroyd-B model of viscoelastic flow. Lions & Masmoudi [30] showed the existence of global-in-time weak solutions to the Oldroyd-B model in a simplified corotational setting (i.e., with $\nabla \boldsymbol{v}$ replaced by $\frac{1}{2}(\nabla \boldsymbol{v} - (\nabla \boldsymbol{v})^{\mathrm{T}})$) by exploiting the propagation in time of the compactness of the solution (i.e., the property that if one takes a sequence of weak solutions that converges weakly and such that the corresponding sequence of initial data converges strongly, then the weak limit is also a solution) and the DiPerna–Lions [19] theory of renormalized solutions to linear hyperbolic equations with nonsmooth transport coefficients. It is not known if an identical global existence result for the Oldroyd-B model also holds in the absence of the crucial assumption that the drag term is corotational. With $\varepsilon > 0$ and constant solvent density ρ, the coupled microscopic-macroscopic model above yields, taking the appropriate moments in the case of Hookean dumbbells, a dissipative version of the Oldroyd-B model. In this sense, the Hookean dumbbell model has a macroscopic closure: it is the Oldroyd-B model when $\varepsilon = 0$, and a dissipative version of Oldroyd-B when $\varepsilon > 0$ (cf. Barrett & Süli [5]). Barrett & Boyaval [3] have proved a global existence result for this dissipative Oldroyd-B model in two space dimensions. In contrast, the FENE model is not known to have an exact closure at the macroscopic level, though Du, Yu & Liu [20] and Yu, Du & Liu [40] have recently considered the analysis of approximate closures of the FENE dumbbell model. Lions & Masmoudi [31] proved the global existence of weak solutions for the *corotational* FENE dumbbell model, once again corresponding to the case of $\varepsilon = 0$, constant solvent density ρ, and $K = 1$, and the Doi model, also called the rod model; see also the work of Masmoudi [32]. Recently, Masmoudi [33] has extended this analysis to the noncorotational case.

Previously, El-Kareh & Leal [23] had proposed a steady macroscopic model, with the *conformation tensor*

$$\int_D \boldsymbol{q}\,\boldsymbol{q}^{\mathrm{T}} U'(\tfrac{1}{2}|\boldsymbol{q}|^2)\,\psi(x, \boldsymbol{q})\,\mathrm{d}\boldsymbol{q}$$

satisfying a transport equation with an added diffusion term, referred to as stress-diffusion, in order to account for Brownian motion across streamlines; the model can be thought of as an approximate macroscopic closure of a FENE-type micro-macro model with centre-of-mass diffusion.

Barrett, Schwab & Süli [4] proved the existence of global weak solutions to the coupled Navier–Stokes–Fokker–Planck system with $\varepsilon = 0$, $K = 1$, constant solvent-density ρ, an x-mollified velocity gradient in the Fokker–Planck equation and an x-mollified probability density function ψ in the Kramers expression, admitting a large class of potentials U (including the Hookean dumbbell model and general FENE-type dumbbell models); in addition to these mollifications, $\boldsymbol{v}$ in the x-convective term $(\boldsymbol{v} \cdot \nabla)\psi$ in the Fokker–Planck equation was also mollified. Unlike Lions & Masmoudi [30], the arguments in Barrett, Schwab & Süli [4] did *not* require that the drag term $\nabla_q \cdot (\nabla \boldsymbol{v}\, \boldsymbol{q}\, \psi)$ in the Fokker–Planck equation was corotational in the FENE case.

In [5], Barrett & Süli derived the Fokker–Planck equation with centre-of-mass diffusion (1.13), in the case of $K = 1$ and constant solvent-density ρ. They established the existence of global-in-time weak solutions to a mollification of the model for a general class of spring-force-potentials including in particular the FENE potential. They justified also, through a rigorous limiting process, certain classical reductions of this model appearing in the literature that exclude the centre-of-mass diffusion term from the Fokker–Planck equation on the grounds that the diffusion coefficient is small relative to other coefficients featuring in the equation. In the case of a corotational drag term they performed a rigorous passage to the limit as the mollifiers in the Kramers expression and the drag term converge to identity operators.

In [6], Barrett & Süli showed the existence of global-in-time weak solutions to the general class of noncorotational FENE type dumbbell models (including the standard FENE dumbbell model) with centre-of-mass diffusion, in the case of $K = 1$ and constant solvent-density ρ, with microsropic cut-off in the drag term. Subsequently, in [7] and [8], they removed the presence of the cut-off by passing to the limit $L \to \infty$ with the cut-off parameter L, with $K \geq 1$, and the solvent density, the viscosity and the drag coefficient kept constant. In a more recent paper, [9], they proved the existence of global-in-time weak solutions to a general class of coupled bead-spring chain models that arise from the kinetic theory of dilute solutions of *nonhomogeneous* polymeric liquids with noninteracting polymer chains with FENE type potentials. The class of models under consideration involves the unsteady incompressible Navier–Stokes equations with variable density and density-dependent dynamic viscosity in a bounded domain in $\mathbb{R}^d$, $d = 2$ or 3, for the density, the velocity and the pressure of the fluid, with an elastic extra-stress tensor appearing on the right-hand side of the momentum equation. Crucial features of the Fokker–Planck equation in the model are the presence of a centre-of-mass diffusion term and a nonlinear density-dependent drag coefficient. With initial density $\rho_0 \in [\rho_{\min}, \rho_{\max}]$ for the continuity equation, where $\rho_{\min} > 0$; a square-integrable and divergence-free initial velocity datum $\boldsymbol{v}_0$ for the Navier–Stokes equation; and a nonnegative initial probability density function ψ_0 for the Fokker–Planck equation, which has finite relative entropy with respect to the Maxwellian M associated with the spring potential in the model, they proved, by a limiting procedure on certain regularization parameters, the existence of a global-in-time weak solution $t \mapsto (\rho(t), \boldsymbol{v}(t), \psi(t))$ to the coupled Navier–Stokes–Fokker–Planck system, satisfying the initial condition $(\rho(0), \boldsymbol{v}(0), \psi(0)) = (\rho_0, \boldsymbol{v}_0, \psi_0)$, such that $t \mapsto \rho(t) \in [\rho_{\min}, \rho_{\max}]$, $t \mapsto \boldsymbol{v}(t)$ belongs to the classical Leray space and $t \mapsto \psi(t)$ has bounded relative entropy with respect to M and $t \mapsto \psi(t)/M$ has integrable Fisher information over any time interval $[0, T]$, $T > 0$. The paper also includes a careful derivation of the Fokker–Planck equation with centre-of-mass diffusion, in the case of general bead-spring chains, and admitting nonlinear dependence of the drag coefficient on the density of the fluid.

The key feature of our work here, which makes it different from previous studies of Navier–Stokes–Fokker–Planck systems in the literature, is that we admit a nonlinear (non-Newtonian) constitutive relation between the dissipative part $\mathbf{S}_v$ of the Cauchy stress and the symmetric part of the velocity gradient $\mathbf{D}(\boldsymbol{v})$. Thus, the fluid in which the polymer macromolecules are dissolved is not necessarily Newtonian but may exhibit shear-thinning or shear-thickening with the possibility of being suitably activated or deactivated. In order to establish the existence of a global weak solution (i.e., to prove that an appropriately defined (weak) solution exists for any size of the data including T, Ω, D_i, $\boldsymbol{v}_0$, ψ_0, etc., measured in norms in which the a priori estimates are available) we rely on recent advances in large-data existence theory for global weak solutions to power-law type (implicitly constituted) incompressible fluid flow models, [18] and [15]. In those papers, the existence of a weak solution to the problem (1.1)–(1.7) (with $\mathbf{S}_e \equiv 0$ in (1.6)) was proved for power-law indices $r > \frac{2d}{d+2}$ and for source terms of the form $\boldsymbol{f} = \operatorname{div} \mathbf{F}$, with $\mathbf{F} \in L^1(0, T; \Omega)^{d \times d}$, using two properties: the fact that the principal part of the differential operator in the balance of linear momentum equation generates a maximal monotone graph; and properties of Lipschitz approximations of Sobolev-space-valued Bochner spaces. In particular, we can apply the results presented in [11, 18] for a no-slip boundary condition, and in [14] for Navier's slip boundary condition, once we have identified a suitable approximation scheme, indexed by the parameter ℓ (say), and established the strong convergence of the corresponding sequence of approximations $\{\mathbf{S}_e^\ell\}_{\ell \geq 1}$ to $\mathbf{S}_e$ in $L^1(\Omega \times (0, T))^{d \times d}$ as $\ell \to \infty$. Proving the latter is one of the key contributions of the present paper, and in the proof of our main result, Theorem 1.1, we shall focus mainly on proving this fact. To this end, we use the Div-Curl lemma, which has a further advantage in comparison with earlier approaches: it allows us to weaken the demands on the rate of decay of the Maxwellian M near the boundary of the configuration space domain D, see (1.21). In particular, our results are applicable to FENE-type models, see Example 1.2, with any FENE exponent $b > 0$ (while earlier results, based on the use of Schauder's Fixed Point Theorem to show the existence of an approximate solution at every time level to a temporally semi-discrete approximation scheme, used compact embeddings of Maxwellian-weighted Sobolev spaces, which then demanded larger values of b; e.g. $b > 2$ in the case of a FENE bead-spring-chain model or $b > 3$ for Cohen's Padé Approximation to the Inverse Langevin function (CPAIL model)); see [4]–[9]. In contrast with [4]–[9], we use a Galerkin method here to construct a sequence of spatially semi-discrete approximations to the initial-boundary-value problem, and we do not require compact embedding theorems in Maxwellian-weighted Sobolev spaces.

For the physical background of the theory of implicitly constituted models we refer to the papers by Rajagopal [35, 36] and Rajagopal & Srinivasa [37], and for a survey of analytical contributions to the PDE theory of implicitly constituted fluid flow models the interested reader should consult the recent articles by Bulíček, Gwiazda, Málek & Świerczewska-Gwiazda [11, 14, 15]. Those papers also contain detailed surveys of earlier results concerning the analysis of mathematical models for incompressible power-law type fluids.

The paper is structured as follows. In the next subsection we shall introduce the necessary function spaces together with our assumptions on the data. In Subsection 1.2 we shall state the main result of the paper, concerning the existence of global weak solutions to the class of kinetic models under consideration. The rest of the paper is then devoted to the proof of the theorem.

We begin, in Section 2, by introducing an approximate problem based on truncating the probability density function in the drag term in the Fokker–Planck equation. In order to maintain energy balance in the resulting coupled (truncated) Navier–Stokes–Fokker–Planck system, the probability density function has to be truncated in the Kramers expression as well; we also truncate the initial condition for the Fokker–Planck equation accordingly. We shall ultimately let the truncation parameter ℓ pass to ∞. In Section 2 we state, in Theorem 2.1, the existence of global weak solutions to this truncated Navier–Stokes–Fokker–Planck system. The proof of Theorem 2.1 is based on performing a spatial Galerkin semidiscretization of the truncated Navier–Stokes–Fokker–Planck system with the velocity expanded in terms of $n \in \mathbb{N}$ divergence-free Galerkin basis functions, and the probability density function expanded in terms of $m \in \mathbb{N}$ (different) Galerkin basis functions. In Subsection 2.2 we derive n-independent a priori estimates, which then allow us to pass to the limit $n \to \infty$ in Subsection 2.3. Passage to the limit $m \to \infty$ requires the nonnegativity of the Galerkin approximations $\widehat{\psi}^m$ to the probability density function $\widehat{\psi}^\ell$ satisfying the truncated Fokker–Planck equation. Strictly speaking the Galerkin approximations to $\widehat{\psi}^\ell$ should have been denoted by $\widehat{\psi}^{\ell,m}$, but since the value of ℓ is fixed in this part of the proof, we omit the letter ℓ and use the single superscript ^{m} instead of the double superscript $^{\ell,m}$. The nonnegativity of the functions $\widehat{\psi}^m$ is proved in Subsection 2.4. In Subsection 2.5 we derive m-independent a priori bounds on the sequence $\{\widehat{\psi}^m\}_{m\geq 1}$, which then allow us to pass to the limit $m \to \infty$ in Subsection 2.6. The most technical part of the argument in Subsection 2.6 is the proof of strong convergence of the sequence $\{\widehat{\psi}^m\}_{m\geq 1}$ to $\widehat{\psi}^\ell$ in the Maxwellian-weighted L^1 space $L^1_M(\mathcal{O}\times(0,T))$ (recall that $\mathcal{O} := \Omega \times D$). It involves a chain of theorems, including the Div-Curl Lemma, to deduce from the weak convergence of the sequence $\{\widehat{\psi}^m\}_{m\geq 1}$ in $L^1(\mathcal{O}_n\times(0,T))$, where $\mathcal{O}_n$ is any open subset of $\mathcal{O}$, such that $\overline{\mathcal{O}_n} \subset \mathcal{O}$, the weak convergence of the sequence $\{(1+\widehat{\psi}^m)^{1+\alpha}\}_{m\geq 1}$, $\alpha \in (0, \frac{1}{2})$, in $L^1(\mathcal{O}_n \times (0,T))$, by noting that thanks to the energy estimates and various bounds that result from them *via* function-space interpolation on $\mathcal{O}_n \times (0,T)$ in standard (non-weighted) Lebesgue and Sobolev spaces, the sequences $\{\widehat{\psi}^m\}_{m\geq 1}$ and $\{(1+\widehat{\psi}^m)^{\alpha}\}_{m\geq 1}$ are weakly convergent in $L^p(\mathcal{O}_n \times (0,T))$ and $L^q(\mathcal{O}_n \times (0,T))$ for certain $p > 1$ and $q > 1$. As $s \in [0,\infty) \to (1+s)^{1+\alpha}$ is a continuous strictly convex function, one can employ Theorem 10.20 in [24] to deduce that (up to the extraction of a subsequence), $\{\widehat{\psi}^m\}_{m\geq 1}$ converges almost everywhere on $\mathcal{O}_n \times (0,T)$. By applying a diagonal procedure, we then extract a subsequence that converges to $\widehat{\psi}^\ell$ almost everywhere on $\mathcal{O} \times (0,T)$. Since our energy estimates imply that the sequence $\{G(\widehat{\psi}^m)\}_{m\geq 1}$ is bounded in $L^\infty(0,T;L^1_M(\mathcal{O}))$, with $G(s) := s\ln s + e^{-1}$, it follows by de la Vallée Poussin's Theorem that the sequence $\{\widehat{\psi}^m\}_{m\geq 1}$ is equi-integrable in $L^1_M(\mathcal{O}\times(0,T))$. Thus, by noting that $\mathrm{d}\mu := M\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t$ defines a finite measure on $\mathcal{O}\times(0,T)$, almost everywhere convergence of a (sub)sequence of $\{\widehat{\psi}^m\}_{m\geq 1}$ and its equi-integrability finally imply, by Vitali's Convergence Theorem (cf. Theorem 2.24 in [26]), strong convergence of the (sub)sequence to $\widehat{\psi}^\ell$ in $L^1_M(\mathcal{O}\times(0,T))$.

In Section 3 we embark on the proof of the main theorem, Theorem 1.1. In Subsection 3.1 we collect the necessary weak and strong convergence results that arise from the energy estimates. In Subsection 3.2 we pass to the limit $\ell \to \infty$ in the truncation parameter ℓ in the various approximating sequences, and in Subsection 3.3 we pass to the limit $\ell \to +\infty$ in the Kramers expression. The attainment of the initial conditions is verified in Subsection 3.4, while the identification of the weak limit of the viscous stress part $\mathbf{S}_v$ of the Cauchy stress tensor $\mathbf{T}$ is carried out in Subsection 3.5 by using the parabolic Lipschitz truncation method of Kinnunen & Lewis [28], nontrivially adjusted to incompressible power-law type fluid flow problems in Diening, Růžička & Wolf [18] in the case of a Dirichlet boundary condition and in [15] to implicitly constituted models with a Navier boundary condition, in conjunction with Chacon's Biting Lemma.

1.1. **Function spaces and assumptions on the data.** We shall use standard notation for Lebesgue, Sobolev and Bochner spaces. In order to distinguish between scalar-, vector- and tensor-valued functions in a Banach space X we use the abbreviation

$$X^m := \underbrace{X\times\cdots\times X}_{m-times}.$$

Since we shall need to work with Maxwellian-weighted spaces, we define, for any measurable set $O \subset \mathbb{R}^m$, any nonnegative $N \in \mathcal{C}(O)$ and any $r \in [1,\infty)$, the weighted spaces

$$\begin{aligned} L^r_N(O) &:= \{u \in L^r_{loc}(O) \,:\, \int_O N(z)\,|u(z)|^r\;\mathrm{d}z < \infty\},\\ W^{1,r}_N(O) &:= \{u \in W^{1,r}_{loc}(O) \,:\, \int_O N(z)\,(|\nabla_z u(z)|^r + |u(z)|^r)\;\mathrm{d}z < \infty\}. \end{aligned}$$

Further, for any pair of functions u, v, with $u \in L^r(O)$ and $v \in L^{r'}(O)$, with $1/r + 1/r' = 1$ and $r, r' \in [1,\infty]$, we set

$$(u,v)_O := \int_O u(z)\,v(z)\,\mathrm{d}z,$$

with analogous notation for vector- and tensor-valued functions. If $O = \Omega$, then we shall for simplicity omit the subscript Ω from the inner product $(u,v)_O$. We define the following function spaces:

$$
\begin{aligned}
W^{1,r}_{\boldsymbol{n}} &:= \overline{\{\boldsymbol{v} \in \mathcal{C}^\infty(\overline{\Omega})^d \,:\, \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\}}^{\,\|\cdot\|_{W^{1,r}(\Omega)}},\\
W^{1,r}_{\boldsymbol{n},\mathrm{div}} &:= \overline{\{\boldsymbol{v} \in \mathcal{C}^\infty(\overline{\Omega})^d \,:\, \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega,\ \mathrm{div}\,\boldsymbol{v} = 0 \text{ in } \Omega\}}^{\,\|\cdot\|_{W^{1,r}(\Omega)}},\\
W^{1,r}_{0,\mathrm{div}} &:= \overline{\{\boldsymbol{v} \in \mathcal{C}_0^\infty(\overline{\Omega})^d \,:\, \mathrm{div}\,\boldsymbol{v} = 0 \text{ in } \Omega\}}^{\,\|\cdot\|_{W^{1,r}(\Omega)}},\\
L^2_{0,\mathrm{div}} &:= \overline{W^{1,2}_{\boldsymbol{n},\mathrm{div}}}^{\,\|\cdot\|_2},\\
W^{-1,r'}_{0,\mathrm{div}} &:= (W^{1,r}_{0,\mathrm{div}})^*, \qquad W^{-1,r'}_{\boldsymbol{n},\mathrm{div}} := (W^{1,r}_{\boldsymbol{n},\mathrm{div}})^*.
\end{aligned}
$$

We note that $W^{1,r}_{\boldsymbol{n}}$, $W^{1,r}_{0,\mathrm{div}}$ and $W^{1,r}_{\boldsymbol{n},\mathrm{div}}$ are separable and reflexive Banach spaces for any $r \in (1,\infty)$. The equivalent characterizations of these spaces in terms of weak/distributional derivatives are known; we note in particular that since Ω has been assumed to be a Lipschitz domain, the trace operator on $\partial\Omega$ is meaningful.

We end this subsection by introducing our assumptions on the data. First, we identify the implicit relation (1.7) with a graph $\mathcal{A} \subset \mathbb{R}^{d\times d}_{sym} \times \mathbb{R}^{d\times d}_{sym}$, i.e.,

$$(\mathbf{S},\mathbf{D}) \in \mathcal{A} \iff \mathbf{G}(\mathbf{S},\mathbf{D}) = \mathbf{0}.$$

Inspired by [15] we assume that, for some $r \in (1,\infty)$, $\mathcal{A}$ is a maximal monotone r-graph, in the sense that $\mathcal{A}$ satisfies the following assumptions:

(A1) $\mathcal{A}$ *includes the origin;* i.e., $(\mathbf{0},\mathbf{0}) \in \mathcal{A}$;

(A2) $\mathcal{A}$ *is a monotone graph;* i.e.,

$$(\mathbf{S}_1 - \mathbf{S}_2)\cdot(\mathbf{D}_1 - \mathbf{D}_2) \geq 0 \ \text{ for all } (\mathbf{D}_1,\mathbf{S}_1),(\mathbf{D}_2,\mathbf{S}_2) \in \mathcal{A};$$

(A3) $\mathcal{A}$ *is a maximal monotone graph;* i.e., for any $(\mathbf{D},\mathbf{S}) \in \mathbb{R}^{d\times d}_{sym} \times \mathbb{R}^{d\times d}_{sym}$,

$$\text{if } (\bar{\mathbf{S}} - \mathbf{S})\cdot(\bar{\mathbf{D}} - \mathbf{D}) \geq 0 \ \text{ for all } (\bar{\mathbf{D}},\bar{\mathbf{S}}) \in \mathcal{A}, \text{ then } (\mathbf{D},\mathbf{S}) \in \mathcal{A};$$

(A4) $\mathcal{A}$ *is an r-graph;* i.e., there exist positive constants C_1, C_2 such that

$$\mathbf{S}\cdot\mathbf{D} \geq C_1(|\mathbf{D}|^r + |\mathbf{S}|^{r'}) - C_2 \ \text{ for all } (\mathbf{D},\mathbf{S}) \in \mathcal{A}. \tag{1.20}$$

We note here that the arguments in this paper can be extended to a more general setting (cf. [14, 15]), where instead of $|\cdot|^r$ and $|\cdot|^{r'}$ on the right-hand side of (1.20) one has, respectively, a Young function $\Psi(\cdot)$ and its dual $\Psi^*(\cdot)$, and Orlicz and Orlicz–Sobolev spaces throughout instead of Lebesgue spaces and classical Sobolev spaces; and where the maximal monotone graphs are t and x dependent, i.e., the constitutive equation under consideration is of the form $\mathbf{G}(t,x,\mathbf{S},\mathbf{D}) = \mathbf{0}$.

For the Maxwellian M, we assume in what follows that

$$M \in \mathcal{C}(\overline{D}) \cap \mathcal{C}^{0,1}_{loc}(D) \cap W^{1,1}_0(D), \quad M \geq 0, \quad M^{-1} \in \mathcal{C}_{loc}(D). \tag{1.21}$$

Finally, we state our assumptions on the initial conditions. For the initial velocity $\boldsymbol{v}_0$ we assume that

$$\boldsymbol{v}_0 \in L^2_{0,\mathrm{div}}. \tag{1.22}$$

For $\widehat{\psi}_0 := \psi_0/M$, where ψ_0 is the initial value of the probability density function ψ, we assume that

$$\widehat{\psi}_0 \geq 0 \ \text{ a.e. in } \mathcal{O}, \quad \widehat{\psi}_0 \ln \widehat{\psi}_0 \in L^1_M(\mathcal{O}), \tag{1.23}$$

and in addition we require that

$$\varrho_0 \in L^\infty(\Omega), \qquad \text{where} \qquad \varrho_0(x) := \int_D M(\boldsymbol{q})\widehat{\psi}_0(x,\boldsymbol{q})\,\mathrm{d}\boldsymbol{q}. \tag{1.24}$$

1.2. **The main result.** For simplicity we only formulate here the result for homogeneous Dirichlet boundary data for $\boldsymbol{v}$. However, the same result holds also for Navier's boundary condition and the statement of the theorem is valid if one replaces the function spaces $W^{1,r}_{0,\mathrm{div}}$ by $W^{1,r}_{\boldsymbol{n},\mathrm{div}}$ and also includes into the weak formulation (1.27) the boundary term $\frac{(1-\lambda)\gamma_*}{\lambda}(\boldsymbol{v},\boldsymbol{w})_{\partial\Omega}$, see [15] for details. We shall not explicitly discuss the appropriate choice of the function space containing the pressure p, as it is the same as in the papers by Diening et al. [18] and Bulíček et al. [15] in the case of a no-slip boundary condition and Navier's slip boundary condition, respectively. An interesting observation in the case of Navier's slip boundary condition is that the pressure can be shown to be an integrable function on $\Omega \times (0,T)$ provided that the boundary $\partial\Omega$ is $\mathcal{C}^{1,1}$; see, [15] or the earlier studies [12], [13] concerning different classes of incompressible fluid flow problems in which the presence of an integrable pressure is essential. It is not known however if an identical result also holds in the case of a no-slip boundary condition; in that case, the pressure is only known to be a distribution with respect to t.

Theorem 1.1. *Let $K \in \mathbb{N}$ be arbitrary, let $D^j \subset \mathbb{R}^d$, for $d \in \{2,3\}$ and $j = 1,\dots,K$, be bounded open balls centred at the origin in $\mathbb{R}^d$, let $\Omega \subset \mathbb{R}^d$ be a bounded open Lipschitz domain, let $r \in (1,\infty)$ and suppose that $\boldsymbol{f} \in L^{r'}(0,T;W^{-1,r'}_{0,\mathrm{div}})$. Assume that $\mathcal{A}$, given by G, is a maximal monotone r-graph satisfying in particular the assumptions* **(A1)–(A4)**, *the mapping $B \in \mathbb{R}^{d\times K} \mapsto \mathbb{A}(B) \in \mathbb{R}^{d\times K}$ is linear and satisfies the pair of inequalities* (1.14)*, the Maxwellian $M : D \to \mathbb{R}$ satisfies* (1.21)*, and the initial data $(\boldsymbol{v}_0, \widehat{\psi}_0)$ satisfy* (1.22)–(1.24)*. Then, there exist $(\boldsymbol{v}, \mathsf{S}_v, \mathsf{S}_e, \widehat{\psi})$ such that*

$$\begin{aligned}
\boldsymbol{v} &\in L^\infty(0,T;L^2_{0,\mathrm{div}}(\Omega)^d) \cap L^r(0,T;W^{1,r}_0(\Omega)^d) \cap W^{1,r^*}(0,T;W^{-1,r^*}_{0,\mathrm{div}}),\\
\mathsf{S}_v &\in L^{r'}(0,T;L^{r'}(\Omega)^{d\times d}),\\
\mathsf{S}_e &\in L^2(0,T;L^2(\Omega)^{d\times d}),\\
\widehat{\psi} &\in L^\infty(Q;L^1_M(D)) \cap L^2(0,T;W^{1,1}_M(\mathcal{O})), \qquad \widehat{\psi} \geq 0 \ \text{a.e. in } \mathcal{O}\times(0,T),\\
M\widehat{\psi} &\in W^{1,1}(0,T;W^{-1,1}(\mathcal{O})),\\
\widehat{\psi}\ln\widehat{\psi} &\in L^\infty(0,T;L^1_M(\mathcal{O})),
\end{aligned} \tag{1.25}$$

where

$$r^* := \min\left\{r', 2, \frac{(d+2)r}{d}\right\}. \tag{1.26}$$

Moreover, (1.1) *is satisfied in the following sense:*

$$\begin{aligned}\int_0^T \langle \boldsymbol{v}_{,t}, \boldsymbol{w}\rangle \,\mathrm{d}t + \int_0^T \big(-(\boldsymbol{v}\otimes\boldsymbol{v}, \nabla\boldsymbol{w}) + (\mathbf{S}_v, \nabla\boldsymbol{w})\big)\mathrm{d}t \\ = \int_0^T \big(-(\mathbf{S}_e, \nabla\boldsymbol{w}) + \langle \boldsymbol{f}, \boldsymbol{w}\rangle\big)\mathrm{d}t \qquad \text{for all } \boldsymbol{w}\in L^\infty(0,T;W^{1,\infty}_{0,\mathrm{div}}),\end{aligned} \tag{1.27}$$

where

$$(\mathbf{S}_v(x,t), \mathbf{D}(\boldsymbol{v}(x,t))) \in \mathcal{A} \qquad \text{for a.e. } (x,t)\in Q, \tag{1.28}$$

and $\mathbf{S}_e$ *is given by*

$$\begin{aligned}\mathbf{S}_e(x,t) = k\sum_{j=1}^K \int_D M\nabla_{\boldsymbol{q}^j}\widehat{\psi}(x,\boldsymbol{q},t)\otimes\boldsymbol{q}^j \,\mathrm{d}\boldsymbol{q} \\ \text{for a.e. } (x,t)\in Q.\end{aligned} \tag{1.29}$$

In addition the Fokker–Planck equation (1.16) *is satisfied in the following sense:*

$$\begin{aligned}\int_0^T \Big[\langle (M\widehat{\psi})_{,t}, \varphi\rangle - \Big(M\boldsymbol{v}\widehat{\psi}, \nabla\varphi\Big)_{\mathcal{O}} - \Big(M\widehat{\psi}(\nabla\boldsymbol{v})\boldsymbol{q}, \nabla_{\boldsymbol{q}}\varphi\Big)_{\mathcal{O}}\Big]\,\mathrm{d}t \\ + \int_0^T \Big[(\nabla(M\widehat{\psi}), \nabla\varphi)_{\mathcal{O}} + \Big(M\mathbb{A}(\nabla_{\boldsymbol{q}}\widehat{\psi}), \nabla_{\boldsymbol{q}}\varphi\Big)_{\mathcal{O}}\Big]\,\mathrm{d}t = 0 \\ \text{for all } \varphi\in L^\infty(0,T;W^{1,\infty}(\mathcal{O})),\end{aligned} \tag{1.30}$$

and the initial data are attained strongly in $L^2(\Omega)^d\times L^1_M(\mathcal{O})$, *i.e.,*

$$\lim_{t\to 0_+} \|\boldsymbol{v}(\cdot,t)-\boldsymbol{v}_0(\cdot)\|_2^2 + \|\widehat{\psi}(\cdot,t)-\widehat{\psi}_0(\cdot)\|_{L^1_M(\mathcal{O})} = 0. \tag{1.31}$$

Moreover, for all $t\in(0,T)$ *the following energy inequality holds in a weak sense:*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathcal{O}} kM\,\widehat{\psi}\ln\widehat{\psi}\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \tfrac12\|\boldsymbol{v}\|_2^2\right) + (\mathbf{S}_v, \mathbf{D}(\boldsymbol{v})) + 4k\left(M\nabla\sqrt{\widehat{\psi}}, \nabla\sqrt{\widehat{\psi}}\,\right)_{\mathcal{O}} \\ + 4k\left(M\mathbb{A}(\nabla_{\boldsymbol{q}}\sqrt{\widehat{\psi}}\,), \nabla_{\boldsymbol{q}}\sqrt{\widehat{\psi}}\,\right)_{\mathcal{O}} \le \langle \boldsymbol{f}, \boldsymbol{v}\rangle.\end{aligned} \tag{1.32}$$

The purpose of the remaining sections is to provide a proof of this theorem by constructing a sequence of approximating sequences for $\boldsymbol{v}$ and $\widehat{\psi}$ and passing to the respective limits in these.

2. Approximate problem

In this section we introduce an approximate problem for which the analysis of existence of solutions is relatively easy and can be performed by using the Galerkin method, a generalized version of monotone operator theory, and suitable a priori entropy estimates. In this section we mainly follow [15] for the theory on maximal monotone graphs, and the papers [7, 8] for the relevant entropy estimates for the function $\widehat{\psi}$.

In order to handle the momentum equation we truncate the convective term and $\mathbf{S}_e$. More precisely, we introduce a smooth nonnegative function $\Gamma \in \mathcal{D}(-2,2)$, such that $\Gamma(s) = 1$ for all $s \in [-1,1]$ and for an arbitrary $\ell \in \mathbb{N}$ we define $\Gamma_\ell(s) := \Gamma(\frac{s}{\ell})$. The primitive function to Γ_ℓ is denoted by

$$T_\ell(s) := \int_0^s \Gamma_\ell(r)\,\mathrm{d}r. \tag{2.1}$$

Next, the ℓ-th approximation of $\mathbf{S}_e$ is defined by

$$\mathbf{S}_e^\ell(x,t) := k\sum_{j=1}^K \int_D M(\boldsymbol{q})\nabla_{\boldsymbol{q}^j} T_\ell(\widehat{\psi}(x,\boldsymbol{q},t)) \otimes \boldsymbol{q}^j\,\mathrm{d}\boldsymbol{q}. \tag{2.2}$$

We note that by formal integration by parts, which can be made rigorous by using Lemma 3.1 in Section 3 of [7] on observing that the boundary term on ∂D vanishes since $M = 0$ on ∂D, we have that

$$\mathbf{S}_e^\ell(x,t) = -k\int_D \left[KM(\boldsymbol{q})T_\ell(\widehat{\psi}(x,\boldsymbol{q},t))\,\mathbf{I} + \sum_{j=1}^K T_\ell(\widehat{\psi}(x,\boldsymbol{q},t))\nabla_{\boldsymbol{q}^j} M(\boldsymbol{q}) \otimes \boldsymbol{q}^j \right] \mathrm{d}\boldsymbol{q}. \tag{2.3}$$

We then define the ℓ-approximation of (1.1) as follows:

$$\boldsymbol{v}^\ell_{,t} + \operatorname{div}(\Gamma_\ell(|\boldsymbol{v}^\ell|^2)\boldsymbol{v}^\ell \otimes \boldsymbol{v}^\ell) - \operatorname{div}\mathbf{S}^\ell_{\boldsymbol{v}} = -\nabla p^\ell + \operatorname{div}\mathbf{S}^\ell_e + \boldsymbol{f} \qquad \text{in } Q, \tag{2.4}$$

with initial and boundary data given by (1.3)–(1.5) with $\boldsymbol{v}$ replaced by $\boldsymbol{v}^\ell$ on the left-hand sides of the equalities (1.3)–(1.5), and with the constitutive relation for $\mathbf{S}_v^\ell$ given by (1.7), with $\mathbf{D}(\boldsymbol{v})$ replaced by $\mathbf{D}(\boldsymbol{v}^\ell)$. In order to preserve the energy identity under this truncation process, we shall also modify (1.16). First, we set

$$\Lambda_\ell(s) := s\Gamma_\ell(s), \tag{2.5}$$

and we then define the ℓ-approximation of (1.16) as

$$\begin{aligned} &(M\widehat{\psi}^\ell)_{,t} + \operatorname{div}\left(M\boldsymbol{v}^\ell\widehat{\psi}^\ell\right) + \operatorname{div}_{\boldsymbol{q}}\left(M\Lambda_\ell(\widehat{\psi}^\ell)(\nabla\boldsymbol{v}^\ell)\boldsymbol{q}\right) - \triangle(M\widehat{\psi}^\ell) \\ &\qquad - \operatorname{div}_{\boldsymbol{q}}\mathbb{A}(M\nabla_{\boldsymbol{q}}\widehat{\psi}^\ell) = 0 \qquad \text{in } \mathcal{O}\times(0,T), \end{aligned} \tag{2.6}$$

supplemented by the Neumann boundary conditions corresponding to (1.17)–(1.18). In order to avoid technical difficulties, we also truncate the initial condition for $\widehat{\psi}^\ell$ as follows:

$$\widehat{\psi}^\ell(x,\boldsymbol{q},0) = T_\ell(\widehat{\psi}_0(x,\boldsymbol{q})). \tag{2.7}$$

For such an approximation, we formulate the main theorem of this subsection corresponding to the case of homogeneous Dirichlet boundary condition for the velocity, i.e., the case $\lambda = 0$ in (1.5).

Theorem 2.1. *Let $K \in \mathbb{N}$ be arbitrary, let $D^i \subset \mathbb{R}^d$, for $d \in \{2,3\}$ and $i = 1,\dots,K$, be bounded open balls centred at the origin in $\mathbb{R}^d$, let $\Omega \subset \mathbb{R}^d$ be a bounded open Lipschitz domain, let $r \in (1,\infty)$ and suppose that $\boldsymbol{f} \in L^{r'}(0,T;W^{-1,r'}_{0,\mathrm{div}})$. Assume that $\mathcal{A}$, given by $\mathbf{G}$, is a maximal monotone r-graph, the mapping $\mathbb{A} : B \in \mathbb{R}^{d\times K} \mapsto \mathbb{A}(B) \in \mathbb{R}^{d\times K}$ is linear and* *satisfies* (1.14)*, the Maxwellian* $M : D \to \mathbb{R}$ *satisfies* (1.21)*, and the initial data* $(\boldsymbol{v}_0, \widehat{\psi}_0)$ *satisfy* (1.22)–(1.24)*. Then, for any* $\ell \in \mathbb{N}$*, there exist* $(\boldsymbol{v}^\ell, \mathbf{S}_v^\ell, \mathbf{S}_e^\ell, \widehat{\psi}^\ell)$ *such that*

$$
\begin{aligned}
&\boldsymbol{v}^\ell \in L^\infty(0,T;L^2_{0,\mathrm{div}}(\Omega)^d) \cap L^r(0,T;W^{1,r}_0(\Omega)^d) \cap W^{1,r'}(0,T;W^{-1,r'}_{0,\mathrm{div}}),\\
&\mathbf{S}_v^\ell \in L^{r'}(0,T;L^{r'}(\Omega)^{d\times d}),\\
&\mathbf{S}_e^\ell \in L^\infty(0,T;L^\infty(\Omega)^{d\times d}),\\
&\widehat{\psi}^\ell \in L^\infty(Q;L^1_M(D)) \cap L^2(0,T;W^{1,1}_M(\mathcal{O})), \qquad \widehat{\psi}^\ell \geq 0 \ \text{a.e. in } \mathcal{O}\times(0,T),\\
&M\widehat{\psi}^\ell \in W^{1,1}(0,T;W^{-1,1}(\mathcal{O})),
\end{aligned}
\tag{2.8}
$$

satisfying the following system of equations:

$$
\begin{aligned}
&\int_0^T \langle \boldsymbol{v}^\ell_{,t}, \boldsymbol{w}\rangle \,\mathrm{d}t + \int_0^T \left[-(\Gamma_\ell(|\boldsymbol{v}^\ell|^2)\boldsymbol{v}^\ell\otimes\boldsymbol{v}^\ell, \nabla\boldsymbol{w}) + (\mathbf{S}_v^\ell, \nabla\boldsymbol{w})\right]\mathrm{d}t\\
&\qquad = \int_0^T \left[-(\mathbf{S}_e^\ell, \nabla\boldsymbol{w}) + \langle \boldsymbol{f}, \boldsymbol{w}\rangle\right]\mathrm{d}t \qquad \text{for all } \boldsymbol{w} \in L^r(0,T;W^{1,r}_{0,\mathrm{div}}),
\end{aligned}
\tag{2.9}
$$

$$
(\mathbf{S}_v^\ell(x,t), \mathbf{D}(\boldsymbol{v}^\ell(x,t))) \in \mathcal{A} \qquad \text{for a.e. } (x,t)\in Q,
\tag{2.10}
$$

$$
\begin{aligned}
\mathbf{S}_e^\ell(x,t) = -k\left(\int_D KM(\boldsymbol{q})T_\ell(\widehat{\psi}(x,\boldsymbol{q},t))\,\mathbf{I} - \sum_{j=1}^K T_\ell(\widehat{\psi}(x,\boldsymbol{q},t))\nabla_{\boldsymbol{q}^j}M(\boldsymbol{q})\otimes\boldsymbol{q}^j\,\mathrm{d}\boldsymbol{q}\right),\\
\text{for a.e. } (x,t)\in Q,
\end{aligned}
\tag{2.11}
$$

$$
\begin{aligned}
&\int_0^T \langle (M\widehat{\psi^\ell})_{,t}, \varphi\rangle - \left(M\boldsymbol{v}^\ell\widehat{\psi}^\ell, \nabla\varphi\right)_{\mathcal{O}} - \left(M\Lambda_\ell(\widehat{\psi}^\ell)(\nabla\boldsymbol{v}^\ell)\boldsymbol{q}, \nabla_{\boldsymbol{q}}\varphi\right)_{\mathcal{O}}\,\mathrm{d}t\\
&\qquad + \int_0^T (M\nabla\widehat{\psi}^\ell, \nabla\varphi)_{\mathcal{O}} + \left(M\mathbb{A}(\nabla_{\boldsymbol{q}}\widehat{\psi}^\ell), \nabla_{\boldsymbol{q}}\varphi\right)_{\mathcal{O}}\,\mathrm{d}t = 0\\
&\qquad\qquad \text{for all } \varphi \in L^\infty(0,T;W^{1,\infty}(\mathcal{O})),
\end{aligned}
\tag{2.12}
$$

attaining the initial conditions in the following sense:

$$
\lim_{t\to 0_+} \|\boldsymbol{v}^\ell(\cdot,t) - \boldsymbol{v}_0(\cdot)\|_2^2 + \|\widehat{\psi}^\ell(\cdot,t) - T_\ell(\widehat{\psi}_0(\cdot))\|_{L^1_M(\mathcal{O})} = 0,
\tag{2.13}
$$

satisfying, for all $t \in (0,T)$*, the energy inequality*

$$
\begin{aligned}
&\int_{\mathcal{O}} MG(\widehat{\psi}^\ell(\cdot,t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \tfrac{1}{2}\int_\Omega |\boldsymbol{v}^\ell(\cdot,t)|^2\,\mathrm{d}x\\
&\quad + 4C_1\int_0^t\int_{\mathcal{O}} M\left|\nabla_{x,\boldsymbol{q}}\sqrt{\widehat{\psi}^\ell}\right|^2\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\tau + \int_0^t (\mathbf{S}_v^\ell, \mathbf{D}(\boldsymbol{v}^\ell))\,\mathrm{d}\tau\\
&\qquad \leq \int_{\mathcal{O}} MG(T_\ell(\widehat{\psi}_0^\ell))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \tfrac{1}{2}\int_\Omega |\boldsymbol{v}_0(\cdot)|^2\,\mathrm{d}x + \int_0^t \langle \boldsymbol{f}, \boldsymbol{v}^\ell\rangle\,\mathrm{d}t,
\end{aligned}
\tag{2.14}
$$

where G *is defined as*

$$
G(s) := s\ln s + e^{-1},
$$

and in particular the following uniform a priori estimate:

$$\sup_{t\in(0,T)}\left(\|\boldsymbol{v}^\ell(\cdot,t)\|_2^2+\|\widehat{\psi}^\ell(\cdot,t)\ln\widehat{\psi}^\ell(\cdot,t)\|_{L^1_M(\mathcal{O})}+\|\varrho^\ell(\cdot,t)\|_\infty\right)$$
$$+\int_0^T\left(\|\boldsymbol{v}^\ell\|_{1,r}^r+\|\mathbf{S}_v^\ell\|_{r'}^{r'}+\|\sqrt{\widehat{\psi}^\ell}\|^2_{W^{1,2}_M(\mathcal{O})}+\|\mathbf{S}_e^\ell\|_2^2+\|\nabla\varrho^\ell\|_2^2\right)\mathrm{d}t \tag{2.15}$$
$$\le C(k,r,\Omega,D,T,\boldsymbol{f},\mathbb{A},\boldsymbol{\mathcal{A}},M,\boldsymbol{v}_0,\widehat{\psi}_0),$$

where

$$\varrho^\ell(x,t):=\int_D M(\boldsymbol{q})\widehat{\psi}^\ell(x,\boldsymbol{q},t)\,\mathrm{d}\boldsymbol{q}. \tag{2.16}$$

The rest of this section is devoted to the proof of Theorem 2.1. In order to simplify the presentation we shall take without loss of generality the constant k in the Kramers expression ((1.11) or, equivalently, (1.15)) to be 1, and we shall write $\boldsymbol{v}$ and $\widehat{\psi}$ instead of $\boldsymbol{v}^\ell$ and $\widehat{\psi}^\ell$, and similarly for all other analogous quantities; the omitted superscript $^\ell$ will be reinstated later on in the paper when we consider the question of passing to the limit $\ell\to\infty$.

2.1. **Galerkin approximation.** In this subsection we introduce a Galerkin approximation of (2.9)–(2.12). First, we fix a sequence of functions $\{\bar{M}^m\}_{m\in\mathbb{N}}\subset\mathcal{C}_0^{0,1}(D)$ such that for each $m\in\mathbb{N}$ the function $\bar{M}^m$ satisfies (1.21), and for each compact set $\kappa\subset D$ the following holds:

$$\lim_{m\to\infty}\|\bar{M}^m-M\|_{\mathcal{C}(\overline{D})\cap W_0^{1,1}(D)}+\|(\bar{M}^m)^{-1}-M^{-1}\|_{\mathcal{C}(\kappa)}=0. \tag{2.17}$$

Then, we define an approximate Maxwellian M^m by

$$M^m:=\bar{M}^m+\frac{1}{m},\qquad\qquad \text{for } m=1,2,\dots. \tag{2.18}$$

The Hilbert space $W^{1,2}_{0,\mathrm{div}}\cap W^{d+1,2}(\Omega)^d$, equipped with the inner product of $W^{d+1,2}(\Omega)^d$ is compactly and densely imbedded in the Hilbert space $L^2_{0,\mathrm{div}}(\Omega)^d$. Hence, by the version of the Hilbert–Schmidt Theorem stated in Lemma A.4 (with $V=W^{1,2}_{0,\mathrm{div}}\cap W^{d+1,2}(\Omega)^d$ and $H=L^2_{0,\mathrm{div}}(\Omega)^d$ and $a(\cdot,\cdot)$ taken to be the inner product of $W^{d+1,2}(\Omega)^d$), there exists a countable set $\{\boldsymbol{w}_i\}_{i\in\mathbb{N}}$ (of eigenfunctions) in $W^{1,2}_{0,\mathrm{div}}\cap W^{d+1,2}(\Omega)^d$ whose linear span is dense in $L^2_{0,\mathrm{div}}(\Omega)^d$, such that the $\boldsymbol{w}_i$, $i=1,2,\dots$, are orthogonal in the inner product of $W^{d+1,2}(\Omega)^d$ and orthonormal in the inner product of $L^2(\Omega)^d$. Similarly, for each $m\in\mathbb{N}$ we find a countable set $\{\varphi_i^m\}_{i\in\mathbb{N}}$ (of eigenfunctions) in $W^{1,2}(\mathcal{O})$ that are orthogonal in $W^{1,2}_{M^m}(\mathcal{O})$ and orthonormal in $L^2_{M^m}(\mathcal{O})$. Finally, we fix $m,n\in\mathbb{N}$ and look for $(\boldsymbol{v}^{m,n},\widehat{\psi}^{m,n})$

given by

$$\boldsymbol{v}^{m,n}(x,t) := \sum_{i=1}^{m} c_i^{m,n}(t)\boldsymbol{w}_i(x), \tag{2.19}$$

$$\widehat{\psi}^{m,n}(x,\boldsymbol{q},t) := \sum_{i=1}^{n} d_i^{m,n}(t)\varphi_i^m(x,\boldsymbol{q}), \tag{2.20}$$

that solve

$$\begin{aligned}(\boldsymbol{v}^{m,n}_{,t},\boldsymbol{w}_i) - (\Gamma_\ell(|\boldsymbol{v}^{m,n}|^2)\boldsymbol{v}^{m,n}\otimes\boldsymbol{v}^{m,n},\nabla\boldsymbol{w}_i) + (\mathbf{S}_v^{m,n},\nabla\boldsymbol{w}_i)\\ = -(\mathbf{S}_e^{m,n},\nabla\boldsymbol{w}_i) + \langle\boldsymbol{f},\boldsymbol{w}_i\rangle \qquad \text{for all } i=1,\dots,m \text{ and a.e. } t\in(0,T),\end{aligned} \tag{2.21}$$

$$\begin{aligned}&(M^m\widehat{\psi}^{m,n}_{,t},\varphi_i^m)_{\mathcal{O}} - \Big(M^m\boldsymbol{v}^{m,n}\widehat{\psi}^{m,n},\nabla\varphi_i^m\Big)_{\mathcal{O}} - \Big(M\Lambda_\ell(\widehat{\psi}^{m,n})(\nabla\boldsymbol{v}^{m,n})\boldsymbol{q},\nabla_{\boldsymbol{q}}\varphi_i^m\Big)_{\mathcal{O}}\\ &\quad + (M^m\nabla\widehat{\psi}^{m,n},\nabla\varphi_i^m)_{\mathcal{O}} + \Big(M^m\mathbb{A}(\nabla_{\boldsymbol{q}}\widehat{\psi}^{m,n}),\nabla_{\boldsymbol{q}}\varphi_i^m\Big)_{\mathcal{O}} = 0\\ &\quad\quad \text{for all } i=1\dots,n \text{ and a.e. } t\in(0,T),\end{aligned} \tag{2.22}$$

with initial data given by

$$\boldsymbol{v}^{m,n}(x,0) = \boldsymbol{v}_0^m(x) := \sum_{i=1}^{m}(\boldsymbol{v}_0,\boldsymbol{w}_i)\,\boldsymbol{w}_i(x),$$

$$\widehat{\psi}^{m,n}(x,\boldsymbol{q},0) = \widehat{\psi}_0^{m,n}(x,\boldsymbol{q}) := \sum_{i=1}^{n}(T_\ell(\widehat{\psi}_0^m),\varphi_i)_{\mathcal{O}}\,\varphi_i^m(x,\boldsymbol{q}),$$

where

$$\widehat{\psi}_0^m := \widehat{\psi}_0\frac{M}{M^m}. \tag{2.23}$$

Furthermore, we require that the expressions $\mathbf{S}_e^{m,n}$ and $\mathbf{S}_v^{m,n}$ appearing in the equations (2.21) and (2.22) satisfy the following properties:

$$(\mathbf{S}_v^{m,n},\mathbf{D}(\boldsymbol{v}^{m,n}))\in\mathcal{A} \qquad \text{a.e. in } Q, \tag{2.24}$$

$$\begin{aligned}\mathbf{S}_e^{m,n} = -\int_D KMT_\ell(\widehat{\psi}^{m,n})\,\mathbf{I} + \sum_{j=1}^{K} T_\ell(\widehat{\psi}^{m,n})\nabla_{\boldsymbol{q}^j}M\otimes\boldsymbol{q}^j\,\mathrm{d}\boldsymbol{q},\\ \text{a.e. in } (x,t)\in Q.\end{aligned} \tag{2.25}$$

The local in time existence of $\boldsymbol{v}^{m,n}$ and $\widehat{\psi}^{m,n}$ for fixed m,n follows from Carathéodory's theory in the case when $\mathcal{A}$ can be rewritten as $\mathbf{S}=\mathbf{S}^*(\mathbf{D})$, where $\mathbf{S}^*$ is a continuous function. For the more general setting when $\mathcal{A}$ gives only an implicit relation between $\mathbf{S}$ and $\mathbf{D}$ we refer the interested reader to [15, 14]. Moreover, using the estimates established below we can extend the solution onto the whole time interval $(0,T)$.

2.2. **n-independent a priori estimates.** Our objective in this subsection is to derive estimates that do not depend on n. To do so, we first note that, thanks to our assumptions on M and M^m stated in (1.21) and (2.17) and because of the presence of the cut-off function T_ℓ in (2.25), we have that

$$|\mathbf{S}_e^{m,n}| \le C\ell. \tag{2.26}$$

Next, we multiply the i-th equation in (2.21) by $c_i^{m,n}(t)$ and sum with respect to $i = 1,\dots,m$ to deduce that (note that the convective term vanishes since $\operatorname{div} \boldsymbol{v}^{m,n} = 0$)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}^{m,n}\|_2^2 + (\mathbf{S}_v^{m,n}, \mathbf{D}(\boldsymbol{v}^{m,n})) = -(\mathbf{S}_e^{m,n}, \mathbf{D}(\boldsymbol{v}^{m,n})) + \langle \boldsymbol{f}, \boldsymbol{v}^{m,n}\rangle. \tag{2.27}$$

Hence, using **(A4)** and the Korn and Young inequalities, we find that

$$\sup_{t\in(0,T)} \|\boldsymbol{v}^{m,n}\|_2^2 + \int_0^T \left(\|\boldsymbol{v}^{m,n}\|_{1,r}^r + \|\mathbf{S}_v^{m,n}\|_{r'}^{r'}\right) \mathrm{d}t \le C(M,\ell) + C(\boldsymbol{v}_0, \boldsymbol{f}). \tag{2.28}$$

In particular, using the definition of $\boldsymbol{v}^{m,n}$ and the orthogonality of the basis we get that

$$\sup_{t\in(0,T);\, i=1,\dots,m} |c_i^{m,n}(t)| + \left|\frac{\mathrm{d}c_i^{m,n}(t)}{\mathrm{d}t}\right| \le C(m,\ell,\boldsymbol{v}_0,\boldsymbol{f},M). \tag{2.29}$$

Similarly, multiplying the i-th equation in (2.22) by $d_i^{m,n}(t)$ and summing with respect to $i = 1,\dots,n$, we find (using again the property $\operatorname{div} \boldsymbol{v}^{m,n} = 0$) that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\widehat{\psi}^{m,n}\|^2_{L^2_{M^m}(\mathcal{O})} + \|\nabla\widehat{\psi}^{m,n}\|^2_{L^2_{M^m}(\mathcal{O})^d} + (M^m\mathbb{A}(\nabla_{\boldsymbol{q}}\widehat{\psi}^{m,n}), \nabla_{\boldsymbol{q}}\widehat{\psi}^{m,n})_{\mathcal{O}} \\ &\quad = \left(M\Lambda_\ell(\widehat{\psi}^{m,n})(\nabla\boldsymbol{v}^{m,n})\boldsymbol{q}, \nabla_{\boldsymbol{q}}\widehat{\psi}^{m,n}\right)_{\mathcal{O}}. \end{aligned} \tag{2.30}$$

Next, noting the definitions of Λ_ℓ and M^m together with the Young and Hölder inequalities we get that

$$\begin{aligned} \left(M\Lambda_\ell(\widehat{\psi}^{m,n})(\nabla\boldsymbol{v}^{m,n})\boldsymbol{q}, \nabla_{\boldsymbol{q}}\widehat{\psi}^{m,n}\right)_Q &\le \tfrac{1}{2}C_1 \int_Q M^m |\nabla_{\boldsymbol{q}}\widehat{\psi}^{m,n}|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \\ &\quad + C(\ell)\|\nabla\boldsymbol{v}^{m,n}\|_\infty^2 \|\widehat{\psi}^{m,n}\|^2_{L^2_{M^m}(\mathcal{O})}. \end{aligned} \tag{2.31}$$

Further, using the smoothness of the basis (note that $W^{d+1,2} \hookrightarrow W^{1,\infty}$) and the estimate (2.29), we get that

$$\|\nabla\boldsymbol{v}^{m,n}\|_\infty \le C(m,\ell).$$

Therefore, inserting (2.31) into (2.30) and using (1.14), we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\widehat{\psi}^{m,n}\|^2_{L^2_{M^m}(\mathcal{O})} + \int_Q M^m |\nabla_{x,\boldsymbol{q}}\widehat{\psi}^{m,n}|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \le C(\ell,m)\|\widehat{\psi}^{m,n}\|^2_{L^2_{M^m}(\mathcal{O})}. \tag{2.32}$$

Thus, the application of Gronwall's lemma, the fact that $M^m \ge \frac{1}{m}$ and the definition of $\widehat{\psi}_0^{m,n}$ imply that

$$\sup_{t\in(0,T)} \|\widehat{\psi}^{m,n}\|^2_{L^2(\mathcal{O})} + \int_0^T \|\widehat{\psi}^{m,n}\|^2_{W^{1,2}(\mathcal{O})} \,\mathrm{d}t \le C(\ell,m). \tag{2.33}$$

In addition, since M^m is Lipschitz continuous, it is easy to deduce from (2.33) that

$$\int_0^T \|M^m \widehat{\psi}^{m,n}\|^2_{W^{1,2}(\mathcal{O})} \, \mathrm{d}t \le C(\ell, m). \tag{2.34}$$

Finally, it follows from (2.29), (2.33) and (2.22) that

$$\int_0^T \|(M^m \widehat{\psi}^{m,n})_{,t}\|^2_{W^{-1,2}(\mathcal{O})} \, \mathrm{d}t \le C(m, \ell). \tag{2.35}$$

2.3. **The limit $n \to \infty$.** Here, we let $n \to \infty$ in (2.21)–(2.25). Thus, from the n-independent estimates (2.29), (2.33), (2.34), (2.35) and by using the Aubin–Lions Lemma we see that there exist subsequences, which we do not relabel, such that

$$c_i^{m,n} \rightharpoonup^* c_i^n \qquad \text{weak}^* \text{ in } W^{1,\infty}(0,T), \tag{2.36}$$

$$c_i^{m,n} \to c_i^n \qquad \text{strongly in } \mathcal{C}[0,T], \tag{2.37}$$

$$\boldsymbol{v}^{m,n} \to \boldsymbol{v}^m \qquad \text{strongly in } \mathcal{C}(0,T; W^{1,r}_{0,\mathrm{div}}(\Omega)^d \cap W^{d+1,2}(\Omega)^d), \tag{2.38}$$

$$\widehat{\psi}^{m,n} \rightharpoonup \widehat{\psi}^m \qquad \text{weakly in } L^2(0,T; W^{1,2}(\mathcal{O})), \tag{2.39}$$

$$(M^m \widehat{\psi}^{m,n})_{,t} \rightharpoonup (M^m \widehat{\psi}^m)_{,t} \qquad \text{weakly in } L^2(0,T; W^{-1,2}(\mathcal{O})), \tag{2.40}$$

$$\widehat{\psi}^{m,n} \to \widehat{\psi}^m \qquad \text{strongly in } L^2(0,T; L^2(\mathcal{O})), \tag{2.41}$$

$$\mathbf{S}_e^{m,n} \rightharpoonup \mathbf{S}_e^m \qquad \text{weak}^* \text{ in } L^\infty(0;T; L^\infty(\Omega)^{d\times d}), \tag{2.42}$$

$$\mathbf{S}_v^{m,n} \rightharpoonup \mathbf{S}_v^m \qquad \text{weak}^* \text{ in } L^\infty(0;T; L^{r'}(\Omega)^{d\times d}). \tag{2.43}$$

In the light of these convergence results it is now standard to pass to the limit $n \to \infty$ in (2.21)–(2.22) to deduce that

$$\begin{aligned} &(\boldsymbol{v}^m_{,t}, \boldsymbol{w}_i) - (\Gamma_\ell(|\boldsymbol{v}^m|^2)\boldsymbol{v}^m \otimes \boldsymbol{v}^m, \nabla \boldsymbol{w}_i) + (\mathbf{S}_v^m, \nabla \boldsymbol{w}_i) \\ &\quad = -(\mathbf{S}_e^m, \nabla \boldsymbol{w}_i) + \langle \boldsymbol{f}, \boldsymbol{w}_i \rangle \qquad \text{for all } i = 1, \dots, m \text{ and a.e. } t \in (0,T), \end{aligned} \tag{2.44}$$

$$\begin{aligned} &\langle M^m \widehat{\psi}^m_{,t}, \varphi \rangle_{\mathcal{O}} - \Big(M^m \boldsymbol{v}^m \widehat{\psi}^m, \nabla \varphi \Big)_{\mathcal{O}} - \Big(M \Lambda_\ell(\widehat{\psi}^m)(\nabla \boldsymbol{v}^m)\boldsymbol{q}, \nabla_{\boldsymbol{q}} \varphi \Big)_{\mathcal{O}} \\ &\quad + (M^m \nabla \widehat{\psi}^m, \nabla \varphi)_{\mathcal{O}} + \Big(M^m \mathbb{A}(\nabla_{\boldsymbol{q}} \widehat{\psi}^m), \nabla_{\boldsymbol{q}} \varphi \Big)_{\mathcal{O}} = 0 \\ &\qquad \text{for all } \varphi \in W^{1,2}(\mathcal{O}) \text{ and a.e. } t \in (0,T). \end{aligned} \tag{2.45}$$

Moreover, it is obvious that $\boldsymbol{v}^m(x,0) = \boldsymbol{v}_0^m(x)$ and it is completely standard to show that

$$\lim_{t \to 0_+} \|\widehat{\psi}^m(\cdot,t) - T_\ell(\widehat{\psi}_0^m(\cdot))\|_{L^2(\mathcal{O})} = 0.$$

Finally, we also let $n \to \infty$ in (2.24) and (2.25). First, using (2.38) and (2.43) we see that

$$\lim_{n \to \infty} (\mathbf{S}_v^{m,n}, \mathbf{D}(\boldsymbol{v}^{m,n}))_Q = (\mathbf{S}_v^m, \mathbf{D}(\boldsymbol{v}^m))_Q.$$

Thus, having in addition (2.24), (2.38) and (2.43), we can use Lemma A.2 to deduce that

$$(\mathbf{S}_v^m, \mathbf{D}(\boldsymbol{v}^m)) \in \mathcal{A} \qquad \text{a.e. in } Q. \tag{2.46}$$

Then, using (2.41) and the Lebesgue Dominated Convergence Theorem, we can take the limit in (2.25) and deduce that

$$\mathsf{S}_e^m = -\int_D \left[KMT_\ell(\widehat{\psi}^m)\,\mathsf{I} + \sum_{j=1}^K T_\ell(\widehat{\psi}^m)\nabla_{\boldsymbol{q}^j} M \otimes \boldsymbol{q}^j \right] \mathrm{d}\boldsymbol{q} \qquad \text{a.e. in } Q. \tag{2.47}$$

2.4. **Minimum principle for $\widehat{\psi}^m$.** In this subsection, we show rigorously that $\widehat{\psi}^m \geq 0$ a.e. in $\mathcal{O} \times (0,T)$. To do so, we set $\varphi := (\widehat{\psi}^m)_- := \min(0, \widehat{\psi}^m)$ in (2.45) and derive by using the fact that $\operatorname{div} \boldsymbol{v}^m = 0$ and by a similar procedure as in (2.30)–(2.32), that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{O}} M^m((\widehat{\psi}^m)_-)^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \int_{\mathcal{O}} |\nabla_{x,\boldsymbol{q}}(\widehat{\psi}^m)_-|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \\ &\qquad \leq C(m,\ell) \int_{\mathcal{O}} M^m((\widehat{\psi}^m)_-)^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}. \end{aligned} \tag{2.48}$$

Since $\widehat{\psi}^m(0) = T_\ell(\widehat{\psi}_0^m) \geq 0$, Gronwall's inequality implies that $(\widehat{\psi}^m)_- \equiv 0$ in $\mathcal{O} \times (0,T)$. Thus we deduce that

$$\widehat{\psi}^m \geq 0 \qquad \text{a.e. in } \mathcal{O} \times (0,T). \tag{2.49}$$

2.5. **Uniform m-independent estimates.** This subsection is devoted to deriving a priori estimates that are independent of m. As a matter of fact, most of the estimates will also be independent of ℓ, and this will be clearly highlighted in the text. First, we set $\varphi \equiv 1$ in (2.45). Thus, using (2.49) and (2.23) we deduce that

$$\begin{aligned} \int_{\mathcal{O}} M^m(\boldsymbol{q})\,|\widehat{\psi}^m(x,\boldsymbol{q},t)| \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} &= \int_{\mathcal{O}} M^m(\boldsymbol{q})\,|T_\ell(\widehat{\psi}_0^m(x,\boldsymbol{q}))| \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \\ &\leq \int_{\mathcal{O}} M^m(\boldsymbol{q})\,|\widehat{\psi}_0^m(x,\boldsymbol{q})| \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} = \int_{\mathcal{O}} M(\boldsymbol{q})\,|\widehat{\psi}_0(x,\boldsymbol{q})| \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \leq C. \end{aligned} \tag{2.50}$$

Next, setting $\varphi(x,\boldsymbol{q},t) := \bar{\varphi}(x,t)$ in (2.45) and defining

$$\varrho^m(x,t) := \int_D M^m(\boldsymbol{q})\,\widehat{\psi}^m(x,\boldsymbol{q},t)\,\mathrm{d}\boldsymbol{q}, \tag{2.51}$$

we see that

$$\begin{aligned} &\langle \varrho^m_{,t}, \bar{\varphi} \rangle - (\boldsymbol{v}^m \varrho^m, \nabla\bar{\varphi}) + (\nabla \varrho^m, \nabla\bar{\varphi}) = 0 \\ &\qquad \text{for all } \bar{\varphi} \in W^{1,2}(\Omega) \text{ and a.e. } t \in (0,T), \end{aligned} \tag{2.52}$$

supplemented by the initial condition $\varrho^m(\cdot,0) = \varrho_0^m$, where

$$0 \leq \varrho_0^m(x) := \int_D M^m(\boldsymbol{q})\,T_\ell(\widehat{\psi}_0^m(x,\boldsymbol{q}))\,\mathrm{d}\boldsymbol{q} \leq \int_D M(\boldsymbol{q})\,\widehat{\psi}_0(x,\boldsymbol{q})\,\mathrm{d}\boldsymbol{q} = \varrho_0(x). \tag{2.53}$$

Consequently, since $\operatorname{div} \boldsymbol{v}^m = 0$ we can use the maximum principle and (1.24) to deduce that

$$
\begin{aligned}
\|\varrho^m\|_{L^\infty(Q)} &\le \|\varrho_0\|_\infty \le C,\\
\int_0^T \|\nabla \varrho^m\|_2^2 \,\mathrm{d}t &\le \tfrac{1}{2}\|\varrho_0\|_2^2 \le C.
\end{aligned} \tag{2.54}
$$

Finally, to obtain m-independent estimates for the velocity field $\boldsymbol{v}^m$ and for the x- and $\boldsymbol{q}$-gradients of $\widehat{\psi}^m$, we set $\varphi := \ln(\widehat{\psi}^m + \delta) + 1$ in (2.45), where $\delta > 0$ is arbitrary. Note that such a choice is legitimate. Hence, by defining

$$
\begin{aligned}
G_\delta(s) &:= (s+\delta)\ln(s+\delta) + \mathrm{e}^{-1}, \qquad\qquad G(s) := s\ln s + \mathrm{e}^{-1},\\
T_{\delta,\ell}(s) &:= \int_0^s \frac{\Lambda_\ell(t)}{t+\delta}\,\mathrm{d}t = \int_0^s \frac{t\Gamma_\ell(t)}{t+\delta}\,\mathrm{d}t,
\end{aligned}
$$

(note that $G_\delta \ge 0$, and $T_{\delta,\ell} \overset{\delta\to 0_+}{\to} T_\ell$ in $\mathcal{C}([0,\infty))$), we obtain from equation (2.45) with $\varphi := \ln(\widehat{\psi}^m + \delta) + 1$ the following identity

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{O}} M^m\, G_\delta(\widehat{\psi}^m)\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} - \Big(M^m\,\boldsymbol{v}^m, \nabla G_\delta(\widehat{\psi}^m)\Big)_{\mathcal{O}}\\
&\qquad + \left(\frac{M^m}{\widehat{\psi}^m + \delta}\,\nabla\widehat{\psi}^m, \nabla\widehat{\psi}^m\right)_{\mathcal{O}} + \left(\frac{M^m}{\widehat{\psi}^m + \delta}\,\mathbb{A}(\nabla_{\boldsymbol{q}}\widehat{\psi}^m), \nabla_{\boldsymbol{q}}\widehat{\psi}^m\right)_{\mathcal{O}}\\
&\qquad\qquad = \Big(M(\nabla\boldsymbol{v}^m)\boldsymbol{q}, \nabla_{\boldsymbol{q}} T_{\delta,\ell}(\widehat{\psi}^m)\Big)_{\mathcal{O}}.
\end{aligned} \tag{2.55}
$$

We begin by observing that the second term on the left-hand side vanishes thanks to the divergence-free constraint on $\boldsymbol{v}^m$. Next, integrating (2.55) with respect to time over $(0,t)$ and using the assumption (1.14), we get that

$$
\begin{aligned}
&\int_{\mathcal{O}} M^m\, G_\delta(\widehat{\psi}^m(\cdot,t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + C_1\int_0^t\int_{\mathcal{O}} \frac{M^m}{\widehat{\psi}^m+\delta}\,|\nabla_{x,\boldsymbol{q}}\widehat{\psi}^m|^2\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\tau\\
&\quad \le \int_{\mathcal{O}} M^m\, G_\delta(T_\ell(\widehat{\psi}_0^m)\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \int_0^t \Big(M(\nabla\boldsymbol{v}^m)\boldsymbol{q}, \nabla_{\boldsymbol{q}} T_{\delta,\ell}(\widehat{\psi}^m)\Big)_{\mathcal{O}}\,\mathrm{d}\tau.
\end{aligned} \tag{2.56}
$$

Now we let $\delta \to 0_+$ in (2.56). It is easy to identify the limit in the first term on the left-hand side and the first term on the right-hand side. Moreover, using the Monotone Convergence Theorem, we also easily identify the limit in the second term on the left-hand side. We thus obtain

$$
\begin{aligned}
&\int_{\mathcal{O}} M^m\, G(\widehat{\psi}^m(\cdot,t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + 4C_1\int_0^t\int_{\mathcal{O}} M^m\left|\nabla_{x,\boldsymbol{q}}\sqrt{\widehat{\psi}^m}\right|^2\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\tau\\
&\quad \le \int_{\mathcal{O}} M^m\, G(T_\ell(\widehat{\psi}_0^m))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \limsup_{\delta\to 0_+}\int_0^t \Big(M(\nabla\boldsymbol{v}^m)\boldsymbol{q}, \nabla_{\boldsymbol{q}} T_{\delta,\ell}(\widehat{\psi}^m)\Big)_{\mathcal{O}}\,\mathrm{d}\tau.
\end{aligned} \tag{2.57}
$$

Finally, we focus on the last term on the right-hand side. Using integration by parts (the boundary term vanishes using Lemma 3.1 in Section 3 of [7] again since $M = 0$ on ∂D), we find, on noting the property $\operatorname{div} \boldsymbol{v}^m = 0$, that

$$\begin{aligned}\int_0^t \Big(M\,(\nabla \boldsymbol{v}^m)\boldsymbol{q}, \nabla_{\boldsymbol{q}} T_{\delta,\ell}(\widehat{\psi}^m)\Big)_{\mathcal{O}}\,\mathrm{d}\tau &= -\int_0^t \Big(\operatorname{div}_{\boldsymbol{q}}(M\,(\nabla \boldsymbol{v}^m)\boldsymbol{q}), T_{\delta,\ell}(\widehat{\psi}^m)\Big)_{\mathcal{O}}\,\mathrm{d}\tau \\ &= -\sum_{j=1}^K \int_0^t \Big(\nabla \boldsymbol{v}^m, T_{\delta,\ell}(\widehat{\psi}^m)\nabla_{\boldsymbol{q}^j} M \otimes \boldsymbol{q}^j)\Big)_{\mathcal{O}}\,\mathrm{d}\tau.\end{aligned}$$

Thus, since $T_{\delta,\ell}$ converges to T_ℓ in $\mathcal{C}([0,\infty))$, we can easily pass to the limit in the last integral in (2.57) to obtain

$$\begin{aligned}&\int_{\mathcal{O}} M^m\, G(\widehat{\psi}^m(\cdot,t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + 4C_1 \int_0^t \int_{\mathcal{O}} M^m \left|\nabla_{x,\boldsymbol{q}}\sqrt{\widehat{\psi}^m}\right|^2 \mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\tau \\ &\le \int_{\mathcal{O}} M^m\, G(T_\ell(\widehat{\psi}_0^m))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} - \sum_{j=1}^K \int_0^t \Big(\nabla \boldsymbol{v}^m, T_\ell(\widehat{\psi}^m)\nabla_{\boldsymbol{q}^j} M \otimes \boldsymbol{q}^j)\Big)_{\mathcal{O}}\,\mathrm{d}\tau.\end{aligned} \tag{2.58}$$

Finally, we multiply the i-th equation in (2.44) by $c_i^m(t)$ to deduce the following energy identity (note that the convective term vanishes):

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}^m\|_2^2 + (\mathbf{S}_v^m, \mathbf{D}(\boldsymbol{v}^m)) = -(\mathbf{S}_e^m, \nabla \boldsymbol{v}^m) + \langle \boldsymbol{f}, \boldsymbol{v}^m\rangle. \tag{2.59}$$

Using $\operatorname{div} \boldsymbol{v}^m = 0$ and the definition of $\mathbf{S}_e^m$ (cf. (2.47)) we deduce that

$$(\mathbf{S}_e^m, \nabla \boldsymbol{v}^m) = -\sum_{j=1}^K \Big(\nabla \boldsymbol{v}^m, T_\ell(\widehat{\psi}^m)\nabla_{\boldsymbol{q}^j} M \otimes \boldsymbol{q}^j)\Big)_{\mathcal{O}}. \tag{2.60}$$

Hence, using this in (2.59), integrating over $(0,t)$ and adding the result to (2.58), we get

$$\begin{aligned}&\int_{\mathcal{O}} M^m\, G(\widehat{\psi}^m(\cdot,t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \tfrac{1}{2}\int_\Omega |\boldsymbol{v}^m(\cdot,t)|^2\,\mathrm{d}x \\ &+ 4C_1 \int_0^t \int_{\mathcal{O}} M^m \left|\nabla_{x,\boldsymbol{q}}\sqrt{\widehat{\psi}^m}\right|^2 \mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\tau + \int_0^t (\mathbf{S}_v^m, \mathbf{D}(\boldsymbol{v}^m))\,\mathrm{d}\tau \\ &\le \int_{\mathcal{O}} M^m\, G(T_\ell(\widehat{\psi}_0^m))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \tfrac{1}{2}\int_\Omega |\boldsymbol{v}_0^m(\cdot)|^2\,\mathrm{d}x + \int_0^t \langle \boldsymbol{f}, \boldsymbol{v}^m\rangle\,\mathrm{d}t.\end{aligned} \tag{2.61}$$

Thus, using the assumption **(A4)** and the Korn and Young inequalities, we arrive at the following estimate that is uniform with respect to m:

$$\begin{aligned}&\sup_{t\in(0,T)} \Big(\|\boldsymbol{v}^m(\cdot,t)\|_2^2 + \|M^m\, G(\widehat{\psi}^m(\cdot,t))\|_{L^1(\mathcal{O})}\Big) \\ &\quad + \int_0^T \|\boldsymbol{v}^m\|_{1,r}^r + \|\mathbf{S}_v^m\|_{r'}^{r'} + \|\sqrt{M^m}\,\nabla_{x,\boldsymbol{q}}\sqrt{\widehat{\psi}^m}\|^2_{L^2(\mathcal{O})^{d(K+1)}}\,\mathrm{d}t \\ &\le C\left(\|M^m\, G(T_\ell(\widehat{\psi}_0^m))\|_{L^1(\mathcal{O})} + \|\boldsymbol{v}_0^m\|_2^2 + \int_0^T \|\boldsymbol{f}\|^{r'}_{W^{-1,r'}_{0,\mathrm{div}}}\,\mathrm{d}t\right) \\ &\le C(1 + \|M^m\, G(T_\ell(\widehat{\psi}_0^m))\|_{L^1(\mathcal{O})}) \le C(\ell).\end{aligned} \tag{2.62}$$

Standard interpolation inequalities and (2.62) then yield the estimate

$$\int_0^T \|\boldsymbol{v}^m\|_{\frac{r(d+2)}{d}}^{\frac{r(d+2)}{d}} \,\mathrm{d}t \le C(\ell). \tag{2.63}$$

It is evident from the definition (2.47) of S_e^m and from the assumption (1.21) that

$$|\mathsf{S}_e^m| \le C\ell. \tag{2.64}$$

Consequently, thanks to the presence of Γ_ℓ in the convective term, it directly follows from (2.62), (2.64) and (2.44) that

$$\int_0^T \|\boldsymbol{v}^m_{,t}\|^{r'}_{W^{-1,r}_{0,\mathrm{div}}} \,\mathrm{d}t \le C(\ell). \tag{2.65}$$

2.6. **The limit $m \to \infty$.** In this final subsection of Section 2 we let $m \to \infty$ to establish the existence of a weak solution stated in Theorem 2.1. First, using (2.62)–(2.65) and the Aubin–Lions Lemma we deduce the existence of a subsequence that we do not relabel, and $(\boldsymbol{v}, \mathsf{S}_v, \mathsf{S}_e)$, such that

$$\boldsymbol{v}^m \rightharpoonup^* \boldsymbol{v} \qquad \text{weak}^* \text{ in } L^\infty(0,T;L^2_{0,\mathrm{div}}), \tag{2.66}$$

$$\boldsymbol{v}^m \rightharpoonup \boldsymbol{v} \qquad \text{weakly in } L^r(0,T;W^{1,r}_{0,\mathrm{div}}), \tag{2.67}$$

$$\boldsymbol{v}^m_{,t} \rightharpoonup \boldsymbol{v}_{,t} \qquad \text{weakly in } L^{r'}(0,T;W^{-1,r'}_{0,\mathrm{div}}), \tag{2.68}$$

$$\boldsymbol{v}^m \rightharpoonup \boldsymbol{v} \qquad \text{weakly in } L^{\frac{r(d+2)}{d}}(0,T;L^{\frac{r(d+2)}{d}}(\Omega)^d), \tag{2.69}$$

$$\boldsymbol{v}^m \to \boldsymbol{v} \qquad \text{strongly in } L^1(0,T;L^1(\Omega)^d), \tag{2.70}$$

$$\mathsf{S}_v^m \rightharpoonup \mathsf{S}_v \qquad \text{weakly in } L^{r'}(0,T;L^{r'}(\Omega)^{d\times d}), \tag{2.71}$$

$$\mathsf{S}_e^m \rightharpoonup^* \mathsf{S}_e \qquad \text{weak}^* \text{ in } L^\infty(0,T;L^\infty(\Omega)^{d\times d}). \tag{2.72}$$

With these convergence results it is then standard to let $m \to \infty$ in (2.44) to deduce (2.9). Also, one can show the attainment of the initial condition for the velocity $(2.13)_1$. In order to prove that S_v and $\mathsf{D}(\boldsymbol{v})$ fulfill (2.10), we assume for a moment that

$$\mathsf{S}_e^m \to \mathsf{S} \qquad \text{strongly in } L^{r'}(0,T;L^{r'}(\Omega)^{d\times d}). \tag{2.73}$$

Then, integrating (2.59) with respect to $t \in (0,T)$, letting $m \to \infty$, using the weak lower semicontinuity of norm and combining (2.67) and (2.73) to identify the limit of the term on the right-hand side of (2.59), we find that

$$\limsup_{m\to\infty} (\mathsf{S}_v^m, \mathsf{D}(\boldsymbol{v}^m))_Q \le \tfrac{1}{2}(\|\boldsymbol{v}_0\|_2^2 - \|\boldsymbol{v}(T,\cdot)\|_2^2) - (\mathsf{S}_e, \mathsf{D}(\boldsymbol{v}))_Q + \int_0^T \langle \boldsymbol{f}, \boldsymbol{v}\rangle \,\mathrm{d}t. \tag{2.74}$$

By setting $\boldsymbol{w} := \boldsymbol{v}$ in (2.9) and comparing the result with (2.74), we then obtain

$$\limsup_{m\to\infty} (\mathsf{S}_v^m, \mathsf{D}(\boldsymbol{v}^m))_Q \le (\mathsf{S}_v, \mathsf{D}(\boldsymbol{v}))_Q. \tag{2.75}$$

Hence, by applying Lemma A.2 to the sequence $(\mathsf{S}_v^m, \mathsf{D}(\boldsymbol{v}^m))$ we see that all assumptions are satisfied and consequently (2.10) holds.

Further, in order to identify all limits in (2.45) and also to show (2.73) we focus on the convergence properties of $\widehat{\psi}^m$. First, we define $\psi^m := M^m \widehat{\psi}^m$; using the definition of G we then deduce from (2.62) and the inequality $M^m \le C$ that

$$\sup_{t\in(0,T)} \int_{\mathcal{O}} \psi^m(x,\boldsymbol{q},t) \ln(1+\psi^m(x,\boldsymbol{q},t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \le C(\ell). \tag{2.76}$$

Since (2.76) implies the uniform equi-integrability of the sequence ψ^m, i.e.,

$$\forall\,\epsilon>0 \quad \exists\,\delta>0 \quad \forall\, m\in\mathbb{N} \quad \forall\, U\subset Q\times D: \qquad |U|\le\delta \implies \int_U \psi^m \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \le \epsilon, \tag{2.77}$$

it directly follows from the characterization of weakly compact sets in L^1 that there exists a $\psi\in L^1(\mathcal{O}\times(0,T))$ and a subsequence that we do not relabel such that

$$\psi^m \rightharpoonup \psi \qquad \text{weakly in } L^1(\mathcal{O}\times(0,T)). \tag{2.78}$$

Since M^m converges to M uniformly in $\mathcal{C}(\overline{D})$, we directly deduce that

$$\widehat{\psi}^m \rightharpoonup \widehat{\psi} \qquad \text{weakly in } L^1_{loc}(\mathcal{O}\times(0,T)). \tag{2.79}$$

Next, we show that there is a subsequence (again not relabelled) such that

$$\widehat{\psi}^m \to \widehat{\psi} \qquad \text{a.e. in } \mathcal{O}\times(0,T). \tag{2.80}$$

Hence, let $\mathcal{O}_0 \subset \overline{\mathcal{O}_0} \subset \mathcal{O}$ be an arbitrary Lipschitz domain. It then follows from (2.62) and from the properties of M and M^m that

$$\sup_{t\in(0,T)} \|\sqrt{\widehat{\psi}^m(\cdot,t)}\|^2_{L^2(\mathcal{O}_0)} + \int_0^T \|\sqrt{\widehat{\psi}^m}\|^2_{W^{1,2}(\mathcal{O}_0)}\,\mathrm{d}t \le C(\mathcal{O}_0). \tag{2.81}$$

Using standard interpolation inequalities we then deduce from (2.81) that

$$\int_0^T \int_{\mathcal{O}_0} |\widehat{\psi}^m|^{\frac{(K+1)d+2}{d(K+1)}}\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t = \int_0^T \int_{\mathcal{O}_0} |\sqrt{\widehat{\psi}^m}|^{\frac{2((K+1)d+2)}{d(K+1)}}\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t \le C(Q_0). \tag{2.82}$$

One can further interpolate using (2.81)–(2.82) and the Hölder inequality to obtain, for any $p\in[1,2)$, that

$$\begin{aligned} &\int_0^T \int_{\mathcal{O}_0} |\nabla_{x,\boldsymbol{q}} \widehat{\psi}^m|^p \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t = C\int_0^T \int_{\mathcal{O}_0} |\nabla_{x,\boldsymbol{q}} \sqrt{\widehat{\psi}^m}|^p |\sqrt{\widehat{\psi}^m}|^p \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t \\ &\quad\le C\left(\int_0^T \int_{\mathcal{O}_0} |\nabla_{x,\boldsymbol{q}}\sqrt{\widehat{\psi}^m}|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t\right)^{\frac{p}{2}} \left(\int_0^T\int_{\mathcal{O}_0} |\sqrt{\widehat{\psi}^m}|^{\frac{2p}{2-p}}\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t\right)^{\frac{2-p}{p}} \le C(\mathcal{O}_0) \end{aligned} \tag{2.83}$$

provided that

$$\frac{2p}{2-p} \le \frac{2((K+1)d+2)}{d(K+1)}.$$

Thus, by selecting the 'optimal' value $p := \frac{(K+1)d+2}{(K+1)d+1}$ that maximizes the power p on the left-hand side of the last inequality, we finally obtain

$$\int_0^T \int_{\mathcal{O}_0} |\nabla_{x,\boldsymbol{q}} \widehat{\psi}^m|^{\frac{(K+1)d+2}{(K+1)d+1}} \, \mathrm{d}x \, \mathrm{d}\boldsymbol{q} \, \mathrm{d}t \le C(\mathcal{O}_0). \tag{2.84}$$

The final improvement, the integrability of ψ^m, will follow from the estimates on ϱ^m. Indeed, it obviously follows from (2.51) and (2.54) that

$$\|\psi^m\|_{L^\infty(Q;L^1(D))} \le C. \tag{2.85}$$

Thus, interpolating between this and (2.82) and using the properties of M^m, we see that for any $q_1 \in (1,\infty)$ there exists a $q_2 > 1$ such that

$$\|\widehat{\psi}^m\|_{L^{q_1}(\Omega_0\times(0,T);L^{q_2}(D_0))} \le C(\mathcal{O}_0), \tag{2.86}$$

where we have used the notation $\mathcal{O}_0 := \Omega_0 \times D_0$. Consequently, using (2.63) and Hölder's inequality, there exists a $\delta > 0$ such that

$$\|\boldsymbol{v}^m \widehat{\psi}^m\|_{L^{1+\delta}(\mathcal{O}_0\times(0,T))^d} \le C(\mathcal{O}_0). \tag{2.87}$$

Next, we use all of the above auxiliary estimates over $\mathcal{O}_0$ to deduce pointwise convergence of $\widehat{\psi}^m$ by means of the Div-Curl Lemma (cf. Lemma A.1). To this end, for some $\alpha \in (0, \frac{1}{2})$ that will be specified later, we define two $(1+d+Kd)$-component vector fields (now vector means vector in all variables $x, \boldsymbol{q}, t$) as follows:

$$\begin{aligned}
\mathsf{H}^m &:= (M^m\widehat{\psi}^m, M^m\widehat{\psi}^m\boldsymbol{v}^m + M\nabla_x\widehat{\psi}^m, M\Lambda_\ell(\widehat{\psi}^m)\boldsymbol{v}^m\boldsymbol{q} + M^m\nabla_{\boldsymbol{q}}\boldsymbol{\psi}^m),\\
\mathsf{Q}^m &:= ((1+\widehat{\psi}^m)^\alpha, \underbrace{0,\dots,0}_{(d+Kd)-\text{times}}).
\end{aligned}$$

Consequently, using (2.63), (2.84) and (2.87), we deduce the existence of a subsequence (not relabelled) such that

$$\begin{aligned}
\mathsf{H}^m &\rightharpoonup \mathsf{H} && \text{weakly in } L^{1+\delta}(\mathcal{O}_0\times(0,T))^{1+d+Kd},\\
\mathsf{Q}^m &\rightharpoonup \mathsf{Q} && \text{weakly in } L^{\frac{1}{\alpha}}(\mathcal{O}_0\times(0,T))^{1+d+Kd},
\end{aligned}$$

where (we use uniform convergence of M^m and strong convergence of $\boldsymbol{v}^m$)

$$\begin{aligned}
\mathsf{H} &:= (M\widehat{\psi}, M\widehat{\psi}\boldsymbol{v} + M\nabla_x\widehat{\psi}, M\overline{\Lambda_\ell(\widehat{\psi})}\boldsymbol{v}\boldsymbol{q} + M\nabla_{\boldsymbol{q}}\boldsymbol{\psi}),\\
\mathsf{Q} &:= (\overline{(1+\widehat{\psi})^\alpha}, 0,\dots,0).
\end{aligned}$$

It remains to check the assumptions of the Div-Curl Lemma. First, it follows from (2.45) that

$$\mathrm{div}_{t,x,\boldsymbol{q}}\,\mathsf{H}^m = 0 \qquad \text{in } \mathcal{O}_0\times(0,T).$$

Moreover, we get by using (2.81) and the fact that $\alpha \in (0, \frac{1}{2})$ that

$$\begin{aligned}
&\int_{\mathcal{O}_0\times(0,T)} |\nabla_{t,x,\boldsymbol{q}}\mathsf{Q}^m - (\nabla_{t,x,\boldsymbol{q}}\mathsf{Q}^m)^T|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t \\
&\qquad \le C \int_{\mathcal{O}_0\times(0,T)} |\nabla_{x,\boldsymbol{q}}(1+\widehat{\psi}^m)^\alpha|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t \\
&\qquad \le C \int_{\mathcal{O}_0\times(0,T)} |\nabla_{x,\boldsymbol{q}}\sqrt{\widehat{\psi}^m}|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t \le C(\mathcal{O}_0).
\end{aligned} \tag{2.88}$$

Hence, the divergence of H^m is precompact in $W^{-1,2}(\mathcal{O}_0 \times (0,T))$ and the curl of Q^m is precompact in $W^{-1,2}(\mathcal{O}_0 \times (0,T))$. Consequently, by choosing $\alpha < \frac{\delta}{1+\delta}$, we deduce that all assumptions of the Div-Curl Lemma are satisfied, and therefore

$$\mathsf{H}^m \cdot \mathsf{Q}^m \rightharpoonup \mathsf{H}\cdot\mathsf{Q} \qquad \text{weakly in } L^1(\mathcal{O}_0 \times (0,T)). \tag{2.89}$$

In particular, we have that

$$M^m\widehat{\psi}^m(1+\widehat{\psi}^m)^\alpha \rightharpoonup M\widehat{\psi}\,\overline{(1+\widehat{\psi})^\alpha};$$

because of the uniform convergence of M^m to M this then implies that

$$(1+\widehat{\psi}^m)^{\alpha+1} \rightharpoonup (1+\widehat{\psi})\,\overline{(1+\widehat{\psi})^\alpha}.$$

Thanks to the convexity of the function $s \in [0,\infty) \mapsto s^{\alpha+1} \in [0,\infty)$ it follows that

$$(1+\widehat{\psi})^{1+\alpha} \le (1+\widehat{\psi})\,\overline{(1+\widehat{\psi})^\alpha}, \qquad \text{and therefore} \qquad (1+\widehat{\psi})^\alpha \le \overline{(1+\widehat{\psi})^\alpha}.$$

On the other hand, the function $s \in [0,\infty) \mapsto s^\alpha \in [0,\infty)$ is concave, and therefore we immediately have that

$$(1+\widehat{\psi})^\alpha = \overline{(1+\widehat{\psi})^\alpha}$$

and consequently, since the function $s \in [0,\infty) \mapsto s^\alpha \in [0,\infty)$ is strictly concave, thanks to Theorem 10.20 in [24] there exists a subsequence (not relabelled) such that

$$\widehat{\psi}^m \to \widehat{\psi} \qquad \text{a.e. in } \mathcal{O}_0 \times (0,T). \tag{2.90}$$

Hence (by the uniform convergence of M^m to M)

$$\psi^m \to \psi \qquad \text{a.e. in } \mathcal{O}_0 \times (0,T). \tag{2.91}$$

Finally, we select a nondecreasing sequence of nested sets $\{\mathcal{O}_0^k\}_{k\in\mathbb{N}}$ such that $\bigcup_{k=1}^\infty \mathcal{O}_0^k = \mathcal{O}$ and for each k we deduce pointwise convergence on $\mathcal{O}_0^k$. Thus, using a diagonal procedure, we finally find a subsequence (that is, once again, not relabelled) such that (2.80) holds.

Hence, by combining (2.77) and (2.80) and recalling Vitali's Convergence Theorem (cf. Theorem 2.24 in [26]), we obtain that

$$\psi^m \to \psi \qquad \text{strongly in } L^1(0,T;L^1(\mathcal{O})). \tag{2.92}$$

Therefore, using Lebesgue's Dominated Convergence Theorem (here we rely on the presence of the truncation T_ℓ) we can let $m \to \infty$ in (2.47) to deduce that

$$\mathsf{S}_e^m \to \mathsf{S}_e \qquad \text{strongly in } L^1(0,T;L^1(\Omega)^{d\times d}), \tag{2.93}$$

where

$$\mathsf{S}_e = -\int_D \left[KMT_\ell(\widehat{\psi})\,\mathsf{I} + \sum_{j=1}^K T_\ell(\widehat{\psi})\nabla_{\boldsymbol{q}^j} M \otimes \boldsymbol{q}^j \right] \mathrm{d}\boldsymbol{q} \qquad \text{a.e. in } Q. \tag{2.94}$$

Thus, by interpolating between (2.72) and (2.93), we find (2.73).

In the rest of this subsection, we focus on passing to the limit $m \to \infty$ in (2.45). First, by interpolating between (2.54) and (2.92) we get that

$$\psi^m \to \psi \qquad \text{strongly in } L^q(Q;L^1(D)) \qquad \text{for all } q \in [1,\infty). \tag{2.95}$$

Next, for any measurable $U \subset (Q \times D)$ we use Hölder's inequality to deduce that

$$\begin{aligned}
\int_U M^m\, |\nabla_{x,\boldsymbol{q}} \widehat{\psi}^m| \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t &= 2\int_U M^m\, |\nabla_{x,\boldsymbol{q}} \sqrt{\widehat{\psi}^m}| \sqrt{\widehat{\psi}^m}\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t \\
&\le 2\left(\int_U M^m\, |\nabla_{x,\boldsymbol{q}} \sqrt{\widehat{\psi}^m}|^2 \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t\right)^{\frac12} \left(\int_U \psi^m \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t\right)^{\frac12} \\
&\overset{(2.77),(2.62)}{\le} C\epsilon^{\frac12},
\end{aligned} \tag{2.96}$$

provided that $|U| \le \delta$. This then implies that we can extract a subsequence such that

$$M^m\, \nabla_{x,\boldsymbol{q}} \widehat{\psi}^m \rightharpoonup M\, \nabla_{x,\boldsymbol{q}} \widehat{\psi} \qquad \text{weakly in } L^1(Q \times \mathcal{O})^{d(K+1)}, \tag{2.97}$$

where for the identification of the weak limit we used the fact that M^m converges uniformly to M and $\nabla_{x,\boldsymbol{q}} \widehat{\psi}^m$ converges locally in L^1, which follows from (2.84). In addition, by noting (2.95) it also follows from (2.62) that

$$\sqrt{M^m}\, \nabla_{x,\boldsymbol{q}} \sqrt{\widehat{\psi}^m} \rightharpoonup \sqrt{M}\, \nabla_{x,\boldsymbol{q}} \sqrt{\widehat{\psi}} \qquad \text{weakly in } L^2(Q \times \mathcal{O})^{d(K+1)}. \tag{2.98}$$

By using the same procedure as in (2.96) we also see that

$$\begin{aligned}
&\int_Q \left(\int_D M^m\, |\nabla_{x,\boldsymbol{q}} \widehat{\psi}^m| \,\mathrm{d}\boldsymbol{q}\right)^2 \mathrm{d}x\,\mathrm{d}t \\
&\qquad \le C\int_Q \|\sqrt{M^m}\, \nabla_{x,\boldsymbol{q}} \sqrt{\widehat{\psi}^m}\|^2_{L^2(D)} \|\psi^m\|^2_{L^1(D)} \,\mathrm{d}x\,\mathrm{d}t \overset{(2.54),(2.62)}{\le} C
\end{aligned} \tag{2.99}$$

and we can then strengthen (2.97) as follows:

$$M^m\, \nabla_{x,\boldsymbol{q}} \widehat{\psi}^m \rightharpoonup M\, \nabla_{x,\boldsymbol{q}} \widehat{\psi} \qquad \text{weakly in } L^2(Q;L^1(D)^{d(K+1)}). \tag{2.100}$$

Finally, using (2.67), (2.69), (2.70) and (2.95) we deduce that, for all $q \in [1, \frac{r(d+2)}{d})$,

$$M^m \,\widehat{\psi}^m \boldsymbol{v}^m \to \psi \boldsymbol{v} \qquad \text{strongly in } L^q(Q; L^1(D)^{d(K+1)}), \tag{2.101}$$

$$\Lambda_\ell(\widehat{\psi}^m)\, \nabla \boldsymbol{v}^m \rightharpoonup \Lambda_\ell(\widehat{\psi})\, \nabla \boldsymbol{v} \qquad \text{weakly in } L^r(Q \times D)^{d\times d}. \tag{2.102}$$

Consequently, using (2.45), and the convergence results (2.100)–(2.102) it follows that

$$(M^m \,\widehat{\psi}^m)_{,t} \rightharpoonup \psi_{,t} \qquad \text{weakly in } L^1(Q; W^{-1,1}(D)^{d(K+1)}). \tag{2.103}$$

Thus, using the linearity of the mapping $B \in \mathbb{R}^{d\times K} \mapsto \mathbb{A}(B) \in \mathbb{R}^{d\times K}$ it is easy to let $m \to \infty$ in (2.45) to deduce (2.12). Moreover, one can also show $(2.13)_2$ by using standard arguments. Finally, to derive (2.15) and (2.14) we let $m \to \infty$ in (2.54), (2.61) and (2.62). To pass to the limit in all terms on the left-hand side, we use either weak lower semicontinuity of norms or Fatou's Lemma, and for the critical term on the right-hand side of (2.62) we have that

$$\|M^m\, G(T_\ell(\widehat{\psi}_0^m))\|_{L^1(\mathcal{O})} \overset{m\to\infty}{\to} \|M\, G(T_\ell(\widehat{\psi}_0))\|_{L^1(\mathcal{O})} \overset{(1.23)}{\leq} C.$$

Thus, the proof of Theorem 2.1 is complete.

3. Proof of the main theorem

This final section is devoted to the proof of Theorem 1.1. We use the sequence of approximate solutions $(\boldsymbol{v}^\ell, \mathbf{S}_v^\ell, \mathbf{S}_e^\ell, \widehat{\psi}^\ell)$ constructed in Theorem 2.1 and let $\ell \to \infty$.

3.1. **Weak/strong convergence results for $\boldsymbol{v}^\ell$.** First, we recall the uniform estimate (2.15) (from now, C signifies a generic positive constant that may depend on the data but is independent of ℓ)

$$\begin{aligned}&\sup_{t\in(0,T)} \Big(\|\boldsymbol{v}^\ell(\cdot,t)\|_2^2 + \|\widehat{\psi}^\ell(\cdot,t) \ln \widehat{\psi}^\ell(\cdot,t)\|_{L^1_M(\mathcal{O})} + \|\varrho^\ell(\cdot,t)\|_\infty \Big) \\ &\qquad + \int_0^T \|\boldsymbol{v}^\ell\|_{1,r}^r + \|\mathbf{S}_v^\ell\|_{r'}^{r'} + \|\sqrt{\widehat{\psi}^\ell}\|^2_{W^{1,2}_M(\mathcal{O})} + \|\mathbf{S}_e^\ell\|_2^2 + \|\nabla \varrho^\ell\|_2^2 \,\mathrm{d}t \leq C,\end{aligned} \tag{3.1}$$

where

$$\varrho^\ell(x,t) := \int_D M(\boldsymbol{q})\, \widehat{\psi}^\ell(x,\boldsymbol{q},t)\,\mathrm{d}\boldsymbol{q}. \tag{3.2}$$

Next, using function space interpolation one can deduce from (3.1) that

$$\int_0^T \|\boldsymbol{v}^\ell\|_{\frac{(d+2)r}{d}}^{\frac{(d+2)r}{d}} \,\mathrm{d}t \leq C. \tag{3.3}$$

Hence, using (3.1), (3.3), the definition of r^* and the identity (2.9), we have that

$$\int_0^T \|\boldsymbol{v}^\ell_{,t}\|^{r^*}_{W^{-1,r^*}_{0,\mathrm{div}}} \mathrm{d}t \leq C. \tag{3.4}$$

Consequently, using the uniform estimates above in conjunction with the Aubin–Lions Lemma we deduce the existence of subsequences, which we do not relabel, and an associated triple $(\boldsymbol{v}, \mathbf{S}_v, \mathbf{S}_e)$ such that

$$\boldsymbol{v}^\ell \rightharpoonup^* \boldsymbol{v} \qquad \text{weak}^* \text{ in } L^\infty(0,T;L^2_{0,\mathrm{div}}), \tag{3.5}$$

$$\boldsymbol{v}^\ell \rightharpoonup \boldsymbol{v} \qquad \text{weakly in } L^r(0,T;W^{1,r}_{0,\mathrm{div}}), \tag{3.6}$$

$$\boldsymbol{v}^\ell_{,t} \rightharpoonup \boldsymbol{v}_{,t} \qquad \text{weakly in } L^{r^*}(0,T;W^{-1,r^*}_{0,\mathrm{div}}), \tag{3.7}$$

$$\boldsymbol{v}^\ell \rightharpoonup \boldsymbol{v} \qquad \text{weakly in } L^{\frac{r(d+2)}{d}}(0,T;L^{\frac{r(d+2)}{d}}(\Omega)^d), \tag{3.8}$$

$$\boldsymbol{v}^\ell \to \boldsymbol{v} \qquad \text{strongly in } L^1(0,T;L^1(\Omega)^d), \tag{3.9}$$

$$\mathbf{S}^\ell_v \rightharpoonup \mathbf{S}_v \qquad \text{weakly in } L^{r'}(0,T;L^{r'}(\Omega)^{d\times d}), \tag{3.10}$$

$$\mathbf{S}^\ell_e \rightharpoonup \mathbf{S}_e \qquad \text{weakly in } L^2(0,T;L^2(\Omega)^{d\times d}). \tag{3.11}$$

Using the above convergence results and assuming that $r > \frac{2d}{d+2}$ in order to handle the convective term, it is now standard to let $\ell \to \infty$ in (2.9) to deduce (1.27). In the case of a Newtonian fluid this would suffice to show (1.30) and identify $\mathbf{S}_v$ and $\mathbf{S}_e$, i.e., to show (1.29). However, in the case of a nonlinear constitutive law described by a graph $\mathcal{A}$, we need to show strong convergence of $\mathbf{S}^\ell_e$ to establish the validity of (1.28).

3.2. **Weak/strong convergence results for $\widehat{\psi}^\ell$ and $\mathbf{S}^\ell_e$.** Here, we mimic the procedure described in Subsection 2.6 and derive strong convergence results for $\widehat{\psi}^\ell$ that are necessary for proving strong convergence of $\mathbf{S}^\ell_e$. Since almost all steps are identical to those in Subsection 2.6, we proceed here by omitting some of the details and refer to Subsection 2.6. First, from (3.1), namely from the estimate

$$\sup_{t\in(0,T)} \int_{\mathcal{O}} M(\boldsymbol{q})\, \widehat{\psi}^\ell(x,\boldsymbol{q},t) \ln(1+\widehat{\psi}^\ell(x,\boldsymbol{q},t))\, \mathrm{d}x\, \mathrm{d}\boldsymbol{q} \le C, \tag{3.12}$$

we deduce the uniform equi-integrability of the sequence $\widehat{\psi}^\ell$ (similarly to (2.77)), and the existence of a $\widehat{\psi} \in L^1_M(\mathcal{O}\times(0,T))$ and of a subsequence that we do not relabel such that

$$\widehat{\psi}^\ell \rightharpoonup \psi \qquad \text{weakly in } L^1_M(\mathcal{O}\times(0,T)). \tag{3.13}$$

The next step is to show that for a subsequence, which we do not relabel, we have that

$$\widehat{\psi}^\ell \to \widehat{\psi} \qquad \text{a.e. in } \mathcal{O}\times(0,T). \tag{3.14}$$

We proceed here in the same way as in Subsection 2.6 and use the fact that (3.1) is ℓ-independent. Hence, we let $\mathcal{O}_0 \subset \overline{\mathcal{O}_0} \subset \mathcal{O}$ be an arbitrary Lipschitz domain. Since $M > \delta$ in $\mathcal{O}_0$ for some δ, we have that

$$\sup_{t\in(0,T)} \|\sqrt{\widehat{\psi}^\ell(\cdot,t)}\|^2_{L^2(\mathcal{O}_0)} + \int_0^T \|\sqrt{\widehat{\psi}^\ell(\cdot,t)}\|^2_{W^{1,2}(\mathcal{O}_0)}\, \mathrm{d}t \le C(\mathcal{O}_0). \tag{3.15}$$

Using standard interpolation inequalities we then deduce from (2.81) that

$$\int_0^T \int_{\mathcal{O}_0} |\widehat{\psi}^\ell|^{\frac{(K+1)d+2}{d(K+1)}} \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t \le C(Q_0). \tag{3.16}$$

Hence, using the procedure described in Subsection 2.6 one can interpolate using this estimate to deduce that

$$\int_0^T \int_{\mathcal{O}_0} |\nabla_{x,\boldsymbol{q}}\widehat{\psi}^\ell|^{\frac{(K+1)d+2}{(K+1)d+1}} \,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}t \le C(\mathcal{O}_0) \tag{3.17}$$

and also, using the fact that

$$\|\psi^\ell\|_{L^\infty(Q;L^1(D))} \le C, \tag{3.18}$$

we see that for any $q_1 \in (1,\infty)$ there exists a $q_2 > 1$ such that

$$\|\widehat{\psi}^\ell\|_{L^{q_1}(\Omega_0\times(0,T);L^{q_2}(D_0))} \le C(\mathcal{O}_0), \tag{3.19}$$

where we have used the notation $\mathcal{O}_0 := \Omega_0 \times D_0$. Consequently, using (3.1), (3.19), the definition of Λ_ℓ, the fact that D_0 is bounded and Hölder's inequality, there exists a $\delta > 0$ such that

$$\|\boldsymbol{v}^\ell\widehat{\psi}^\ell\|_{L^{1+\delta}(\mathcal{O}_0\times(0,T))^d} + \|\Lambda_\ell(\widehat{\psi}^\ell)(\nabla\boldsymbol{v}^\ell)\boldsymbol{q}\|_{L^{1+\delta}(\mathcal{O}_0\times(0,T))^{Kd}} \le C(\mathcal{O}_0). \tag{3.20}$$

Next, similarly as before, we define

$$\begin{aligned}
\mathsf{H}^\ell &:= (M\widehat{\psi}^\ell, M\widehat{\psi}^\ell\boldsymbol{v}^\ell + M\nabla_x\widehat{\psi}^\ell, M\Lambda_\ell(\widehat{\psi}^\ell)\boldsymbol{v}^\ell\boldsymbol{q} + M\nabla_{\boldsymbol{q}}\widehat{\psi}^\ell),\\
\mathsf{Q}^\ell &:= ((1+\widehat{\psi}^\ell)^\alpha, \underbrace{0,\dots,0}_{(d+Kd)-\text{times}}),
\end{aligned}$$

with some $\alpha > 0$ such that $\alpha < \frac{\delta}{1+\delta}$. Consequently, using (3.1), (3.17) and (3.20), we deduce the existence of subsequences, which we do not relabel, such that

$$\begin{aligned}
\mathsf{H}^\ell &\rightharpoonup \mathsf{H} && \text{weakly in } L^{1+\delta}(\mathcal{O}_0\times(0,T))^{1+d+Kd},\\
\mathsf{Q}^\ell &\rightharpoonup \mathsf{Q} && \text{weakly in } L^{\frac{1}{\alpha}}(\mathcal{O}_0\times(0,T))^{1+d+Kd},
\end{aligned}$$

where (by noting the strong convergence of $\boldsymbol{v}^\ell$)

$$\begin{aligned}
\mathsf{H} &:= (M\widehat{\psi}, M\widehat{\psi}\boldsymbol{v} + M\nabla_x\widehat{\psi}, M\widehat{\psi}\boldsymbol{v}\boldsymbol{q} + M\nabla_{\boldsymbol{q}}\widehat{\psi}),\\
\mathsf{Q} &:= (\overline{(1+\widehat{\psi})^\alpha}, 0,\dots,0).
\end{aligned}$$

Thus, by applying the Div-Curl Lemma (cf. Lemma A.1 in the Appendix), all assumptions of which are valid thanks to our choice of α and the above estimates (see Subsection 2.6 for details), we deduce that

$$\mathsf{H}^\ell\cdot\mathsf{Q}^\ell \rightharpoonup \mathsf{H}\cdot\mathsf{Q} \qquad \text{weakly in } L^1(\mathcal{O}_0\times(0,T)). \tag{3.21}$$

In particular, we have that

$$\widehat{\psi}^\ell(1+\widehat{\psi}^\ell)^\alpha \rightharpoonup \widehat{\psi}\,\overline{(1+\widehat{\psi})^\alpha},$$

which then implies that

$$\widehat{\psi}^{\ell} \to \widehat{\psi} \qquad \text{a.e. in } \mathcal{O}_0 \times (0,T). \tag{3.22}$$

Next, we select a nondecreasing sequence of nested sets $\{\mathcal{O}_0^k\}_{k\in\mathbb{N}}$ such that $\bigcup_{k=1}^{\infty} \mathcal{O}_0^k = \mathcal{O}$ and for each k we deduce pointwise convergence on $\mathcal{O}_0^k$. Thus, using a diagonal procedure, we finally find a subsequence (that is, once again, not relabelled) such that (3.14) holds.

Hence, by combining (3.13) and (3.14), we obtain that

$$\widehat{\psi}^{m} \to \widehat{\psi} \qquad \text{strongly in } L^1(0,T;L^1_M(\mathcal{O})). \tag{3.23}$$

Moreover, using (3.1) and standard function space interpolation, we also deduce that

$$\widehat{\psi}^{\ell} \to \widehat{\psi} \qquad \text{strongly in } L^q(Q;L^1_M(D)) \qquad \text{for all } q \in [1,\infty). \tag{3.24}$$

Having shown the uniform estimate (3.1) and the strong convergence (3.24) we can now follow the argument in Subsection 2.6 to deduce the following convergence results:

$$\nabla_{x,\boldsymbol{q}}\widehat{\psi}^{\ell} \rightharpoonup \nabla_{x,\boldsymbol{q}}\widehat{\psi} \qquad \text{weakly in } L^1(0,T;L^1_M(\mathcal{O})^{d(K+1)}), \tag{3.25}$$

$$\nabla_{x,\boldsymbol{q}}\sqrt{\widehat{\psi}^{\ell}} \rightharpoonup \nabla_{x,\boldsymbol{q}}\sqrt{\widehat{\psi}} \qquad \text{weakly in } L^2(0,T;L^2(\mathcal{O})^{d(K+1)}), \tag{3.26}$$

$$\nabla_{x,\boldsymbol{q}}\widehat{\psi}^{\ell} \rightharpoonup \nabla_{x,\boldsymbol{q}}\widehat{\psi} \qquad \text{weakly in } L^2(Q;L^1_M(D)^{d(K+1)}). \tag{3.27}$$

Moreover, using the definition of Λ_ℓ, (3.1), (3.6) and (3.24) we deduce that

$$\Lambda_\ell(\widehat{\psi}^{\ell})\nabla\boldsymbol{v}^{\ell} \rightharpoonup \widehat{\psi}\,\nabla\boldsymbol{v} \qquad \text{weakly in } L^r(Q;L^1_M(D)^{d\times d}). \tag{3.28}$$

In addition, by combining (3.9) and (3.24) we see that, up to a subsequence, $\widehat{\psi}^{\ell}\boldsymbol{v}^{\ell}$ converges to $\widehat{\psi}\boldsymbol{v}$ almost everywhere in $Q \times D$. Thus, using the fact that $\boldsymbol{v}$ is independent of $\boldsymbol{q}$, the uniform equi-integrability of $\widehat{\psi}^{\ell}$, which follows from (3.24) and the a priori bound (3.1), it follows that the sequence $\widehat{\psi}^{\ell}\boldsymbol{v}^{\ell}$ is also uniformly equi-integrable, and then Vitali's Convergence Theorem (cf. Theorem 2.24 in [26]) directly implies that

$$\widehat{\psi}^{\ell}\boldsymbol{v}^{\ell} \to \widehat{\psi}\boldsymbol{v} \qquad \text{strongly in } L^1(Q;L^1_M(D)^{d(K+1)})\,.$$

Finally, using this convergence result, (3.8) and (3.24) we observe that

$$\widehat{\psi}^{\ell}\boldsymbol{v}^{\ell} \to \widehat{\psi}\boldsymbol{v} \qquad \text{strongly in } L^q(Q;L^1_M(D)^{d(K+1)}). \tag{3.29}$$

Consequently, using the identity (2.12), the convergence results (3.27)–(3.29) and the boundedness of D it follows that

$$(M\widehat{\psi}^{\ell})_{,t} \rightharpoonup M\widehat{\psi}_{,t} \qquad \text{weakly in } L^1(Q;W^{-1,1}(D)^{d(K+1)}). \tag{3.30}$$

Thus, using the linearity of the mapping $B \in \mathbb{R}^{d\times K} \mapsto \mathbb{A}(B) \in \mathbb{R}^{d\times K}$ it is easy to let $\ell \to \infty$ in (2.12) to deduce (1.30).

3.3. **Convergence properties of S_e^ℓ.** In this subsection, we focus on (2.11) and let $\ell \to \infty$ to deduce (1.29). To this, using partial integration and the fact that M has zero trace, we rewrite S_e^ℓ as follows (see also Lemma 3.1 in Section 3 of [7]):

$$
\begin{aligned}
\mathsf{S}_e^\ell(x,t) &:= -k\left(\int_D KM(\boldsymbol{q})T_\ell(\widehat{\psi}^\ell(x,\boldsymbol{q},t))\,\mathsf{I} - \sum_{j=1}^K T_\ell(\widehat{\psi}^\ell(x,\boldsymbol{q},t))\nabla_{\boldsymbol{q}^j} M(\boldsymbol{q}) \otimes \boldsymbol{q}^j \,\mathrm{d}\boldsymbol{q}\right) \\
&= k\sum_{j=1}^K \int_D M\,\nabla_{\boldsymbol{q}^j} T_\ell(\widehat{\psi}^\ell(x,\boldsymbol{q},t)) \otimes \boldsymbol{q}^j \,\mathrm{d}\boldsymbol{q}.
\end{aligned} \tag{3.31}
$$

Thus, we can now apply (3.24) and (3.25) and let $\ell \to \infty$ in (3.31) to deduce (1.29). In what follows, we prove the strong convergence of S_e^ℓ, which is needed for proving (1.28). Hence, by recalling the definition of S_e, we have that

$$
\mathsf{S}_e^\ell(x,t) - \mathsf{S}_e(x,t) := k\sum_{j=1}^K \int_D M\,\nabla_{\boldsymbol{q}^j}(T_\ell(\widehat{\psi}^\ell(x,\boldsymbol{q},t)) - \widehat{\psi}(x,\boldsymbol{q},t)) \otimes \boldsymbol{q}^j \,\mathrm{d}\boldsymbol{q}. \tag{3.32}
$$

Let $\Omega_0 \subset \overline{\Omega_0} \subset \Omega$ be an arbitrary Lipschitz domain. We then have that

$$
\begin{aligned}
&\int_0^T \int_\Omega |\mathsf{S}_e^\ell(x,t) - \mathsf{S}_e(x,t)|\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T \int_{\Omega\setminus\Omega_0} |\mathsf{S}_e^\ell(x,t) - \mathsf{S}_e(x,t)|\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\Omega_0} |\mathsf{S}_e^\ell(x,t) - \mathsf{S}_e(x,t)|\,\mathrm{d}x\,\mathrm{d}t \\
&\le C|\Omega\setminus\Omega_0|^{\frac{1}{2}} + \int_0^T \int_{\Omega_0} |\mathsf{S}_e^\ell(x,t) - \mathsf{S}_e(x,t)|\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{3.33}
$$

where in the transition to the right-hand side of the last inequality we have used Hölder's inequality and the a priori estimate (3.1). Similarly, let $D_0 \subset \overline{D_0} \subset D$ be an arbitrary Lipschitz domain. We can then decompose $\mathsf{S}_e^\ell - \mathsf{S}_e$ as follows:

$$
\begin{aligned}
\mathsf{S}_e^\ell(x,t) - \mathsf{S}_e(x,t) &:= k\sum_{j=1}^K \int_{D\setminus D_0} M\,\nabla_{\boldsymbol{q}^j}(T_\ell(\widehat{\psi}^\ell(x,\boldsymbol{q},t)) - \widehat{\psi}(x,\boldsymbol{q},t)) \otimes \boldsymbol{q}^j \,\mathrm{d}\boldsymbol{q} \\
&\quad + k\sum_{j=1}^K \int_{D_0} M\,\nabla_{\boldsymbol{q}^j}(T_\ell(\widehat{\psi}^\ell(x,\boldsymbol{q},t)) - \widehat{\psi}(x,\boldsymbol{q},t)) \otimes \boldsymbol{q}^j \,\mathrm{d}\boldsymbol{q}.
\end{aligned} \tag{3.34}
$$

Consequently, by inserting this decomposition into (3.33) and using the fact that D is bounded, we deduce that

$$\begin{aligned}
&\int_0^T \int_\Omega |\mathbf{S}_e^\ell(x,t) - \mathbf{S}_e(x,t)| \,\mathrm{d}x\,\mathrm{d}t \\
&\le C|\Omega\setminus\Omega_0|^{\frac12} + C\int_0^T\int_{\Omega_0}\int_{D\setminus D_0} M\left(|\nabla_{\boldsymbol{q}}\widehat{\psi}^\ell(x,\boldsymbol{q},t)| + |\nabla_{\boldsymbol{q}}\widehat{\psi}(x,\boldsymbol{q},t)|\right)\mathrm{d}\boldsymbol{q}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + k\int_0^T\int_{\Omega_0}\left|\sum_{j=1}^K\int_{D_0} M\,\nabla_{\boldsymbol{q}^j}(T_\ell(\widehat{\psi}^\ell(x,\boldsymbol{q},t)) - \widehat{\psi}(x,\boldsymbol{q},t))\otimes\boldsymbol{q}^j\,\mathrm{d}\boldsymbol{q}\right|\mathrm{d}x\,\mathrm{d}t.
\end{aligned}\tag{3.35}$$

Finally, in the last term we use an integration by parts (we emphasize that the boundary term does *not* vanish here) to get

$$\begin{aligned}
&\sum_{j=1}^K\int_{D_0} M\,\nabla_{\boldsymbol{q}^j}(T_\ell(\widehat{\psi}^\ell(x,\boldsymbol{q},t)) - \widehat{\psi}(x,\boldsymbol{q},t))\otimes\boldsymbol{q}^j\,\mathrm{d}\boldsymbol{q} \\
&\quad = -\int_{D_0} KM(\boldsymbol{q})(T_\ell(\widehat{\psi}^\ell(x,\boldsymbol{q},t)) - \widehat{\psi}(x,\boldsymbol{q},t))\,\mathbf{I}\,\mathrm{d}\boldsymbol{q} \\
&\qquad - \sum_{j=1}^K\int_{D_0}(T_\ell(\widehat{\psi}^\ell(x,\boldsymbol{q},t)) - \widehat{\psi}(x,\boldsymbol{q},t))\nabla_{\boldsymbol{q}^j}M(\boldsymbol{q})\otimes\boldsymbol{q}^j\,\mathrm{d}\boldsymbol{q} \\
&\qquad + \sum_{j=1}^K\int_{\partial\overline{D_0}^j} M\,(T_\ell(\widehat{\psi}^\ell(x,\boldsymbol{q},t)) - \widehat{\psi}(x,\boldsymbol{q},t))\boldsymbol{n}^j\otimes\boldsymbol{q}^j\,\mathrm{d}S(\boldsymbol{q}^j).
\end{aligned}\tag{3.36}$$

Thus, by inserting this into (3.35), denoting $\mathcal{O}_0 := \Omega_0\times D_0$ and using the fact that D is bounded we deduce that

$$\begin{aligned}
&\int_0^T \int_\Omega |\mathbf{S}_e^\ell(x,t) - \mathbf{S}_e(x,t)| \,\mathrm{d}x\,\mathrm{d}t \\
&\le C|\Omega\setminus\Omega_0|^{\frac12} + C\int_0^T\int_{\Omega_0}\int_{D\setminus D_0} M\left(|\nabla_{\boldsymbol{q}}\widehat{\psi}^\ell(x,\boldsymbol{q},t)| + |\nabla_{\boldsymbol{q}}\widehat{\psi}(x,\boldsymbol{q},t)|\right)\mathrm{d}\boldsymbol{q}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + C\int_0^T\int_{\mathcal{O}_0}|T_\ell(\widehat{\psi}^\ell(x,\boldsymbol{q},t)) - \widehat{\psi}(x,\boldsymbol{q},t)|\,(|\nabla_{\boldsymbol{q}}M(\boldsymbol{q})| + M(\boldsymbol{q}))\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + C\int_0^T\int_{\partial\mathcal{O}_0} M(\boldsymbol{q})\,|T_\ell(\widehat{\psi}^\ell(x,\boldsymbol{q},t)) - \widehat{\psi}(x,\boldsymbol{q},t)|\,\mathrm{d}S.
\end{aligned}\tag{3.37}$$

Next, we specify our choice of Ω_0 and D_0. First, for any $\epsilon > 0$ we choose a Lipschitz subdomain Ω_0 of Ω such that $C|\Omega\setminus\Omega_0| \le \frac12\epsilon$. Similarly, from the weak convergence result (3.25) we deduce uniform equi-integrability of the sequence $\{M\nabla_{\boldsymbol{q}}\widehat{\psi}^\ell\}_{\ell\in\mathbb{N}}$, which implies that for any $\epsilon > 0$ there is $\delta > 0$ such that if $|D \setminus D_0| \le \delta$ then

$$C \int_0^T \int_{\Omega_0} \int_{D \setminus D_0} M (|\nabla_{\boldsymbol{q}} \widehat{\psi}^\ell(x, \boldsymbol{q}, t)| + |\nabla_{\boldsymbol{q}} \widehat{\psi}(x, \boldsymbol{q}, t)|) \,\mathrm{d}\boldsymbol{q} \,\mathrm{d}x \,\mathrm{d}t \le \tfrac{1}{2}\epsilon.$$

Hence, for given $\epsilon > 0$ we find $\delta > 0$ and a Lipschitz subdomain D_0 of D such that $|D \setminus D_0| \le \delta$ and with this choice and by using the assumption (1.21) on the Maxwellian, which implies that $|\nabla_{\boldsymbol{q}} M| + M \le C(D_0)$ in D_0, the inequality (3.37) reduces to the following (from now on $\mathcal{O}_0$ is fixed):

$$\begin{aligned} &\int_0^T \int_\Omega |\mathbf{S}_e^\ell(x,t) - \mathbf{S}_e(x,t)| \,\mathrm{d}x \,\mathrm{d}t \\ &\le C(D_0) \int_0^T \int_{\mathcal{O}_0} |T_\ell(\widehat{\psi}^\ell(x,\boldsymbol{q},t)) - \widehat{\psi}(x,\boldsymbol{q},t)| \,\mathrm{d}\boldsymbol{q} \,\mathrm{d}x \,\mathrm{d}t \\ &\quad + C(D_0) \int_0^T \int_{\partial\mathcal{O}_0} |T_\ell(\widehat{\psi}^\ell(x,\boldsymbol{q},t)) - \widehat{\psi}(x,\boldsymbol{q},t)| \,\mathrm{d}S(x,\boldsymbol{q}) \,\mathrm{d}t + \epsilon. \end{aligned} \tag{3.38}$$

Thus, using (3.23), the uniform convergence of T_ℓ and the fact that M^{-1} is bounded in D_0 (which is a consequence of (1.21)), we find that

$$\lim_{\ell\to\infty} \int_0^T \int_{\mathcal{O}_0} |T_\ell(\widehat{\psi}^\ell(x,\boldsymbol{q},t)) - \widehat{\psi}(x,\boldsymbol{q},t)| \,\mathrm{d}\boldsymbol{q} \,\mathrm{d}x \,\mathrm{d}t = 0. \tag{3.39}$$

In order to pass to the limit in the boundary integral, we first recall that (3.17) implies that

$$\widehat{\psi}^\ell \rightharpoonup \widehat{\psi} \qquad \text{weakly in } L^{\frac{(K+1)d+2}{(K+1)d+1}}(0,T; W^{1,\frac{(K+1)d+2}{(K+1)d+1}}(\mathcal{O}_0)).$$

Hence, by using function space interpolation and the strong convergence (3.24), we deduce, for any $0 < \gamma \le 1$, that

$$\widehat{\psi}^\ell \to \widehat{\psi} \qquad \text{strongly in } L^{\frac{(K+1)d+2}{(K+1)d+1}}(0,T; W^{1-\gamma,\frac{(K+1)d+2}{(K+1)d+1}}(\mathcal{O}_0)).$$

Finally, the Trace Theorem (see, for example, [1]) immediately gives

$$\widehat{\psi}^\ell \to \widehat{\psi} \qquad \text{strongly in } L^1(0,T; L^1(\partial\mathcal{O}_0)),$$

and consequently

$$\lim_{\ell\to\infty} \int_0^T \int_{\partial\mathcal{O}_0} |T_\ell(\widehat{\psi}^\ell(x,\boldsymbol{q},t)) - \widehat{\psi}(x,\boldsymbol{q},t)| \,\mathrm{d}S(x,\boldsymbol{q}) \,\mathrm{d}t = 0. \tag{3.40}$$

Hence, inserting (3.39) and (3.40) into (3.38) we have that

$$\limsup_{\ell\to\infty} \int_0^T \int_\Omega |\mathbf{S}_e^\ell(x,t) - \mathbf{S}_e(x,t)| \,\mathrm{d}x \,\mathrm{d}t \le \epsilon. \tag{3.41}$$

Since ϵ was an arbitrary positive real number, it follows from (3.41) and (3.11) that

$$\mathbf{S}_e^\ell \to \mathbf{S}_e \qquad \text{strongly in } L^q(0,T; L^q(\Omega)^{d\times d}) \quad \text{for all } q \in [1,2), \tag{3.42}$$

which completes the proof of the strong convergence of the sequence $\mathbf{S}_e^\ell$ as $\ell \to \infty$.

3.4. **Attainment of the initial conditions.** Here we focus on the proof of (1.31). We note that the standard procedure for showing the attainment of the initial conditions, i.e., arguing separately for $\boldsymbol{v}_0$ and $\widehat{\psi}_0$, would work only in the case of $r \geq 2$. In order to also cover the case of $1 < r \leq 2$ we must proceed more carefully and the proof below is therefore based on the energy inequality for the complete system. First, we recall the standard procedure for proving attainment of the initial data in a weak sense. To this end, we first set $\boldsymbol{w}(x,t) := \chi_{[0,t]}\boldsymbol{u}(x)$ in (2.9), where $\boldsymbol{u} \in W^{1,\infty}_{0,\mathrm{div}}$ is arbitrary, to deduce that

$$
\begin{aligned}
(\boldsymbol{v}^\ell(t), \boldsymbol{u}) + \int_0^t \big[- (\Gamma_\ell(|\boldsymbol{v}^\ell|^2)\boldsymbol{v}^\ell \otimes \boldsymbol{v}^\ell, \nabla \boldsymbol{u}) + (\mathbf{S}_v^\ell, \nabla \boldsymbol{u})\big] \mathrm{d}\tau \\
= \int_0^t \big[- (\mathbf{S}_e^\ell, \nabla \boldsymbol{u}) + \langle \boldsymbol{f}, \boldsymbol{u}\rangle\big] \mathrm{d}\tau + (\boldsymbol{v}_0, \boldsymbol{u}).
\end{aligned}
\tag{3.43}
$$

Consequently, letting $\ell \to \infty$ in (3.43) and using the convergence properties established in the preceding subsections we deduce that, for almost all $t \in (0,T)$,

$$
(\boldsymbol{v}(t), \boldsymbol{u}) + \int_0^t [-(\boldsymbol{v} \otimes \boldsymbol{v}, \nabla \boldsymbol{u}) + (\mathbf{S}_v, \nabla \boldsymbol{u})] \,\mathrm{d}\tau = \int_0^t [-(\mathbf{S}_e, \nabla \boldsymbol{u}) + \langle \boldsymbol{f}, \boldsymbol{u}\rangle] \,\mathrm{d}\tau + (\boldsymbol{v}_0, \boldsymbol{u}).
\tag{3.44}
$$

Hence, we see that after a possible redefinition of $\boldsymbol{v}$ on a set of measure zero, the identity (3.44) holds for all $t \in (0,T)$. It then directly follows that

$$
\lim_{t \to 0_+} (\boldsymbol{v}(t), \boldsymbol{u}) = (\boldsymbol{v}_0, \boldsymbol{u}) \qquad \text{for all } \boldsymbol{u} \in W^{1,\infty}_{0,\mathrm{div}},
$$

and consequently, thanks to (1.25), we deduce that

$$
\boldsymbol{v}(t) \overset{t \to 0_+}{\rightharpoonup} \boldsymbol{v}_0 \qquad \text{weakly in } L^2(\Omega)^d.
\tag{3.45}
$$

Similarly, setting $\varphi(x, \boldsymbol{q}, t) := \chi_{[0,t]}\phi(x, \boldsymbol{q})$ in (2.12), where $\phi \in W^{1,\infty}(\mathcal{O})$ is arbitrary, we deduce that

$$
\begin{aligned}
(M\widehat{\psi^\ell}(t), \phi)_{\mathcal{O}} - \int_0^t \Big[\Big(M\boldsymbol{v}^\ell\widehat{\psi}^\ell, \nabla\phi\Big)_{\mathcal{O}} - \Big(M\Lambda_\ell(\widehat{\psi}^\ell)(\nabla\boldsymbol{v}^\ell)\boldsymbol{q}, \nabla_{\boldsymbol{q}}\phi\Big)_{\mathcal{O}}\Big] \,\mathrm{d}\tau \\
+ \int_0^t \Big[(M\nabla\widehat{\psi}^\ell, \nabla\phi)_{\mathcal{O}} + \Big(M\mathbb{A}(\nabla_{\boldsymbol{q}}\widehat{\psi}^\ell), \nabla_{\boldsymbol{q}}\phi\Big)_{\mathcal{O}}\Big] \,\mathrm{d}\tau = (MT_\ell(\widehat{\psi}_0), \phi)_{\mathcal{O}}.
\end{aligned}
\tag{3.46}
$$

Hence, letting $\ell \to \infty$ and using the above convergence results we deduce that, for all $t \in (0,T)$ (after a possible redefinition of $\widehat{\psi}$ on a set of zero measure), we have that

$$
\begin{aligned}
(M\widehat{\psi}(t), \phi)_{\mathcal{O}} - \int_0^t \Big[\Big(M\boldsymbol{v}\widehat{\psi}, \nabla\phi\Big)_{\mathcal{O}} - \Big(M\widehat{\psi}(\nabla\boldsymbol{v})\boldsymbol{q}, \nabla_{\boldsymbol{q}}\phi\Big)_{\mathcal{O}}\Big] \,\mathrm{d}\tau \\
+ \int_0^t \Big[(M\nabla\widehat{\psi}, \nabla\phi)_{\mathcal{O}} + \Big(M\mathbb{A}(\nabla_{\boldsymbol{q}}\widehat{\psi}), \nabla_{\boldsymbol{q}}\phi\Big)_{\mathcal{O}}\Big] \,\mathrm{d}\tau = (M\widehat{\psi}_0, \phi)_{\mathcal{O}}.
\end{aligned}
\tag{3.47}
$$

As a direct consequence of (3.47) we then have that

$$\lim_{t\to 0_+}(M\widehat{\psi}(t),\phi)_{\mathcal{O}} = (M\widehat{\psi}_0,\phi)_{\mathcal{O}} \qquad \text{for all } \phi\in W^{1,\infty}(\mathcal{O}). \tag{3.48}$$

Next, we invoke $(1.25)_6$, which implies that the family $\{M\widehat{\psi}(t)\}$ is uniformly equi-integrable and, therefore, that it has a weakly convergent subsequence in $L^1(\mathcal{O})$. By the uniqueness of the weak limit we then deduce from (3.48) that

$$\widehat{\psi}(t) \overset{t\to 0_+}{\rightharpoonup} \widehat{\psi}_0 \qquad \text{weakly in } L^1_M(\mathcal{O}). \tag{3.49}$$

Our next objective is to show that the weak convergence results (3.45) and (3.49) can be strengthened to (1.31). To do so, we first let $\ell\to\infty$ in (2.14); neglecting the integrals on the left-hand side that are nonnegative and using the above weak convergence results we deduce, for all $t\in(0,T)$, that

$$\begin{aligned}&\int_{\mathcal{O}} M\,G(\widehat{\psi}(\cdot,t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \tfrac{1}{2}\int_{\Omega}|\boldsymbol{v}(\cdot,t)|^2\,\mathrm{d}x\\ &\leq \int_{\mathcal{O}} M\,G(\widehat{\psi}_0)\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \tfrac{1}{2}\int_{\Omega}|\boldsymbol{v}_0(\cdot)|^2\,\mathrm{d}x + \int_0^t\langle \boldsymbol{f},\boldsymbol{v}\rangle\,\mathrm{d}t.\end{aligned} \tag{3.50}$$

Recall here that $G(s) := s\ln s + e^{-1}$ is a nonnegative strictly convex continuous function on $(0,\infty)$. Hence, from (3.50) we have that

$$\limsup_{t\to 0_+}\left[\int_{\mathcal{O}} M\,G(\widehat{\psi}(\cdot,t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \tfrac{1}{2}\|\boldsymbol{v}(t)\|_2^2\right] \leq \int_{\mathcal{O}} M\,G(\widehat{\psi}_0)\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \tfrac{1}{2}\|\boldsymbol{v}_0\|_2^2\,. \tag{3.51}$$

On the other hand, using the weak convergence results (3.45) and (3.49) and the convexity of G, we see that also

$$\liminf_{t\to 0_+}\left[\int_{\mathcal{O}} M\,G(\widehat{\psi}(\cdot,t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \tfrac{1}{2}\|\boldsymbol{v}(t)\|_2^2\right] \geq \int_{\mathcal{O}} M\,G(\widehat{\psi}_0)\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \tfrac{1}{2}\|\boldsymbol{v}_0\|_2^2 \tag{3.52}$$

and consequently

$$\lim_{t\to 0_+}\left[\int_{\mathcal{O}} M\,G(\widehat{\psi}(\cdot,t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \tfrac{1}{2}\|\boldsymbol{v}(t)\|_2^2\right] = \int_{\mathcal{O}} M\,G(\widehat{\psi}_0)\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} + \tfrac{1}{2}\|\boldsymbol{v}_0\|_2^2. \tag{3.53}$$

Next, we split the information coming from (3.53). Assume (for contradiction) that

$$\limsup_{t\to 0_+}\ \|\boldsymbol{v}(t)\|_2^2 > \|\boldsymbol{v}_0\|_2^2. \tag{3.54}$$

It then follows from (3.53) that

$$\liminf_{t\to 0_+}\int_{\mathcal{O}} M\,G(\widehat{\psi}(\cdot,t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} < \int_{\mathcal{O}} M\,G(\widehat{\psi}_0)\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}. \tag{3.55}$$

However, using the convexity of G, we also have that

$$\liminf_{t\to 0_+}\int_{\mathcal{O}} M\,G(\widehat{\psi}(\cdot,t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q} \geq \int_{\mathcal{O}} M\,G(\widehat{\psi}_0)\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}, \tag{3.56}$$

which is a contradiction. Thus we obtain that

$$\lim_{t\to 0_+}\|\boldsymbol{v}(t)\|_2^2=\|\boldsymbol{v}_0\|_2^2, \qquad \lim_{t\to 0_+}\int_{\mathcal{O}} M\,G(\widehat{\psi}(\cdot,t))\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}=\int_{\mathcal{O}} M\,G(\widehat{\psi}_0)\,\mathrm{d}x\,\mathrm{d}\boldsymbol{q}. \tag{3.57}$$

Consequently, combining (3.45) and (3.57) we deduce the first part of (1.31). Similarly, since G is strictly convex, it follows from (3.49) and (3.57) that

$$M\widehat{\psi}(\cdot,t)\overset{t\to 0_+}{\to} M\widehat{\psi}_0(\cdot) \qquad \text{almost everywhere in } \mathcal{O}.$$

Thus, on combining this with (3.49) we immediately arrive at the second part of (1.31).

3.5. **Identification of $\mathbf{S}_v$.** Here, we finally prove (1.28). Since we have already established the strong convergence of $\mathbf{S}_e^\ell$ and $\boldsymbol{v}^\ell$ we are in a position to apply the method of parabolic Lipschitz truncation developed in [18]. Indeed, following the arguments of [18, Section 4] we deduce[2] that there is a subsequence that we do not relabel such that

$$\lim_{\ell\to\infty}\int_Q \left|(\mathbf{S}_v^\ell-\mathbf{S}^*(\mathbf{D}(\boldsymbol{v})))\cdot\mathbf{D}(\boldsymbol{v}^\ell-\boldsymbol{v})\right|^\alpha \,\mathrm{d}x\,\mathrm{d}t=0 \tag{3.58}$$

for any $\alpha\in(0,1)$. Here, we have denoted by $\mathbf{S}^*$ a (measurable) selection mapping such that for any $\mathbf{D}$ we have $(\mathbf{S}^*(\mathbf{D}),\mathbf{D})\in\mathcal{A}$. In particular, it follows (for a subsequence) from (3.58) that

$$(\mathbf{S}_v^\ell-\mathbf{S}^*(\mathbf{D}(\boldsymbol{v})))\cdot\mathbf{D}(\boldsymbol{v}^\ell-\boldsymbol{v})\to 0 \qquad \text{almost everywhere in } Q. \tag{3.59}$$

Moreover, using (3.1) we see that

$$\int_Q \left|(\mathbf{S}_v^\ell-\mathbf{S}^*(\mathbf{D}(\boldsymbol{v})))\cdot\mathbf{D}(\boldsymbol{v}^\ell-\boldsymbol{v})\right| \,\mathrm{d}x\,\mathrm{d}t\le C. \tag{3.60}$$

Thus, we can apply Chacon's Biting Lemma A.3 to find a nondecreasing countable sequence of measurable sets $Q_1\subset\cdots\subset Q_k\subset Q_{k+1}\subset\cdots\subset Q$ such that

$$\lim_{k\to\infty}|Q\setminus Q_k|\to 0 \tag{3.61}$$

and such that for any k there is a subsequence such that

$$(\mathbf{S}_v^\ell-\mathbf{S}^*(\mathbf{D}(\boldsymbol{v})))\cdot\mathbf{D}(\boldsymbol{v}^\ell-\boldsymbol{v}) \qquad \text{converges weakly in } L^1(Q_k). \tag{3.62}$$

Hence, using the characterization of weakly convergent sequences in L^1, the monotonicity of $\mathcal{A}$ and the pointwise convergence result (3.59) we deduce that

$$(\mathbf{S}_v^\ell-\mathbf{S}^*(\mathbf{D}(\boldsymbol{v})))\cdot\mathbf{D}(\boldsymbol{v}^\ell-\boldsymbol{v})\to 0 \qquad \text{strongly in } L^1(Q_k). \tag{3.63}$$

Now, using (3.6) and (3.10), we deduce from (3.63) that

$$\lim_{\ell\to\infty}(\mathbf{S}_v^\ell,\mathbf{D}(\boldsymbol{v}^\ell))_{Q_k}=(\mathbf{S}_v,\mathbf{D}(\boldsymbol{v}))_{Q_k}. \tag{3.64}$$

Therefore, by applying Lemma A.2 we see that $(\mathbf{S}_v,\mathbf{D}(\boldsymbol{v}))\in\mathcal{A}$ almost everywhere in Q_k. Finally, using a diagonal procedure and (3.61) we arrive at (1.28), which completes the proof of Theorem 1.1.

[2]In the case of a Navier boundary condition we refer the reader to [15].

4. Conclusions

We have established long-time large-data existence of weak solutions to a general class of kinetic models of homogeneous incompressible dilute polymers, the main new feature of the model being the presence of a general implicit constitutive equation relating the viscous part $\mathbf{S}_v$ of the Cauchy stress and the symmetric part $\mathbf{D}$ of the velocity gradient. We have considered implicit relations that generate maximal monotone (possibly multivalued) graphs, and the corresponding rate of dissipation was characterized by the sum of a Young function and its conjugate depending on $\mathbf{D}$ and $\mathbf{S}_v$, respectively. The elastic properties of the flow, characterizing the response of polymer macromolecules in the viscous solvent, have been modelled by the elastic part $\mathbf{S}_e$ of the Cauchy stress tensor, whose divergence appears on the right-hand side of the momentum equation, and which is defined by the Kramers expression involving the probability density function, associated with the random motion of the polymer molecules in the solvent. The probability density function satisfies a Fokker–Planck equation, which is nonlinearly coupled to the momentum equation. A possible extension of the analysis presented here would be to admit a nonhomoheneous solvent, with variable density. In the case of a coupled Navier–Stokes–Fokker–Planck system with variable density and density-dependent dynamic viscosity and drag coefficients the existence of global weak solutions was shown in the recent paper [9]. The main theoretical hurdle in extending the results of [9] to nonhomogeneous fluid flow models, where instead of a linear relationship between $\mathbf{S}_v$ and $\mathbf{D}$ these quantities are related through an implicit relationship, is that currently the parabolic Lipschitz truncation method of Diening, Růžička & Wolf [18] and Bulíček, Gwiazda, Málek & Świerczewska-Gwiazda [15] is not available for such models.

References

[1] R. A. Adams and J. J. F. Fournier. *Sobolev Spaces.* Pure and Applied Mathematics, 140. Academic Press, Amsterdam, 2nd edition, 2003.

[2] J. M. Ball and F. Murat. Remarks on Chacon's biting lemma. *Proc. Amer. Math. Soc.*, 107(3):655–663, 1989.

[3] J. W. Barrett and S. Boyaval. Existence and approximation of a (regularized) Oldroyd-B model. *Math. Models Methods Appl. Sci.*, 21:1783–1837, 2011.

[4] J. W. Barrett, Ch. Schwab, and E. Süli. Existence of global weak solutions for some polymeric flow models. *Math. Models Methods Appl. Sci.*, 15:939–983, 2005.

[5] J. W. Barrett and E. Süli. Existence of global weak solutions to some regularized kinetic models of dilute polymers. *Multiscale Model. Simul.*, 6:506–546, 2007.

[6] J. W. Barrett and E. Süli. Existence of global weak solutions to dumbbell models for dilute polymers with microscopic cut-off. *Math. Models Methods Appl. Sci.*, 18:935–971, 2008.

[7] J. W. Barrett and E. Süli. Existence and equilibration of global weak solutions to kinetic models for dilute polymers I: Finitely extensible nonlinear bead-spring chains. *Math. Models Methods Appl. Sci.*, 21(6):1211–1289, 2011.

[8] J. W. Barrett and E. Süli. Existence and equilibration of global weak solutions to hookean-type bead-spring chain models for dilute polymers II: Hookean-type bead-spring chains. *Math. Models Methods Appl. Sci.*, 22(5), 2012. 1150024 (84 pages).

[9] J. W. Barrett and E. Süli. Existence of global weak solutions to finitely extensible nonlinear bead-spring chain models for dilute polymers with variable density and viscosity. *J. Differential Equations*, 2012. (Submitted).

[10] H. Brezis. *Analyse fonctionnelle: Théorie et applications.* Collection Mathématiques Appliquées pour la Maîtrise. Masson, Paris, 1983.

[11] M. Bulíček, P. Gwiazda, J. Málek, and A. Świerczewska-Gwiazda. On steady flows of incompressible fluids with implicit power-law-like rheology. *Adv. Calc. Var.*, 2:109–136, 2009.

[12] M. Bulíček, J. Málek, and K. R. Rajagopal. Navier's slip and evolutionary Navier-Stokes-like systems with pressure and shear-rate dependent viscosity. *Indiana Univ. Math. J.*, 56(1):51–85, 2007.

[13] M. Bulíček, J. Málek, and K. R. Rajagopal. Mathematical analysis of unsteady flows of fluids with pressure, shear-rate, and temperature dependent material moduli that slip at solid boundaries. *SIAM J. Math. Anal.*, 41(2):665–707, 2009.

[14] M. Bulíček, P. Gwiazda, J. Málek, K. R. Rajagopal, and A. Świerczewska-Gwiazda. On flows of fluids described by an implicit constitutive equation characterized by a maximal monotone graph. *J.C. Robinson, J.L. Rodrigo & W. Sadowski, Eds., LMS Lecture Note Series, CUP.* (To appear). Preprint of NCMM, no. 2011-007, 2011.

[15] M. Bulíček, P. Gwiazda, J. Málek, and A. Świerczewska-Gwiazda. On unsteady flows of implicitly constituted incompressible fluids. *SIAM J. Math. Anal.* (Accepted). Preprint of NCMM, no. 2011-008, 2011.

[16] P. Constantin. Nonlinear Fokker–Planck Navier–Stokes systems. *Commun. Math. Sci.*, 3:531–544, 2005.

[17] P. Constantin and G. Seregin. Global regularity of solutions of coupled Navier–Stokes equations and nonlinear Fokker–Planck equations. *Discrete Contin. Dyn. Syst.*, 26(4):1185–1196, 2010.

[18] L. Diening, M. Růžička, and J. Wolf. Existence of weak solutions for unsteady motions of generalized Newtonian fluids. *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, IX(1):1–46, 2010.

[19] R. J. DiPerna and P.-L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, 98:511–547, 1989.

[20] Q. Du, C. Liu, and P. Yu. FENE dumbbell models and its several linear and nonlinear closure approximations. *Multiscale Model. Simul.*, 4:709–731, 2005.

[21] G. Duvaut and J.-L. Lions. *Inequalities in Mechanics and Physics.* Springer, Berlin, 1976.

[22] W. E, T. J. Li, and P.-W. Zhang. Well-posedness for the dumbbell model of polymeric fluids. *Comm. Math. Phys.*, 248:409–427, 2004.

[23] A. W. El-Kareh and L. G. Leal. Existence of solutions for all Deborah numbers for a non-Newtonian model modified to include diffusion. *J. Non-Newtonian Fluid Mech.*,

33:257–287, 1989.
[24] E. Feireisl and A. Novotný. *Singular Limits in Thermodynamics of Viscous Fluids.* Advances in Mathematical Fluid Mechanics. Birkhauser, Basel, Boston, Berlin, 2009.
[25] L. Figueroa and E. Süli. Greedy approximation of high-dimensional Ornstein–Uhlenbeck operators with unbounded drift. *Foundations of Computational Mathematics.* (In print). Preprint: `arXiv`:1103.0726v1 [math.NA].
[26] I. Fonseca and G. Leoni. *Modern Methods in the Calculus of Variations: L^p Spaces.* Springer Monographs in Mathematics. Springer, New York, 2007.
[27] B. Jourdain, T. Lelièvre, and C. Le Bris. Existence of solution for a micro-macro model of polymeric fluid: the FENE model. *J. Funct. Anal.*, 209:162–193, 2004.
[28] J. Kinnunen and J. L. Lewis. Very weak solutions of parabolic systems of p-Laplacian type. *Ark. Mat.*, 40(1):105–132, 2002.
[29] T. Li, H. Zhang, and P.-W. Zhang. Local existence for the dumbbell model of polymeric fuids. *Comm. Partial Differential Equations*, 29:903–923, 2004.
[30] P.-L. Lions and N. Masmoudi. Global solutions for some Oldroyd models of non-Newtonian flows. *Chin. Ann. Math. Ser. B*, 21:131–146, 2000.
[31] P.-L. Lions and N. Masmoudi. Global existence of weak solutions to some micro-macro models. *C. R. Math. Acad. Sci. Paris*, 345:15–20, 2007.
[32] N. Masmoudi. Well posedness of the FENE dumbbell model of polymeric flows. *Comm. Pure Appl. Math.*, 61:1685–1714, 2008.
[33] N. Masmoudi. Global existence of weak solutions to the FENE dumbbell model of polymeric flows. *Invent. Math.*, 2012 (DOI: 10.1007/s00222-012-0399-y).
[34] F. Otto and A. Tzavaras. Continuity of velocity gradients in suspensions of rod-like molecules. *Comm. Math. Phys.*, 277:729–758, 2008.
[35] K. R. Rajagopal. On implicit constitutive theories. *Appl. Math.*, 48(4):279–319, 2003.
[36] K. R. Rajagopal. On implicit constitutive theories for fluids. *J. Fluid Mech.*, 550:243–249, 2006.
[37] K. R. Rajagopal and A. R. Srinivasa. On the thermodynamics of fluids defined by implicit constitutive relations. *Z. Angew. Math. Phys.*, 59(4):715–729, 2008.
[38] M. Reed and B. Simon. *Methods of Modern Mathematical Physics. I.* Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, second edition, 1980. Functional analysis.
[39] M. Renardy. An existence theorem for model equations resulting from kinetic theories of polymer solutions. *SIAM J. Math. Anal.*, 22:313–327, 1991.
[40] P. Yu, Q. Du, and C. Liu. From micro to macro dynamics via a new closure approximation to the FENE model of polymeric fluids. *Multiscale Model. Simul.*, 3:895–917, 2005.
[41] E. Zeidler. *Applied Functional Analysis. Applications to Mathematical Physics.* Number 108 in Applied Mathematical Sciences. Springer-Verlag, New York, 1995.
[42] H. Zhang and P. Zhang. Local existence for the FENE-dumbbell model of polymeric fluids. *Arch. Ration. Mech. Anal.*, 181(2):373–400, 2006.

Appendix A. Auxiliary tools

For the reader's convenience we recall here some of the technical tools that were required in the paper.

Lemma A.1 (Div-Curl Lemma [24]). *Suppose that $\mathfrak{D} \subset \mathbb{R}^{\mathfrak{N}}$ is a bounded open Lipschitz domain and $\mathfrak{N} \in \mathbb{N}_{\geq 2}$. Let, for any real number $s > 1$, $W^{-1,s}(\mathfrak{D})$ and $W^{-1,s}(\mathfrak{D};\mathbb{R}^{\mathfrak{N}\times\mathfrak{N}})$ denote the duals of the Sobolev spaces $W_0^{1,\frac{s}{s-1}}(\mathfrak{D})$ and $W_0^{1,\frac{s}{s-1}}(\mathfrak{D};\mathbb{R}^{\mathfrak{N}\times\mathfrak{N}})$, respectively. Assume that*

$$\left.\begin{array}{ll} \mathbf{H}_n \to \mathbf{H} & \text{weakly in } L^p(\mathfrak{D};\mathbb{R}^{\mathfrak{N}}), \\ \mathbf{Q}_n \to \mathbf{Q} & \text{weakly in } L^q(\mathfrak{D};\mathbb{R}^{\mathfrak{N}}), \end{array}\right\}$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1$. Suppose also that there exists a real number $s > 1$ such that

$$\left.\begin{array}{lcll} \operatorname{div} \mathbf{H}_n & \equiv & \nabla \cdot \mathbf{H}_n & \text{is precompact in } W^{-1,s}(\mathfrak{D}), \text{ and} \\ \operatorname{curl} \mathbf{Q}_n & \equiv & \big(\nabla \mathbf{Q}_n - (\nabla \mathbf{Q}_n)^{\mathrm{T}}\big) & \text{is precompact in } W^{-1,s}(\mathfrak{D};\mathbb{R}^{\mathfrak{N}\times\mathfrak{N}}). \end{array}\right\}$$

Then,

$$\mathbf{H}_n \cdot \mathbf{Q}_n \to \mathbf{H} \cdot \mathbf{Q} \quad \text{weakly in } L^r(\mathfrak{D}).$$

Lemma A.2 (Lemma 2.5 in [15]). *Let $\mathcal{A}$ be a maximal monotone r-graph and suppose that $U \subset Q$ is a bounded measurable set. Assume that there exist sequences $\{\mathsf{S}^n\}_{n\in\mathbb{N}}$ and $\{\mathsf{D}^n\}_{n\in\mathbb{N}}$ such that*

$$(\mathsf{S}^n, \mathsf{D}^n) \in \mathcal{A} \qquad \text{a.e. in } U \tag{A.1}$$

and

$$\begin{aligned} \mathsf{S}^n &\rightharpoonup \mathsf{S} && \text{weakly in } L^{r'}(U)^{d\times d}, \\ \mathsf{D}^n &\rightharpoonup \mathsf{D} && \text{weakly in } L^{r}(U)^{d\times d}. \end{aligned}$$

If in addition

$$\limsup_{n\to\infty} (\mathsf{S}^n, \mathsf{D}^n)_U \leq (\mathsf{S}, \mathsf{D})_U, \tag{A.2}$$

then

$$(\mathsf{S}, \mathsf{D}) \in \mathcal{A} \qquad \text{a.e. in } U. \tag{A.3}$$

Lemma A.3 (Chacon's Biting Lemma, see [2]). *Let Ω be a bounded domain in $\mathbb{R}^d$ and let $\{v^n\}_{n\in\mathbb{N}}$ be a bounded sequence in $L^1(\Omega)$. Then there exists a nonincreasing sequence of measurable subsets $E_j \subset \Omega$ with $|E_j|$ such that $\{v^n\}_{n\in\mathbb{N}}$ is pre-compact in the weak topology of $L^1(\Omega \setminus E_j)$, for each $j \in \mathbb{N}$.*

Lemma A.4 (Lemmas 15 and 16 in [25]). *Let H and V be separable infinite-dimensional Hilbert spaces, with $V \hookrightarrow H$ and $\overline{V} = H$ in the norm of H. Let $a\colon V \times V \to \mathbb{R}$ be a nonzero, symmetric, bounded and coercive bilinear form. Then, there exist sequences of real numbers $\{\lambda_n\}_{n\in\mathbb{N}}$ and unit H-norm members $\{e_n\}_{n\in\mathbb{N}}$ of V, which solve the eigenvalue problem:* Find $\lambda \in \mathbb{R}$ and $e \in H \setminus \{0\}$ such that

$$a(e, v) = \lambda \langle e, v\rangle_H \quad \forall\, v \in V. \tag{A.4}$$

The λ_n, which can be assumed to be in increasing order with respect to n, are positive, bounded from below away from 0*, and* $\lim_{n\to\infty} \lambda_n = \infty$*.*

Additionally, the e_n form an H-orthonormal system whose H-closed span is H and the rescaling $e_n/\sqrt{\lambda_n}$ gives rise to an a-orthonormal system whose a-closed span is V, so we have

$$h = \sum_{n=1}^{\infty} \langle h, e_n\rangle_H e_n \quad \textit{and} \quad \|h\|_H^2 = \sum_{n=1}^{\infty} \langle h, e_n\rangle_H^2 \qquad \forall\, h \in H \tag{A.5}$$

and

$$v = \sum_{n=1}^{\infty} a\left(v, \frac{e_n}{\sqrt{\lambda_n}}\right) \frac{e_n}{\sqrt{\lambda_n}} \quad \textit{and} \quad \|v\|_a^2 = \sum_{n=1}^{\infty} a\left(v, \frac{e_n}{\sqrt{\lambda_n}}\right)^2 \qquad \forall\, v \in V; \tag{A.6}$$

further,

$$h \in H \quad \textit{and} \quad \sum_{n=1}^{\infty} \lambda_n \langle h, e_n\rangle_H^2 < \infty \iff h \in V. \tag{A.7}$$

Proof. The proofs of the stated results can be partially found in textbooks on functional analysis (see, for example, Theorem VI.15 in Reed & Simon [38] or Section 4.2 in Zeidler [41]). A version of the proof for the special case in which V and H are standard Sobolev spaces is contained in Section IX.8 of Brezis [10]; using the abstract results in Chapter VI of [10], the result in Section IX.8 of [10] can be easily adapted to the setting of the present theorem. For a detailed proof we refer to Lemmas 15 and 16 in the, extend, `arXiv` version of the paper [25]. □

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