Modeling Gum Metal and other newly developed titanium alloys within a new class of constitutive relations for elastic bodies [☆]

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Abstract

Many titanium alloys and even materials like concrete exhibit a nonlinear relationship between the strain and the stress, when the strain is small enough in the sense that the square of the norm of the displacement gradient can be ignored in comparison to the norm of the displacement gradient. Such response cannot be described within the classical theory of Cauchy elasticity wherein linearization of the nonlinear strain leads to the classical linearized elastic response. A new framework for elasticity has been put into place wherein one can justify rigorously a nonlinear relationship between the linearized strain and the stress. Here, we consider one such model based on a power-law relationship. Previous attempts at describing such response has been either limited to the response of one particular material, say Gum metal, or involves a model with more material moduli, than the model considered in this work. For the uniaxial response of several metallic alloys, the model that is being considered fits experimental data exceedingly well.

Keywords: Titanium alloys, Modeling, Implicit constitutive theory

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1. Introduction

In material science there is an ongoing effort to develop new titanium alloys in virtue of their beneficial properties. An illustrative example is Gum Metal[1], a material that has been developed by Toyota central R&D labs. Gum Metal is a designation for class of beta titanium alloys with unique elastic properties that include low Young modulus, high strength and high yield strain relative to the other conventionally used materials and titanium alloys, see [1]. Cold swagged Gum Metal has reversible nonlinear elastic response up to the strains of 2.5%, which is refereed to as super elasticity, see [2, 3].

Gum Metal however is not the only titanium alloy that exhibits nonlinear response in the elastic regime that could be considered small strain, there is growing evidence of recently developed titanium alloys behaving nonlinearly when subject to small strains. Such nonlinear elastic behavior seems to be typical for many beta phase titanium alloys, see [4, 5, 6].

In the small displacement gradient range, the square of the norm of the displacement gradient can be neglected in comparison to the norm of the displacement gradient. Such a response cannot be described within the context of any Cauchy elastic model (and hence any Green elastic model) as linearization of a nonlinear constitutive expression for the stress leads to the classical linearized elastic model which is a linear relationship between the stress and the linearized strain. While the response can be curve fitted by using a nonlinear relationship between the stress and the strain, one cannot justify them or show that they follow from the linearization of a model to describe elastic response. Recently, Rajagopal [7] (see also [8]) recognized that the class of bodies that are elastic, if by elastic one understands the body is incapable of dissipation, that is convert mechanical working into thermal energy (heat), is far larger than Cauchy elastic bodies. He proposed a class of implicit relationships between the Cauchy stress and deformation gradient to describe elastic response, Cauchy elasticity being a very small special sub-class of them. Rajagopal and Srinivasa [9, 10] provided a rigorous thermodynamic basis for the same. In the case of isotropic elastic bodies

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[1] GUMMETAL is a trademark owned by the TOYOTSU MATERIAL INCORPORATED company (as of 2017).

described by implicit constitutive relations, we have the relationship

$$\boldsymbol{G}(\rho, \boldsymbol{T}, \boldsymbol{B}) = \boldsymbol{0}. \tag{1.1}$$

A very special sub-class of the bodies defined through the above implicit relation are classical isotropic compressible Cauchy elastic body described through the following constitutive expression for the stress

$$\boldsymbol{T} = \delta_0 \boldsymbol{I} + \delta_1 \boldsymbol{B} + \delta_2 \boldsymbol{B}^2, \tag{1.2}$$

where the material moduli depend on the density and the principal invariants of $\boldsymbol{B}$. We note that the above equation presents an explicit expression for the Cauchy stress in terms of the Cauchy-Green tensor.

A different set of constitutive relations is given by

$$\boldsymbol{B} = \alpha_0 \boldsymbol{I} + \alpha_1 \boldsymbol{T} + \alpha_2 \boldsymbol{T}^2, \tag{1.3}$$

where the material moduli are functions of the density and the principal invariants of the stress. Truesdell and Moon [11] have obtained conditions under which the representation (1.2) is invertible. However, they were considering invertibility of isotropic functions and were not interested in delineating models of the form (1.3) that do not belong to (1.2), or put differently whether there are models of the form (1.3) that are not Cauchy elastic. It is important to recognize that many models belonging to the class defined by (1.3) do not belong to the class defined by (1.2).

While linearization of (1.2) under the assumption that the displacement gradient be small leads to the classical linearized elastic model, linearizing (1.3) under the same assumption leads to the approximation

$$\boldsymbol{\varepsilon} = \beta_0 \boldsymbol{I} + \beta_1 \boldsymbol{T} + \beta_2 \boldsymbol{T}^2, \tag{1.4}$$

and thus it is possible for the linearized strain to bear a nonlinear relationship to the stress. Rajagopal [12] and Devendiran et al. [13] use models belonging to the above class to describe the response of titanium alloys. In this paper we shall also use a power-law model that belongs to the above class of constitutive relations to describe the response of titanium alloys. Our model has fewer material constants than the model used by Devendiran et al. [13] to corroborate experimental data on titanium alloys. While Rajagopal [12] used a model with fewer material moduli to describe the response of GUM metal, he did not corroborate the response of other titanium alloys. Here, we model numerous titanium alloys within the framework of the same form of constitutive relation only difference being the values for the material moduli.

2. Experimental data and the new class of constitutive relations

The recent studies by Rajagopal [12] and Devendiran et al. [13] employ a new class of constitutive relations to corroborate experimental data for the tensile loading of beta phase titanium alloys, see Saito et al. [3], Sakaguch et al. [4], Hao et al. [5], and Hou et al. [6].

Because of the nature of the data describing uniaxial loading experiments we consider the stress tensor of the following form

$$\boldsymbol{T} = (\boldsymbol{e}_1 \otimes \boldsymbol{e}_1)\sigma = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.1}$$

where $\sigma := T_{11}$ is the only nonzero component of the stress tensor[2]. We shall denote by $\eta := \boldsymbol{\varepsilon}_{11}$ the normal strain component along the 1-direction.

The reversible elastic response of cold swagged Gum Metal can be observed up to the strains of 2.5%, see [3] and Figure 1. Since the elastic response of Gum Metal and many other titanium alloys is in the range $|\boldsymbol{\varepsilon}| < 0.025$, such strains could be regarded as large by experimentalists since most metals do not exhibit elastic response for the magnitude of such strains. On the other hand from the modeling point of view we can use the small displacement gradient approximation and model the response using small strain tensor since the displacement gradient is so small that its square can be neglected in comparison to itself, see [12].

Two models were proposed by Rajagopal in [12] to fit the experimental data that is available for cold swagged Gum Metal. The first model is a power law model wherein the strain is given by

$$\boldsymbol{\varepsilon} = \lambda_1 \boldsymbol{I} \operatorname{tr} \boldsymbol{T} + \lambda_2 (1 + \alpha \operatorname{tr} \boldsymbol{T}^2)^n \boldsymbol{T}, \tag{2.2}$$

where λ_1, λ_2, α and n are the material moduli. The second, an exponential model reads

$$\boldsymbol{\varepsilon} = \lambda_1 \boldsymbol{I} \operatorname{tr} \boldsymbol{T} + 2\lambda_2 \boldsymbol{T} \exp(\eta \operatorname{tr} \boldsymbol{T}), \tag{2.3}$$

where λ_1, λ_2 and η are the material moduli. For the uniaxial loading of the form (2.1), when setting $\lambda_1 = 0$ the model (2.3) reduces to

$$\varepsilon_{11} = 2\lambda_2 \sigma \exp(\eta\sigma). \tag{2.4}$$

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[2] Here, $\boldsymbol{e}_1$ denotes the unit vector in the direction of the first Cartesian coordinate and the symbol $\otimes$ denotes the tensor product.

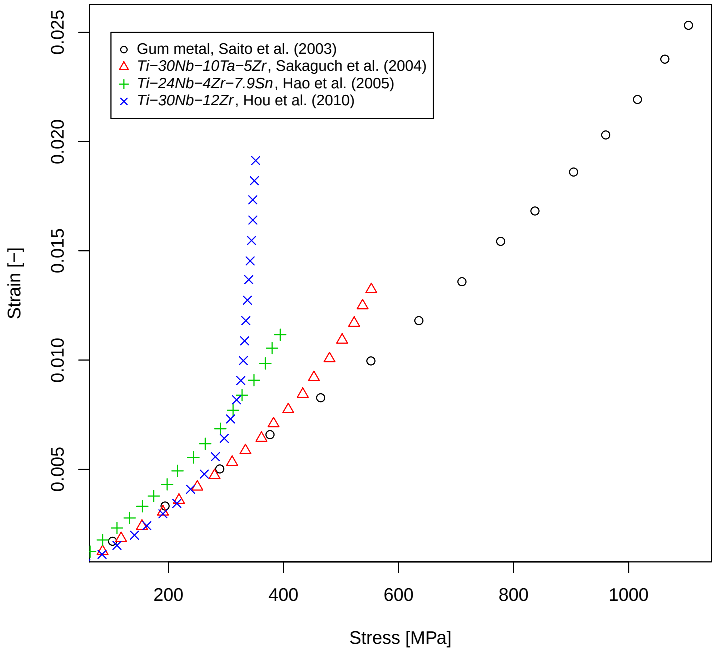

Figure 1: Stress strain response to tensile loading of considered titanium alloys in the elastic regimen.

Actual material parameters that were used in [12] are

$$\lambda_2 = 1.57\,10^{-11} Pa^{-1}, \qquad \eta = 3.22\,10^{-10} Pa^{-1}. \tag{2.5}$$

The model (2.4) using parameters (2.5) provides very good agreement with experimental data. If we linearize this model around $\sigma = 0$ we obtain the value for the Young modulus

$$\frac{1}{E} = \frac{d\varepsilon_{11}}{d\sigma}(0), \tag{2.6}$$

which leads to the value

$$E = \frac{1}{1.57} 10^{11} Pa \approx 64 GPa. \tag{2.7}$$

Devediran et al. [13] use two models to fit the response of beta phase titanium alloys. One of them is a fully implicit model which is given by

$$\begin{aligned}\boldsymbol{\varepsilon} - \alpha_1((\operatorname{tr}\boldsymbol{T} + \operatorname{tr}\boldsymbol{T}\operatorname{tr}\boldsymbol{\varepsilon} - 2\operatorname{tr}(\boldsymbol{T}\boldsymbol{\varepsilon}))\boldsymbol{I} + 2(\operatorname{tr}\boldsymbol{T})\boldsymbol{\varepsilon}) - \\ (\alpha_2 + \alpha_3 \exp(1 + \alpha_4(\operatorname{tr}\boldsymbol{T}^2 + 2\operatorname{tr}(\boldsymbol{T}^2)\operatorname{tr}\boldsymbol{\varepsilon} + 4\operatorname{tr}(\boldsymbol{T}^2\boldsymbol{\varepsilon})))^{\frac{n}{2}})\boldsymbol{T} = \boldsymbol{0},\end{aligned} \tag{2.8}$$

while the second is an explicit representation for the linearized strain

$$\boldsymbol{\varepsilon} = \beta_1(\operatorname{tr}\boldsymbol{T})\boldsymbol{I} + (\beta_2 + \beta_3 \exp(1 + \beta_4 \operatorname{tr}(\boldsymbol{T}^2))^{\frac{n}{2}})\boldsymbol{T}. \tag{2.9}$$

In the tensile loading setting, assuming $\varepsilon_{22} = \varepsilon_{33}$, the model (2.8) reduces to

$$\begin{aligned}\varepsilon_{11} &= (\alpha_1(1 + \varepsilon_{11} + 2\varepsilon_{22}) + \alpha_2)T_{11} \\ &\qquad + \alpha_3 \exp(1 + \alpha_4(1 + 6\varepsilon_{11} + 4\varepsilon_{22})T_{11}^2)^{\frac{n}{2}},\end{aligned} \tag{2.10a}$$

$$\varepsilon_{22} = \alpha_1(1 + 4\varepsilon_{22} - \varepsilon_{11})T_{11}. \tag{2.10b}$$

and the model (2.9) reduces to

$$\varepsilon_{11} = (\beta_1 + \beta_2 + \beta_3 \exp(1 + \beta_4 \operatorname{tr}(T_{11}^2))^{\frac{n}{2}})T_{11}. \tag{2.11}$$

The quantities

$$E_\alpha = \frac{1}{\alpha_1 + \alpha_2} \tag{2.12}$$

and

$$E_\beta = \frac{1}{\beta_1 + \beta_2} \tag{2.13}$$

are expression for the Young modulus for models (2.8) and (2.9) respectively. For actual values of fitted parameters of models (2.8) and (2.9), see [13, Tables 1 and 2].

3. Fitting tensile loading experiments to power law models

In this section we corroborate the experimental data that is available for titanium alloys to a power law model defined through

$$\boldsymbol{\varepsilon} = \frac{1}{9\hat{K}}(\operatorname{tr}\boldsymbol{T})\boldsymbol{I} + \frac{1}{2\hat{\mu}}\boldsymbol{T}^d, \tag{3.1}$$

where

$$\hat{K} = \frac{tr(\boldsymbol{T})}{3tr(\boldsymbol{\varepsilon})} = K\left(\frac{\tau_0^2}{\tau_0^2 + |tr(\boldsymbol{T})|^2}\right)^{\frac{s-2}{2}} \text{ and } \hat{\mu} = \frac{|\boldsymbol{T}^d|}{2|\boldsymbol{\varepsilon}^d|} = \mu\left(\frac{\frac{2}{3}\tau_0^2}{\frac{2}{3}\tau_0^2 + |\boldsymbol{T}^d|^2}\right)^{\frac{q-2}{2}} \tag{3.2}$$

are generalized bulk and shear moduli, $\tau_0 > 0$, $q \in (1, \infty)$, $s \in (1, \infty)$, $\mu > 0$ and $K > 0$ are material moduli. Parameter s and coefficient K describe changes in the volume to the mean normal stress, while the parameter q and coefficient μ describe the isochoric part of deformation. We refer to parameters K and μ as to bulk and shear moduli. Note that we obtain the Hookean model with $\hat{K} = K$ and $\hat{\mu} = \mu$ upon setting $s = q = 2$ in (3.2). Parameter τ_0 determines the magnitude of the stress for which the response is linear. Upon linearizing the model (3.1) around $\boldsymbol{T} = \boldsymbol{0}$ we obtain the classical linearized elastic model as long as

$$|\boldsymbol{T}| \ll \tau_0. \tag{3.3}$$

Since we assume that the Cauchy stress tensor $\boldsymbol{T}$ is of the form (2.1) and $T_{11} = \sigma$ is its only nonzero component, it immediately follows that

$$\boldsymbol{T}^d = \begin{pmatrix} \frac{2}{3}\sigma & 0 & 0 \\ 0 & -\frac{1}{3}\sigma & 0 \\ 0 & 0 & -\frac{1}{3}\sigma \end{pmatrix}, \qquad |\boldsymbol{T}^d| = \sqrt{\frac{2}{3}}\sigma, \qquad \operatorname{tr}\boldsymbol{T} = \sigma. \tag{3.4}$$

When corroborating tensile loading data to the model (3.1) with parameters (τ_0, s, q, K, μ), the model reduces to

$$\eta = \frac{1}{9K}(\frac{1}{\tau_0})^{s-2}(\tau_0^2 + \sigma^2)^{\frac{s-2}{2}}\sigma + \frac{1}{2\mu}(\frac{1}{\tau_0})^{q-2}(\tau_0^2 + \sigma^2)^{\frac{q-2}{2}}\sqrt{\frac{2}{3}}\sigma. \tag{3.5}$$

We use two approaches to fixing τ_0 in (3.5). For all experimental data that we use $|\boldsymbol{T}|$ takes values in the range $10^7 - 10^9$ $(10MPa - 1GPa)$, and for

$|\boldsymbol{T}| > 5.10^8$ the response is nonlinear, see Figure 1. In the first approach we set $\tau_0 = 5.10^8$ for all the materials that were studied. In the second approach we set $\tau_0 = \sigma_{max}$, where σ_{max} represents the maximal loading in the elastic regime; the explicit values for each titanium alloy are specified below in the first column of Table 2. In both approaches we always meet the assumption (3.3) in the linear regime.

Thus, upon fixing τ_0, our model (3.5) is completely characterized by four parameters. Model (2.4) used by Rajagopal [12] to corroborate the experimental data for cold swagged Gum Metal has two parameters but as mentioned earlier the model was not used to corroborate the experiments on other titanium alloys other than Gum metal, while the explicit model used by Devendiran et al. [13] to describe the tensile response of titanium alloys has five parameters.

3.1. Procedure of corroborating experimental data

For corroborating the experimental data that is available we use linear regression. We understand equation (3.5) for particular values of (τ_0, s, q) as a linear model of the form

$$\eta = c_1 f_1(\sigma) + c_2 f_2(\sigma), \tag{3.6}$$

where

$$f_1(\sigma) = \left(\frac{\tau_0^2 + \sigma^2}{\tau_0^2}\right)^{\frac{s-2}{2}} \sigma, \quad f_2(\sigma) = \left(\frac{\tau_0^2 + \sigma^2}{\tau_0^2}\right)^{\frac{q-2}{2}} \sqrt{\frac{2}{3}}\sigma. \tag{3.7}$$

Let σ^i, $i \in 1 \ldots N$, represent the particular experimental tensile stress and η^i, $i \in 1 \ldots N$, represent the observations of the strain. The values of functions $f_1(\sigma^i)$ and $f_2(\sigma^i)$ are understood as independent variables and the value of the strain η^i is understood as the observed value. Using linear regression we obtain estimates for coefficients c_1 and c_2 in (3.6) and derive estimates of the bulk and shear moduli

$$K = \frac{1}{9c_1}, \quad \mu = \frac{1}{2c_2}. \tag{3.8}$$

Using this procedure we can estimate optimal values of the parameters K and μ for a given (τ_0, s, q). Since the parameter τ_0 is fixed, we need to estimate optimal values of the exponents s and q. We decided to perform this estimation based on comparing the quality of fit for different pairs (s, q). For measuring the quality of fit of the model we need the following definitions:

Definition 1 (Mean of observations).

$$\overline{\eta} = \frac{1}{N}\sum_{i=1}^{N}\eta^i. \tag{3.9}$$

Definition 2 (Total sum of squares).

$$S_{tot} = \sum_{i=1}^{N}(\eta^i - \overline{\eta})^2. \tag{3.10}$$

Definition 3 (Residual sum of squares).

$$S_{res} = \sum_{i=1}^{N}(\eta^i - (c_1 f_1(\sigma^i) + c_2 f_2(\sigma^i)))^2. \tag{3.11}$$

Definition 4 (Coefficient of determination R^2).

$$R^2 = 1 - \frac{S_{res}}{S_{tot}}. \tag{3.12}$$

The coefficient of determination R^2 is a standard measure of the quality of fit in linear regression; $R^2 \leq 1$ since the values of $S_{tot} \geq 0$ and $S_{res} \geq 0$. The closer is the value of coefficient of determination to 1 the better the fit is.

3.2. Implementation

We outline the algorithm used for fitting experimental data as follows:

- We fix some particular value of τ_0, derived from a characteristic magnitude of the stress for which the response can be modeled as linear for small strain tending to zero.
- We choose an admissible set of the model parameters (s, q). We use $s \in \{1.01, 1.02, \ldots, 100\}$, $q \in \{1.01, 1.02, \ldots, 100\}$. The values of s and q are discrete values from the finite sequence $\{1.01, 1.02, \ldots, 100\}$ of the numbers incremented by 0.01, we also consider[3] $|s-2| > |q-2|$.

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[3] If we were to require that $|s-2| < |q-2|$, then we would obtain another set of values for the material moduli. In our opinion, the set of values considered here is more realistic from the physical point of view, see [14] for details.

- For each alloy and for each admissible pair of (s, q) using linear regression of the experimental data we obtain estimates of coefficients (c_1, c_2) of the model (3.6) and R^2 using (3.12).

- For each alloy and for each admissible pair of (s, q) we compute the values of the parameters K and μ using (3.8).

- For each alloy we choose parameters (τ_0, s, q, K, μ) that maximize the coefficient of determination R^2 among all admissible pairs of (s, q).

The best fit obtained by this algorithm maximizes the coefficient of determination and minimizes the residual sum of squares among all admissible pairs of (s, q). Linear regression was performed using function `lm` from the R software environment and language, see [15]. For more details about fitting models in R, see [16].

4. Results

In Tables 1 and 2 we present the values for the best fit for each titanium alloy studied. Table 1 lists material moduli when $\tau_0 = 5.10^8$. Table 2 lists material moduli when $\tau_0 = \sigma_{max}$ specified in the first column of Table 2. In Table 3 we compare the quality of the achieved fits with the fits for the models (2.4), (2.10) and (2.11) considered in [12, 13].

Material	$\tau_0[GPa]$	s	q	$K[GPa]$	$\mu[GPa]$	R^2	$E[GPa]$
Gum metal	0.5	7.65	2.23	6223	24.7	0.9999	74.0
Ti-30Nb-10Ta-5Zr	0.5	9.15	2.49	334	27.4	0.9998	79.9
Ti-24Nb-4Zr-7.9Sn	0.5	15.68	2.99	1126	20.2	0.9997	60.3
Ti-30Nb-12Zr	0.5	56.49	4.29	180252	30.8	0.9980	92.3

Table 1: The best fit for the model (3.1) where $\tau_0 = 0.5GPa$. E is the value of the Young's modulus.

In Figures 2, 3, 4, 5 there is a comparison of the best fit of the power law model (3.5) for $\tau_0 = 5.10^8$ with the predictions of the explicit models considered in [12] and [13] when fitting tensile loading experiments.

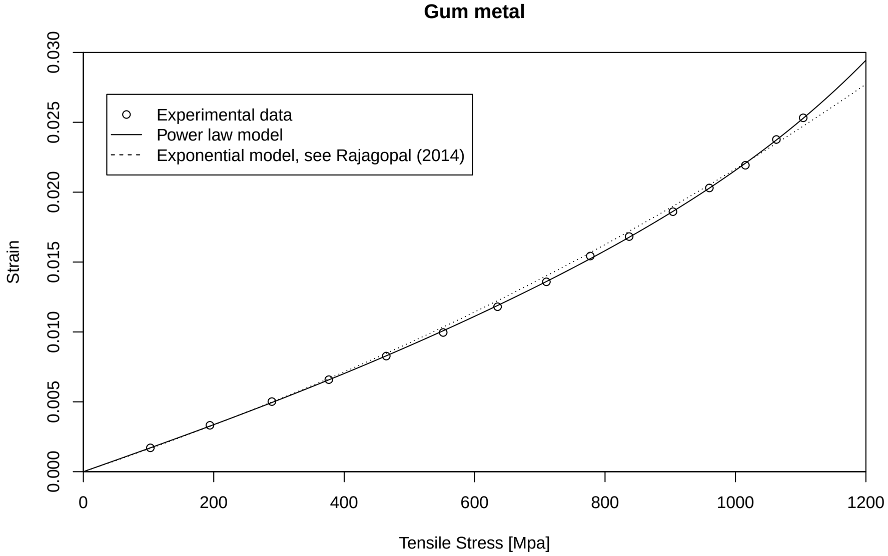

Figure 2: Plot of the best fit for the model (3.5) compared with the exponential model (2.4) for Gum metal, see [12].

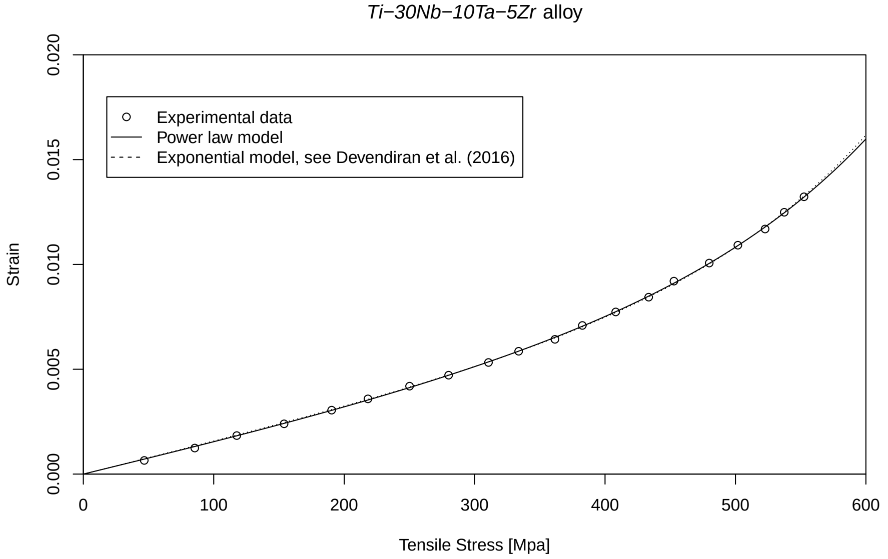

Figure 3: Plot of the best fit for the model (3.5) compared with the exponential model (2.11) for *Ti-30Nb-10Ta-5Zr*, see [13].

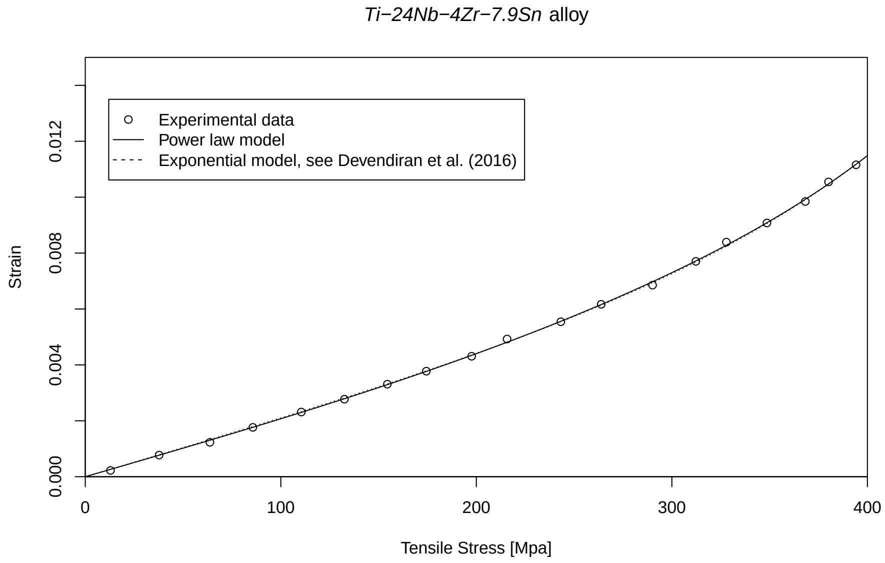

Figure 4: Plot of the best fit for the model (3.5) compared with the explicit exponential model (2.11) for *Ti-24Nb-4Zr-7.9Sn*, see [13].

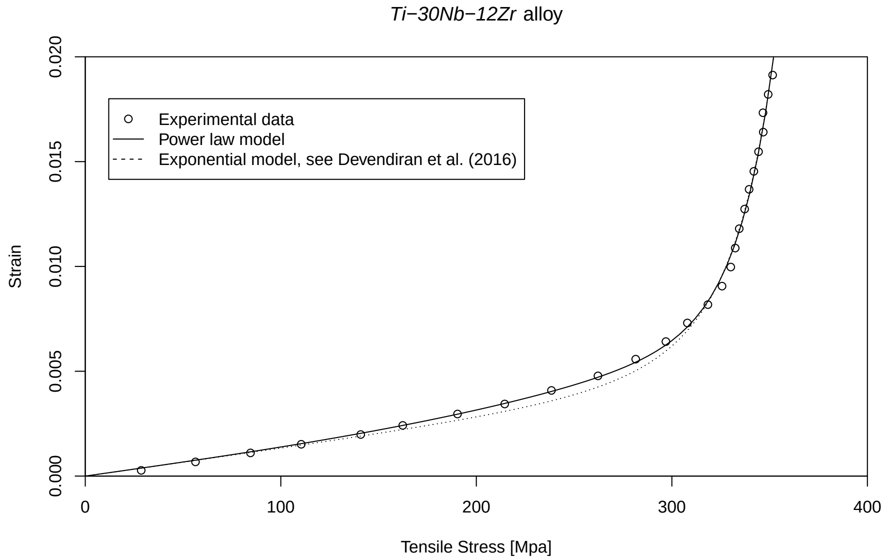

Figure 5: Plot of the best fit for the model (3.5) compared with the explicit exponential model (2.11) for *Ti-30Nb-12Zr*, see [13].

Material	$\tau_0[GPa]$	s	q	$K[GPa]$	$\mu[GPa]$	R^2	$E[GPa]$
Gum metal	1.1	28.50	2.82	1283552	24.5	0.9999	73.6
Ti-30Nb-10Ta-5Zr	0.6	9.81	2.58	291	27.4	0.9998	79.8
Ti-24Nb-4Zr-7.9Sn	0.4	12.13	2.65	1201	20.2	0.9997	60.4
Ti-30Nb-12Zr	0.4	37.89	3.39	804644	31.4	0.9981	94.3

Table 2: The best fit for the model (3.1) where τ_0 is specified in the first column. E is the value of the Young's modulus.

Material	R^2_{exp}	$E_{exp}[GPa]$	R^2_{imp}	$E_{imp}[GPa]$
Gum metal	0.99822	63.7		
Ti-30Nb-10Ta-5Zr	0.99975	67.0	0.99596	67.0
Ti-24Nb-4Zr-7.9Sn	0.99964	50.5	0.99926	50.5
Ti-30Nb-12Zr	0.99663	80.5	0.99437	61.6

Table 3: Comparison of the quality of fit, characterized by the coefficient of determination R and Young's modulus estimates E between the power-law model (3.5) (see the last two columns in Tables 1 and 2) and the models (2.4), (2.10) and (2.11) studied earlier in [12, 13]. Here, R^2_{exp} is the coefficient of determination for particular explicit (exponential) models (2.4) and (2.11), E_{exp} is the estimate of Young's modulus for the explicit (exponential) models (2.4) and (2.11). Finally, R^2_{imp} is the coefficient of determination for the implicit model (2.10) and E_{imp} is the estimate of Young modulus for the implicit model (2.10).

4.1. Predicted bulk and shear responses of power law model

It follows from (3.1) and (3.2) that the bulk and shear response predicted by the power-law model have the following representations

$$\begin{aligned} |\operatorname{tr}\boldsymbol{\varepsilon}| &= \frac{1}{3K}(\frac{1}{\tau_0^2})^{\frac{s-2}{2}}(\tau_0^2+|\operatorname{tr}\boldsymbol{T}|^2)^{\frac{s-2}{2}}|\operatorname{tr}\boldsymbol{T}|,\\ |\boldsymbol{\varepsilon}^d| &= \frac{1}{2\mu}(\frac{3}{2\tau_0^2})^{\frac{q-2}{2}}(\frac{2}{3}\tau_0^2+|\boldsymbol{T}^d|^2)^{\frac{q-2}{2}}|\boldsymbol{T}^d|. \end{aligned} \tag{4.1}$$

In Figures 6, 7, 8, 9 we plot the predicted bulk and shear response for Gum metal, *Ti-30Nb-10Ta-5Zr*, *Ti-24Nb-4Zr-7.9Sn*, *Ti-30Nb-12Zr* respectively using the values for the material moduli listed in Table 1.

4.2. Discussion

The power law models are able to describe tensile loading behavior of Gum metal and other beta phase titanium alloys in the full range of nonlinear elastic response as can be seen from Figures 2, 3, 4, 5. It is also evident from

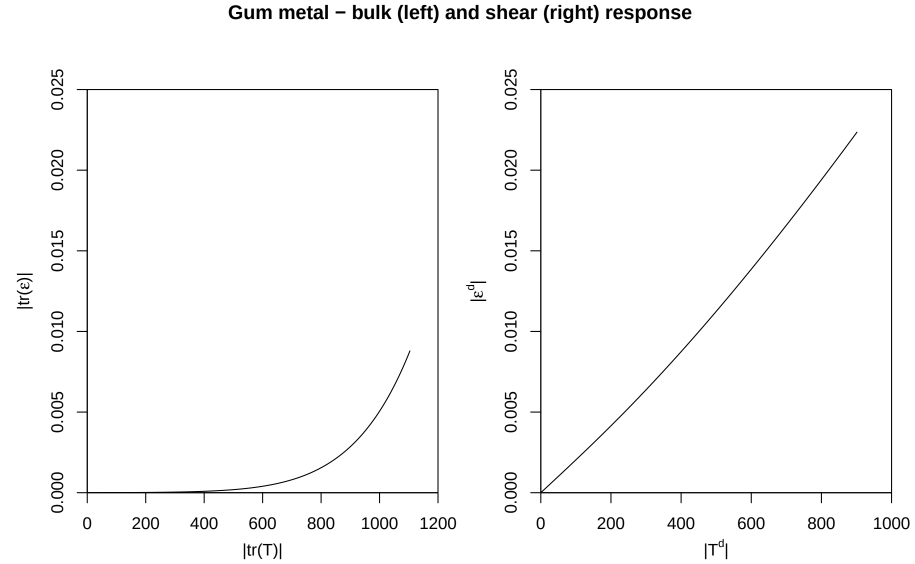

Figure 6: Bulk and shear response (4.1) of the model (3.1)–(3.2) for Gum metal.

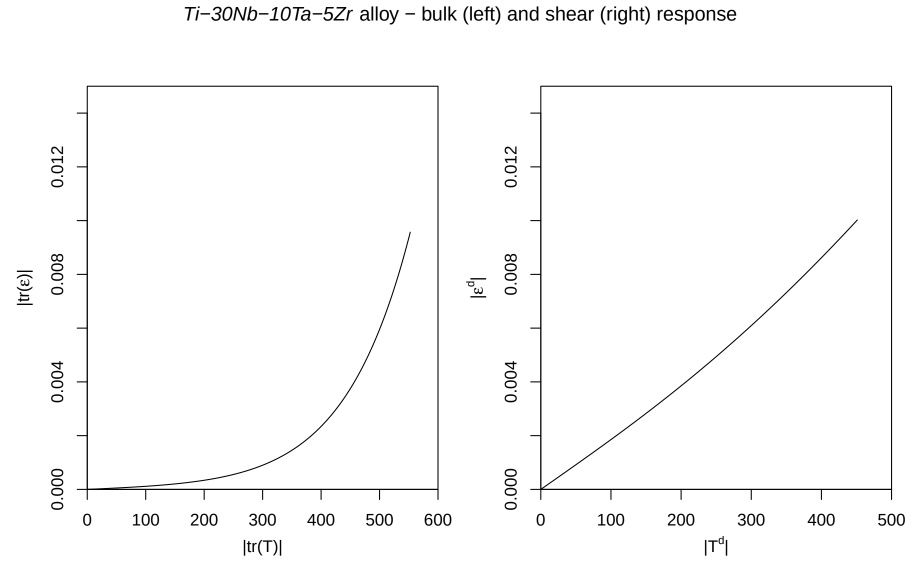

Figure 7: Bulk and shear response (4.1) of the model (3.1)–(3.2) for *Ti-30Nb-10Ta-5Zr* alloy.

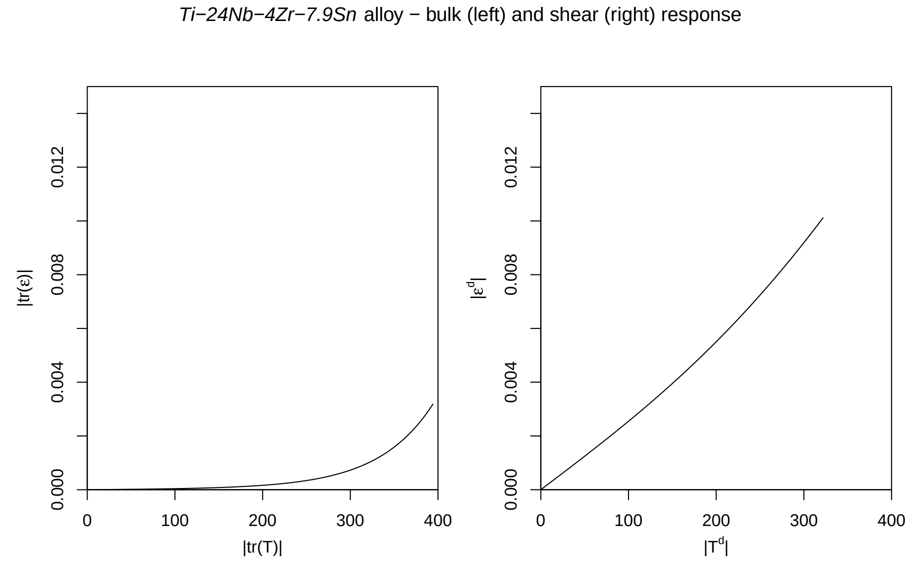

Figure 8: Bulk and shear response (4.1) of the model (3.1)–(3.2) for *Ti-24Nb-4Zr-7.9Sn* alloy.

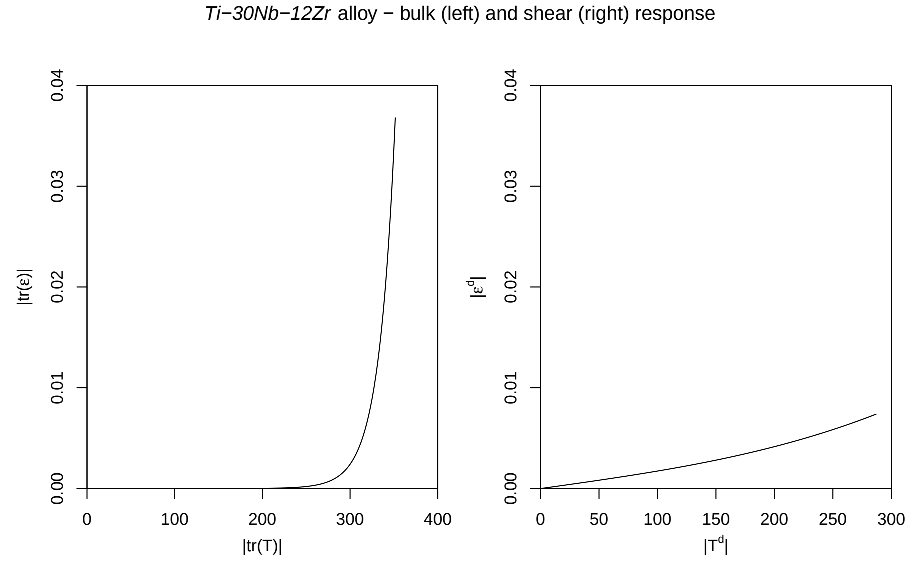

Figure 9: Bulk and shear response (4.1) of the model (3.1)–(3.2) for *Ti-30Nb-12Zr* alloy.

Tables 1, 3 that the power law model (3.1) outperforms or at least is as good as the existing models considered in [12] and [13] for describing tensile loading behavior of the beta phase titanium alloys. The coefficient of determination R^2 is very close to the ideal value of 1 for all the above mentioned models. Of course, one needs to consider more general deformations in order to validate the model, but such experiments are not available with regard to the metallic alloys being considered at this point in time.

All the beta titanium alloys that have been studied behave nonlinearly in their elastic regime. Thus it would be inappropriate to describe them using the linearized elastic model.

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