Relative entropy applied to the stability of shocks for fluid mechanics

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The equation

- Full compressible Euler system:

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- Isentropic gas dynamics:

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- We restrict ourselves to the 1D case.
- We are interested to the "strong" stability of those special solutions.
- It is closely related to the study of asymptotic limits to shocks (for instance, from Navier-Stokes to Euler).
- Remark: We can consider more general systems than the Euler case.

A physical motivation

- Shocks are fundamental solutions in physics. But, the derivation of the macroscopic model is problematic for those solutions (no local thermodynamical equilibrium for the derivation from kinetic equations, for instance).

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- Shocks are fundamental solutions in physics. But, the derivation of the macroscopic model is problematic for those solutions (no local thermodynamical equilibrium for the derivation from kinetic equations, for instance).
- The difficulty come from the production of layers.
- What happens if the system carries too much energy for the stability of the layer ?

Mathematical motivations

- In 1D, Shocks corresponds to the solitons of the equation, where solitons are defined as solutions which are invariant through blow-ups.

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- To be an entropy means that any physical solutions verify:

$$\int \eta(U(t,x))\, dx$$

is not increasing.

Relative entropy

We define the relative entropy between two states $U_1, U_2 \in \mathcal{V}$

$$\eta(U_1|U_2) = \eta(U_1) - \eta(U_2) - \eta'(U_2)(U_1 - U_2).$$

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Dafermos- DiPerna (79'): If U_2 is a Lipschitz solution and U_1 is a weak solution, then

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Especially, if at $t = 0$ $\int_{\mathbb{R}} \eta(U_1|U_2)\,dx \approx \varepsilon^2$, then at t: $\approx e^{Ct}\varepsilon^2$.

Strong stability L^2

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- Can be used for asymptotic limit and hydrodynamic. In other context, see Yau (91'), Bardos Golse Levermore (91'),...
- In this context, the consistence implies the convergence. The nonlinearities are driven by the strong stability of the limit function.

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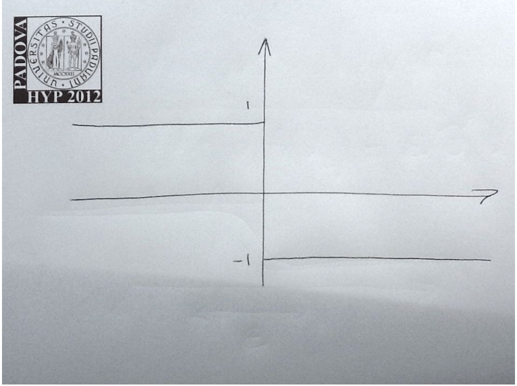
- Example for Burgers:

$$\partial_t u + \partial_x u^2 = 0.$$



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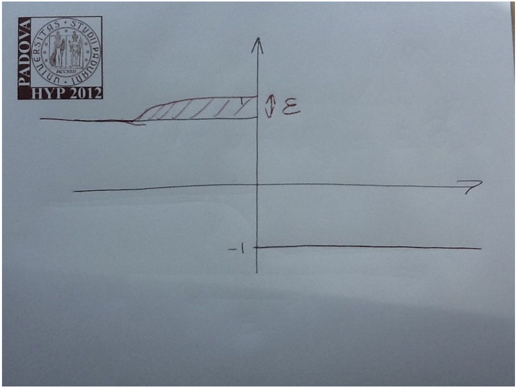

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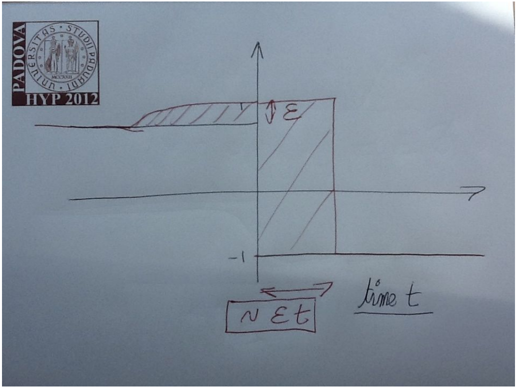

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- An ε perturbation of a shock S at $t = 0$ will give an error $\approx \sqrt{\varepsilon t}$ at time t.
- This is because it perturbs the SPEED of the shock.
- However, the profile of the shock is still VERY stable (up to a translation).

The system case

Theorem

(Leger, V.) Consider (U_L, U_R, σ) *a shock. Then there exist constants* $C > 0$, $\varepsilon_0 > 0$ *such that for any* $0 < \varepsilon < \varepsilon_0$, *and*

$$\int_0^\infty |U_0(x) - S(x)|^2 \, dx \leq \varepsilon,$$

there exists a Lipschitzian map $x(t)$ *such that for any* $0 < t < T$*:*

$$\int_0^\infty |U(t,x) - S(x - x(t))|^2 \, dx \leq C\varepsilon(1+t),$$

$$|x(t) - \sigma t| \leq C\sqrt{\varepsilon t(1+t)}.$$

For $x < 0$, $S(x) = U_L$, for $x > 0$, $S(x) = U_R$.

Remarks

- Provides a stability result in the class of bounded weak solutions having a strong trace property. There is no smallness conditions. We do not need the microstructure of the solutions. The stability is driven by the entropy.
- It is a strong L^2 stability result up to a shift.

Citations

- L^1 theory: Bressan, Liu....
- DiPerna (79'): Uniqueness of shocks (but no stability).
- Chen, Frid, Li (01', 02, 04'): 3×3 Euler with big amplitude. Uniqueness, and asymptotic (in time) L^2 stability.
- Leger (08'): L^2 stability for the scalar case.
- Leger, V. (10'): system case with ϵ/ϵ^4 restriction.

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- It uses only ONE entropy.
- We were able to extend the proof to the system case.
- The additional difficulty was to work with several waves (the scalar case has only one).

A first application

Scalar case:

$$\partial_t U_\varepsilon + \partial_x U_\varepsilon^2 = \varepsilon \partial_{xx} U_\varepsilon.$$

For U_L, U_R, we define $S(x) = U_L$ if $x < 0$, and $S(x) = U_R$ if $x > 0$.

The result

Theorem

(Choi, V.) There exists $\varepsilon_0 > 0$, such that for any U_ε solution to the viscous Burgers equation with $\varepsilon < \varepsilon_0$ and

$$\|(\partial_x U_0)_+\|_{L^2} \leq C,$$

there exists $X(t)$ Lipschitz such that for any time $t > 0$

$$\int \eta(U_\varepsilon(t,x)|S_0(x - X(t)))\,dx$$
$$\leq \int \eta(U_0(x)|S_0(x))\,dx + C\varepsilon(\log^+(1/\varepsilon) + 1)(1 + t).$$

Case with small initial perturbation

- If $\int \eta(U_0(x)|S(x))\, dx \leq C\varepsilon$, Then we can study the layer problem by scaling $V(t,x) = U(\varepsilon t, \varepsilon x)$.

$$\partial_t V + \partial_x V^2 = \partial_{xx} V.$$

- This problem has been extendedly studied (Ilin Oleinik (64'), Osher and Ralston (82'), Goodman (89'), Jones Gardner and Kapitula (93'), Freistuhler and Serre (96'), Kenig and Merle (06'))
- V converges to the layer $Q(x - \sigma t)$ up to a drift (nondependent on time).

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- V converges to the layer $Q(x - \sigma t)$ up to a drift (nondependent on time).
- In this context, our result is weaker (the error is bigger than ε).
- But, in the case $\int \eta(U_0(x)|S(x))\,dx >> \varepsilon$, the layer study collapse. (The layer can be destroyed). Still, we can obtain the expected limit with a precise rate.

Remarks:

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- Contrary to the layer study, the method does NOT use the comparison principles.
- It uses only one entropy.
- Hypothesis are very general. Again, the convergence is driven by the entropy.
- The shift is still imposed by the hyperbolic part. We use some dissipation from the hyperbolic part to control some viscous smoothing of the profile. This provide the rate of convergence in (almost) ε.

Future work

- get asymptotic limits for systems.
- study multi-D stability of 1-D shocks.
- Get more structure on solutions of conservation laws with large initial data (1D case).

Thank you

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- they corresponds to the family of slowest shocks (associated to the smallest eigenvalue of A').
- but they are very powerful at blocking information flowing from the right to left. (all eigenvalues of $A'(U_L)$ are bigger than the speed of the shock).

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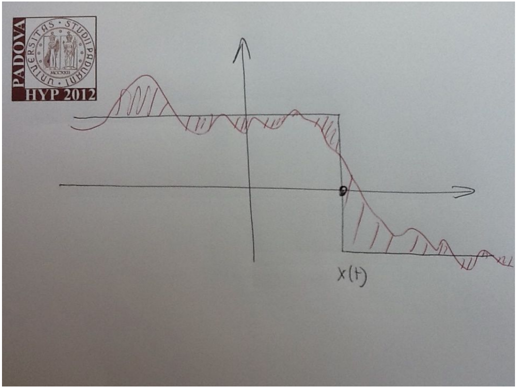

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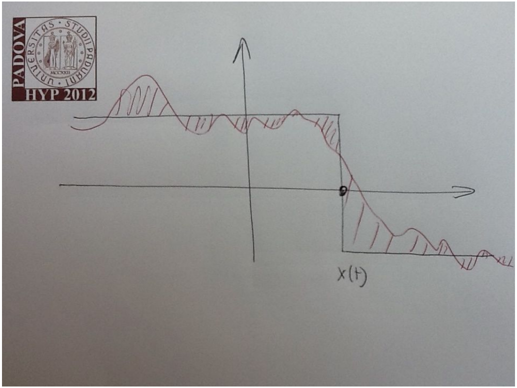

By choosing $x'(t)$ we can change the fluxes of entropy (depending on the "value" of $U(t, x(t))$!).

the drift (2)

- We will solve an ODE with a discontinuous flux. We use Fillipov flow.
- Generically, the interface $x(t)$ is stuck in a shock !

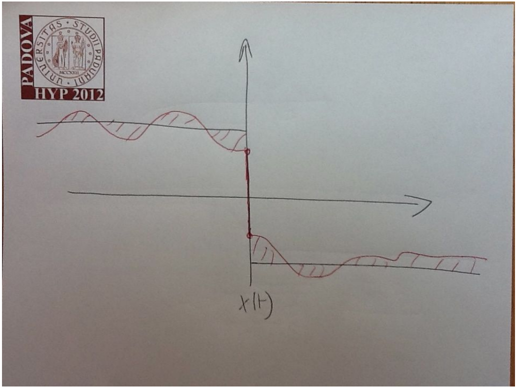

Figure: Drift

Fixing the left side

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  - the left part strictly dissipates some entropy.
  -The interface does not cruise faster than a 1-shock.
- that way we can consider only 1-shock (only one wave as in the scalar case).
- Then the right part takes care of itself ! (based on a nice algebraic structure discovered by DiPerna).

sketch (1)

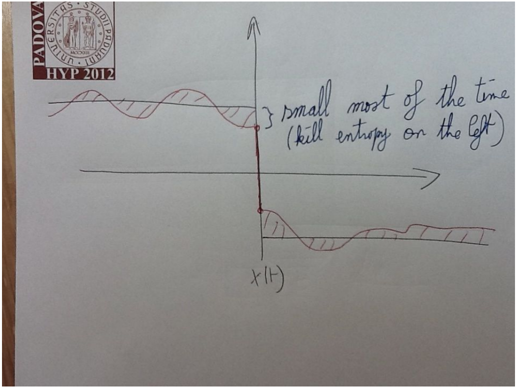
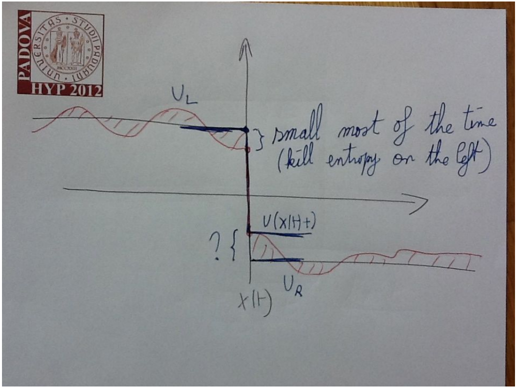


Figure: Proof

sketch (2)



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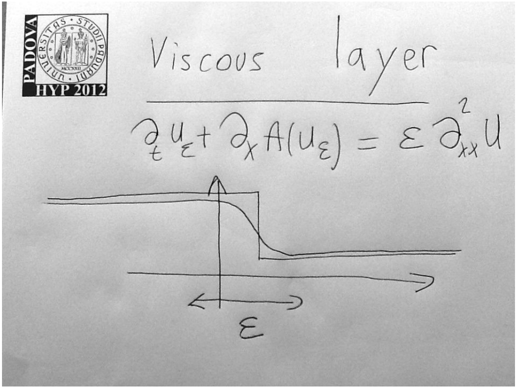

Figure: example of layer

Shocks and shock layers

- Asymptotic limits to shocks involve the production of LAYERS.
- The control of the layers usually involves smallness conditions: Liu Zumbrun, Bressan (L^1 theory)...

Shocks and shock layers (2)

QUESTION:

- Is the whole structure of the layer needed to perform asymptotic limits ?

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- Is the whole structure of the layer needed to perform asymptotic limits ?
- Would the entropy (relative entropy) be enough to drive the convergence, whatever the fine structure in the layer ?
- Do we have enough strong stability on shocks ?