Relative entropies, dissipative solutions, and singular limits of complete fluid systems

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Abstract

We discuss the role of relative entropies in the analysis of complete fluid systems. The relative entropy, or rather relative energy functional measures the "distance" between a weak solution of a given system of equations and any other trajectory ranging in the same function space. We introduce a relative entropy functional for the full Navier-Stokes-Fourier system based on the ballistic free energy and discuss possible applications in the mathematical analysis of singular limits.

Keywords: Relative entropy, Navier-Stokes-Fourier system, low Mach number limit, inviscid limit.

1 Introduction

The method of relative entropies has been widely used in rather different areas of the modern theory of partial differential equations, see Berthelin and Vasseur [3], Carrillo [5], Dafermos [7], Saint-Raymond [31], among others. To introduce the concept of *relative entropy*, we consider an abstract (infinite-dimensional) dynamical system generated by the solution operator of the evolutionary problem

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t) = \mathcal{A}(t, U(t)),\ t > 0,\ U(0) = U_0, \tag{1.1}$$

where $\mathcal{A}$ is a (non-linear) generator. We suppose that the problem (1.1) admits a (not necessarily) unique solution U ranging in a Banach space X. Here, we suppose that U is a kind of generalized (weak) solution and the space X chosen as large as possible. In the applications studied in the present paper, the system (1.1) will be a system of partial differential equations governing the time evolution of a fluid,

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while U is its distributional solution. Besides, we introduce a target space for regular (smooth) solutions $Y \subset X$.

We say that a functional

$$\mathcal{E}\left(U \middle| V\right) : X \times Y,\ Y \subset X \to R \tag{1.2}$$

is a *relative entropy* for the problem (1.1) if $\mathcal{E}$ enjoys the following properties:

- **Distance property.** We have $\mathcal{E}(U|V) \geq 0$ and

$$\mathcal{E}\left(U \middle| V\right) = 0 \text{ only if } U \equiv V.$$

- **Lyapunov functional.** Let V be an *equilibrium solution* of the system (1.1), meaning

$$\mathcal{A}(t, V) = 0 \text{ for all } t.$$

  Then $V \in Y$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}\left(U(t) \middle| V\right) \leq 0 \tag{1.3}$$

  for any (weak) solution U of (1.1).

- **Gronwall inequality.** Let U be a (weak) solution of the system (1.1) ranging in the space X and V a more regular (strong) solution of the same problem ranging in the space Y. Then

$$\mathcal{E}\left(U(\tau) \middle| V(\tau)\right) \leq \mathcal{E}\left(U(0) \middle| V(0)\right) + c\int_0^\tau \mathcal{E}\left(U(t) \middle| V(t)\right)\ \mathrm{d}t \text{ for a.a. } \tau \geq 0. \tag{1.4}$$

Possessing a relative entropy provides a valuable piece of information concerning a given system of equations, in particular in the case when the latter is known to admit only global-in-time weak solutions - the situation typical for the Navier-Stokes system and related problems posed in the natural $3D$-topology, see Fefferman [13]. With a relative entropy at hand, it is possible to introduce the concept of *dissipative* solution and show the principle of *weak-strong uniqueness*. Specifically, the weak (dissipative) and strong solution coincide as long as the latter exists, meaning, the strong solutions are unique in the class of weak solutions - this is a direct consequence of (1.4). Another application of the relative entropy discussed in the present paper is the rigorous justification of several singular limits in fluid mechanics, in particular in the cases where viscosity becomes negligible.

The paper is organized as follows. In the first part, consisting in Sections 2 - 4 we introduce the concept of relative entropy and dissipative solutions to the Navier-Stokes-Fourier system describing the motion of a general compressible, viscous, and heat conducting fluid and compare it to the quantity introduced by Dafermos [7] in the context of hyperbolic conservation laws. Section 5 is devoted to the analysis of singular limits of the scaled problem by means of the method of relative entropies, in particular, the case of the inviscid incompressible limit. We present frequency localized Strichartz estimates for the acoustic equation and extend the result of [16] to more general physical domains.

2 Thermostatics, relative entropies

To begin, we review some basic concepts of continuum fluid mechanics. We suppose that the *state* of a fluid in thermodynamic *equilibrium* is fully determined by its *mass density* ϱ and the *absolute temperature* ϑ. Alternatively, we may also replace ϱ by the *specific volume* $V = 1/\varrho$ and ϑ by the *internal energy* e. The internal energy e, the *pressure* p, and the *entropy* s satisfy *Gibbs' equation*:

$$\vartheta Ds = De + pDV,\ V = \frac{1}{\varrho}. \tag{2.1}$$

In this section, we discuss relative entropies $\eta(\varrho, \vartheta | \tilde{\varrho}, \tilde{\vartheta})$ relating the thermostatic variables ϱ, ϑ to some reference values $\tilde{\varrho}$, $\tilde{\vartheta}$.

2.1 Thermodynamic stability

The concept of relative entropy in hyperbolic systems of conservation laws was proposed by Dafermos [7] in order to study the stability issues. Following [7], we consider first the standard entropy $s = s(V, e)$ expressed as a function of the specific volume V and the internal energy e. Furthermore, we impose the hypothesis of *thermodynamic stability*:

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0,\ \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0, \tag{2.2}$$

where the former condition expresses positive *compressibility* of the fluid, while the latter enforces positivity of the *specific heat at constant volume*. Both conditions are rather natural and form one of the main building blocks of the theory developed below.

Expressing the thermodynamic functions p, s, as well as the absolute temperature ϑ in terms of ϱ, e we deduce from (2.1), (2.2) that the mapping

$$(V, e) \mapsto -s(V, e) \text{ is convex in } (V, e),$$

cf. Bechtel, Rooney, and Forest [2]. Consequently, a natural candidate for the relative entropy evaluated in terms of the thermostatic variables V, e is the quantity η,

$$\eta \left(V, e \middle| \tilde{V}, \tilde{e} \right) = - \left(s(V, e) - \partial_V s(\tilde{V}, \tilde{e})(V - \tilde{V}) - \partial_e s(\tilde{V}, \tilde{e})(e - \tilde{e}) - s(\tilde{V}, \tilde{e}) \right).$$

Going back to the independent variables ϱ, ϑ and using Gibbs' relation (2.1) we obtain

$$\eta \left(\varrho, \vartheta \middle| \tilde{\varrho}, \tilde{\vartheta} \right) = - \left(s(\varrho, \vartheta) - \frac{p(\tilde{\varrho}, \tilde{\vartheta})}{\tilde{\vartheta}} \left(\frac{1}{\varrho} - \frac{1}{\tilde{\varrho}} \right) - \frac{1}{\tilde{\vartheta}} \Big(e(\varrho, \vartheta) - e(\tilde{\varrho}, \tilde{\vartheta}) \Big) - s(\tilde{\varrho}, \tilde{\vartheta}) \right),$$

which may be viewed as a "specific" relative entropy related to unit mass. For applications to conservation laws, it is more convenient to replace $\eta \approx \varrho\eta$, specifically we take

$$\eta \left(\varrho, \vartheta \middle| \tilde{\varrho}, \tilde{\vartheta} \right) = - \varrho \left(s(\varrho, \vartheta) - \frac{p(\tilde{\varrho}, \tilde{\vartheta})}{\tilde{\vartheta}} \left(\frac{1}{\varrho} - \frac{1}{\tilde{\varrho}} \right) - \frac{1}{\tilde{\vartheta}} \Big(e(\varrho, \vartheta) - e(\tilde{\varrho}, \tilde{\vartheta}) \Big) - s(\tilde{\varrho}, \tilde{\vartheta}) \right) \tag{2.3}$$

2.2 Ballistic free energy

In applications to dissipative equations like the Navier-Stokes system, we further modify the functional η by introducing:

$$\xi\left(\varrho,\vartheta\Big|\tilde{\varrho},\tilde{\vartheta}\right)=\tilde{\vartheta}\ \eta\left(\varrho,\vartheta\Big|\tilde{\varrho},\tilde{\vartheta}\right)$$

$$=\Big(\varrho e(\varrho,\vartheta)-\tilde{\vartheta}\varrho s(\varrho,\vartheta)\Big)-\Big(\tilde{\varrho}e(\tilde{\varrho},\tilde{\vartheta})-\tilde{\vartheta}\tilde{\varrho}s(\tilde{\varrho},\tilde{\vartheta})\Big)$$

$$+\left[\frac{p(\tilde{\varrho},\tilde{\vartheta})}{\tilde{\varrho}}+e(\tilde{\varrho},\tilde{\vartheta})-\tilde{\vartheta}s(\tilde{\varrho},\tilde{\vartheta})\right](\tilde{\varrho}-\varrho).$$

Consequently, using once more Gibbs'relation (2.1) we arrive at

$$\xi\left(\varrho,\vartheta\Big|\tilde{\varrho},\tilde{\vartheta}\right)=H_{\tilde{\vartheta}}(\varrho,\vartheta)-\frac{\partial H_{\tilde{\vartheta}}(\tilde{\varrho},\tilde{\vartheta})}{\partial\varrho}(\varrho-\tilde{\varrho})-H_{\tilde{\vartheta}}(\tilde{\varrho},\tilde{\vartheta}),\tag{2.4}$$

where we have introduced another thermodynamic potential called *ballistic free energy* (cf. Ericksen [11]),

$$H_{\tilde{\vartheta}}(\varrho,\vartheta)=\varrho\Big(e(\varrho,\vartheta)-\overline{\vartheta}s(\varrho,\vartheta)\Big),$$

see [17]. Note that ξ has the physical dimension of *energy* rather than *entropy.*

3 Fluids in motion

Up to now, we have considered fluids in thermodynamic equilibrium characterized by the thermostatic variables ϱ, ϑ. Now, we suppose that the fluid moves with a *macroscopic* velocity $\mathbf{u}=\mathbf{u}(t,x)$, which is a function of the time t and the spatial position x. In accordance with the commonly accepted principles of continuum thermodynamics, we assume that the state of the fluid at each instant t is still described by the density $\varrho=\varrho(t,x)$ and the absolute temperature $\vartheta=\vartheta(t,x)$. Thus the trio $[\varrho,\vartheta,\mathbf{u}]$ provides a full description of the fluid at any time and any spatial position of a given physical domain $\Omega\subset R^3$.

3.1 Navier-Stokes-Fourier system

Given the initial state of the fluid

$$\varrho(0,\cdot)=\varrho_0,\ \mathbf{u}(0,\cdot)=\mathbf{u}_0,\ \vartheta(0,\cdot)=\vartheta_0,\tag{3.1}$$

the *time evolution* of the state variables is described by means of the following *Navier-Stokes-Fourier system* that expresses the fundamental physical principles:

MASS CONSERVATION

$$\partial_t\varrho+\mathrm{div}_x(\varrho\mathbf{u})=0;\tag{3.2}$$

MOMENTUM BALANCE

$$\partial_t(\varrho\mathbf{u})+\mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u})+\nabla_x p(\varrho,\vartheta)=\mathrm{div}_x\mathbb{S}(\vartheta,\nabla_x\mathbf{u})+\varrho\mathbf{f};\tag{3.3}$$

ENERGY BALANCE

$$\partial_t \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) \right) + \mathrm{div}_x \left[\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) \right) \mathbf{u} + p(\varrho,\vartheta)\mathbf{u} - \mathbb{S}(\vartheta,\nabla_x\mathbf{u})\cdot\mathbf{u} \right]$$
$$+\mathrm{div}_x \mathbf{q}(\vartheta,\nabla_x\vartheta) = \varrho\mathbf{f}\cdot\mathbf{u}; \tag{3.4}$$

where $\mathbf{f}$ is an external force, $\mathbb{S}(\vartheta,\nabla_x\mathbf{u})$ is the *viscous stress tensor* here determined by

NEWTON'S LAW

$$\mathbb{S}(\vartheta,\nabla_x\mathbf{u}) = \mu(\vartheta)\left(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\right) + \eta(\vartheta)\mathrm{div}_x\mathbf{u}\mathbb{I}; \tag{3.5}$$

and $\mathbf{q}(\vartheta,\nabla_x\vartheta)$ is the heat flux given by

FOURIER'S LAW

$$\mathbf{q}(\vartheta,\nabla_x\vartheta) = -\kappa(\vartheta)\nabla_x\vartheta. \tag{3.6}$$

3.2 Physical domains, boundary conditions

In the case $\partial\Omega \neq \emptyset$, relevant boundary conditions must be prescribed. We focus on the domains with *impermeable* boundaries, both mechanically and thermally. Accordingly, we impose the boundary conditions

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0 \tag{3.7}$$

and

$$\mathbf{q}(\vartheta,\nabla_x\vartheta)\cdot\mathbf{n}|_{\partial\Omega} = 0. \tag{3.8}$$

In addition to (3.7), we suppose that the behavior of the fluid in the tangential direction to $\partial\Omega$ obeys

NAVIER'S SLIP BOUNDARY CONDITION

$$[\mathbb{S}(\vartheta,\nabla_x\mathbf{u})\cdot\mathbf{n}]_{\tan} + \beta[\mathbf{u}]_{\tan}|_{\partial\Omega} = 0, \tag{3.9}$$

where $\beta \in [0,\infty]$ plays the role of a *friction* coefficient. We focus on the two extremal situations where either $\beta = 0$ and (3.9) reduces to the *complete slip* boundary condition

$$[\mathbb{S}(\vartheta,\nabla_x\mathbf{u})\cdot\mathbf{n}]\times\mathbf{n}|_{\partial\Omega} = 0, \tag{3.10}$$

or $\beta = \infty$, for which (3.7), (3.9) give rise to the very common *no slip* condition

$$\mathbf{u}|_{\partial\Omega} = 0. \tag{3.11}$$

The boundary conditions (3.7), (3.8), supplemented with either (3.10) or (3.11), are *conservative* and give rise, by integrating (3.4), to

TOTAL ENERGY BALANCE

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) \right)\,\mathrm{d}x = \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x \tag{3.12}$$

at least if $\Omega \subset R^3$ is a bounded domain.

If $\Omega \subset R^3$ is unbounded, the far-field behavior of the state variables must be prescribed, for instance,

$$\varrho \to \varrho_\infty, \ \vartheta \to \vartheta_\infty, \ \mathbf{u} \to \mathbf{u}_\infty \text{ as } |x| \to \infty, \tag{3.13}$$

and the total energy balance (3.12) must be modified accordingly.

3.3 Equivalent formulation of the energy balance

The energy balance equation (3.4) is very often replaced by another balance law that is equivalent to (3.4) at least in the framework of *classical solutions* to the Navier-Stokes-Fourier system.

3.3.1 Thermal energy

Introducing the *specific heat at constant volume* (cf. (2.2))

$$c_V(\varrho, \vartheta) = \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta}$$

we may rewrite (3.4) in the form of

THERMAL ENERGY EQUATION

$$\varrho c_v(\varrho, \vartheta)\Big(\partial_t \vartheta + \mathbf{u} \cdot \nabla_x \vartheta\Big) - \mathrm{div}_x \Big(\kappa(\vartheta) \nabla_x \vartheta\Big) = \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \vartheta \frac{\partial p(\varrho, \vartheta)}{\partial \vartheta} \mathrm{div}_x \mathbf{u}, \tag{3.14}$$

where, of course, we have exploited several identities resulting from the remaining equations in the Navier-Stokes-Fourier system.

The formulation of the Navier-Stokes-Fourier system by means of the equations (3.2), (3.3), and (3.14) is frequently used in the literature, in particular, the nowadays standard existence theory in the framework of classical solutions developed by Matsumura and Nishida [28], [29], Tani [35], Valli [36], [37], Valli and Zajackowski [38] uses this setting.

3.3.2 Entropy equation and the Second law of thermodynamics

Unlike (3.4), the thermal energy equation (3.14) is not in a divergence form that is more convenient for the *weak formulation*, where the differential operators are typically transferred on suitable smooth *test functions.* To this end, it seems more convenient to use

ENTROPY PRODUCTION EQUATION

$$\partial_t (\varrho s(\varrho, \vartheta)) + \mathrm{div}_x (\varrho s(\varrho, \vartheta) \mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \right) = \sigma, \tag{3.15}$$

with the *entropy production rate*

$$\sigma = \frac{1}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right), \tag{3.16}$$

which can be obtained dividing (3.14) on ϑ and using the continuity equation.

In accordance with the Second law of thermodynamics, the entropy production rate σ must be non-negative. On the other hand, it is difficult to establish (3.16) in the framework of *weak solutions* to the Navier-Stokes-Fourier system. The problem seems to be of the same origin as its counterpart in the theory of incompressible fluid flows discussed by Duchon and Robert [9], Eyink [12], Nagasawa [30], Shvydkoy [32], or, in the context of inviscid incompressible fluids by DeLellis and Székelyhidi [8]. In other words, the weak solutions may, *hypothetically*, dissipate more kinetic energy than expressed by the quantity on the right-hand side of (3.16), specifically

$$\sigma \geq \frac{1}{\vartheta}\left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta}\right). \tag{3.17}$$

On the other hand, under the conservative boundary conditions specified in Section 3.2, the balance of the total energy (3.12) remains valid. Consequently, we may use the equations (3.2), (3.3), together with the entropy production equation (3.15), where σ satisfies (3.17), *and* the total energy balance (3.12) as a new formulation of the Navier-Stokes-Fourier system. It can be shown (see [15, Chapter 2]) that this new formulation is perfectly equivalent to the original system of equations, in particular the entropy production rate is given by (3.16), as soon as the state variables $[\varrho, \vartheta, \mathbf{u}]$ are smooth. As we will see below, the new formulation can be suitably adapted in the context of weak (distributional) solutions to obtain a mathematically tractable object.

4 Weak and dissipative solutions

In accordance with the previous discussion, one of possible *weak formulations* of the Navier-Stokes-Fourier system consists of the equation of continuity (3.2), the momentum equation (3.3), together with entropy production inequality (3.15), (3.17), supplemented with the total energy balance (3.12), where the derivatives as well as the boundary conditions are satisfied in the sense of distributions and their traces, see [15, Chapter 3] for details. Here, we introduce even more general class of the so-called *dissipative solutions* characterized by the satisfaction of the relative entropy inequality specified below.

4.1 Relative entropy

Motivated by the discussion in Section 2.2, specifically by formula (2.4), we introduce a relative entropy

$$\begin{aligned} &\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \,\middle|\, r, \Theta, \mathbf{U}\right) \\ &= \int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + H_\Theta(\varrho,\vartheta) - \frac{\partial H_\Theta(r,\Theta)}{\partial \varrho}(\varrho - r) - H_\Theta(r,\Theta)\right] \, \mathrm{d}x. \end{aligned} \tag{4.1}$$

If $[\varrho, \vartheta, \mathbf{u}]$ is a smooth solution of the Navier-Stokes-Fourier system, supplemented with the no-slip condition (3.11) or the complete slip condition (3.7), (3.10), and if $[r, \Theta, \mathbf{U}]$ is and arbitrary trio of smooth test functions satisfying

$$r > 0,\ \Theta > 0,\ \text{and } \mathbf{U}|_{\partial\Omega} = \ \text{ or } \mathbf{U}\cdot\mathbf{n}|_{\partial\Omega} = 0, \tag{4.2}$$

then it is a routine matter to check that the following *relative entropy inequality* holds:

$$
\begin{aligned}
&\left[\mathcal{E}\left(\varrho,\vartheta,\mathbf{u}\middle|r,\Theta,\mathbf{U}\right)\right]_{t=0}^{t=\tau}+\int_0^\tau\int_\Omega\frac{\Theta}{\vartheta}\left(\mathbb{S}(\vartheta,\nabla_x\mathbf{U}):\nabla_x\mathbf{u}-\frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\right)\,\mathrm{d}x\,\mathrm{d}t \\
&\leq\int_0^\tau\int_\Omega\Big(\varrho(\mathbf{U}-\mathbf{u})\cdot\partial_t\mathbf{U}+\varrho(\mathbf{U}-\mathbf{u})\otimes\mathbf{u}:\nabla_x\mathbf{U}-p(\varrho,\vartheta)\mathrm{div}_x\mathbf{U}\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad+\int_0^\tau\int_\Omega\Big(\mathbb{S}(\vartheta,\nabla_x\mathbf{u}):\nabla_x\mathbf{U}+\varrho\mathbf{f}\cdot(\mathbf{u}-\mathbf{U})\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad-\int_0^\tau\int_\Omega\Big(\varrho\Big(s(\varrho,\vartheta)-s(r,\Theta)\Big)\partial_t\Theta+\varrho\Big(s(\varrho,\vartheta)-s(r,\Theta)\Big)\mathbf{u}\cdot\nabla_x\Theta\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad+\int_0^\tau\int_\Omega\frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)}{\vartheta}\cdot\nabla_x\Theta\,\mathrm{d}x\,\mathrm{d}t \\
&\quad+\int_0^\tau\int_\Omega\left(\left(1-\frac{\varrho}{r}\right)\partial_t p(r,\Theta)-\frac{\varrho}{r}\mathbf{u}\cdot\nabla_x p(r,\Theta)\right)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
\tag{4.3}
$$

for a.a. $\tau\in[0,T]$.

Now, the crucial observation exploited in [17] is that the relative entropy inequality (4.3) remains valid also for any *weak solution* $[\varrho,\vartheta,\mathbf{U}]$ as long as the test functions $[r,\Theta,\mathbf{U}]$ are sufficiently smooth and satisfy the compatibility condition (4.2).

4.2 Dissipative solutions

Following the idea of DiPerna and Lions [25] we say that $[\varrho,\vartheta,\mathbf{u}]$ is a *dissipative solution* of the Navier-Stokes-Fourier system if

$$\varrho\in L^\infty(0,T;L^p(\Omega))\text{ for a certain }p>1,\ \varrho\geq 0\text{ a.a. in }(0,T)\times\Omega,$$

$$\vartheta\in L^\infty(0,T;L^p(\Omega))\cap L^r(0,T;W^{1,r}(\Omega))\text{ for certain }q,r>1,\ \vartheta>0\text{ a.a. in }(0,T)\times\Omega,$$

$$\mathbf{u}\in L^s(0,T;W^{1,s}(\Omega;R^3))\text{ for a certain }s>1,\ \mathbf{u}|_{\partial\Omega}=0\text{ or }\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega}=0,$$

and the relative entropy inequality (4.3) holds for any trio of smooth test functions $[r,\Theta,\mathbf{U}]$ satisfying (4.2). Of course, the exponents p, q, r, and s are not arbitrary and must be adjusted so that all integrals appearing in (4.3) make sense. This issue will be discussed in detail in the following part of the paper.

4.3 Existence theory

The main advantage of the weak formulation of the Navier-Stokes-Fourier system based on the entropy production balance discussed in Section 3.3.2 is that the resulting problem is mathematically tractable, specifically, we can establish an existence theory of global-in-time solutions in the spirit of Leray's seminal paper [24].

4.3.1 Hypotheses

In order to present the main existence result in the framework of weak solutions, the class of thermodynamic functions p, e, and s as well as the transport coefficient μ, η and κ must be restricted.

To begin, we assume that the pressure p obeys a state equation in the form

$$p(\varrho, \vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4, \ a > 0, \tag{4.4}$$

with $P \in C^1[0, \infty)$. The first expression on the right-hand side is a general pressure of a monoatomic gas, while the second one accounts for the effect of radiation, see Eliezer, Ghatak, and Hora [10]. The reader may consult [15, Chapter 1] for details concerning the physical background of (4.4) as well as the other hypotheses introduced below.

The specific internal energy will be taken in the form

$$e(\varrho, \vartheta) = \frac{3}{2}\frac{\vartheta^{5/2}}{\varrho} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{\varrho}\vartheta^4, \tag{4.5}$$

and

$$s(\varrho, \vartheta) = S\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{4a}{3}\frac{\vartheta^3}{\varrho}, \tag{4.6}$$

where

$$S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z^2}. \tag{4.7}$$

In accordance with the hypothesis of thermodynamic stability, we further suppose that

$$P'(Z) > 0 \text{ for any } Z \geq 0, \ \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} > 0 \text{ for any } Z > 0, \tag{4.8}$$

and

$$\lim_{Z\to\infty} \frac{P(Z)}{Z^{5/3}} = p_\infty > 0. \tag{4.9}$$

Finally, we impose technical but physically grounded hypotheses (cf. [15, Chapter 1])

$$P(0) = 0, \ \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} < c \text{ for all } Z > 0. \tag{4.10}$$

The transport coefficients μ, η, and κ are continuously differentiable for $\vartheta \in [0, \infty)$ satisfying

$$\underline{\mu}(1 + \vartheta^\Lambda) \leq \mu(\vartheta) \leq \overline{\mu}(1 + \vartheta^\Lambda), \ |\mu'(\vartheta)| < c \text{ for all } \vartheta \in [0, \infty) \text{ for some } \frac{2}{5} < \Lambda \leq 1, \tag{4.11}$$

$$0 \leq \eta(\vartheta) \leq \overline{\eta}(1 + \vartheta^\Lambda) \text{ for all } \vartheta \in [0, \infty), \tag{4.12}$$

$$\underline{\kappa}(1 + \vartheta^3) \leq \kappa(\vartheta) \leq \overline{\kappa}(1 + \vartheta^3) \text{ for all } \vartheta \in [0, \infty). \tag{4.13}$$

4.3.2 Global-in-time existence

Having specified the basic hypotheses, we are ready to state the following global-in-time existence result for the Navier-Stokes-Fourier system in the framework of weak solutions, see [15, Theorem 3.1].

Theorem 4.1 *Let $\Omega \subset R^3$ be a bounded domain of class $C^{2+\nu}$, $\nu > 0$. Assume that the initial data satisfy*

$$\varrho_0 \in L^\infty(\Omega),\ \vartheta_0 \in L^\infty(\Omega), (\varrho\mathbf{u})_0 \in L^\infty(\Omega; R^3),\ \varrho_0 > 0,\ \vartheta_0 > 0 \text{ a.a. in } \Omega,$$

and let $\mathbf{f} \in L^\infty((0,T) \times \Omega; R^3)$ be given. Let the functions p, e, s and the transport coefficients μ, η, and κ satisfy the hypotheses (4.4 - 4.13).

Then the Navier-Stokes-Fourier system admits a weak solution $[\varrho, \vartheta, \mathbf{u}]$ in the set $(0,T) \times \Omega$ for any $T > 0$.

4.3.3 Weak-strong uniqueness and regularity criterion

As observed in [17], any weak solution satisfies the relative entropy inequality (4.3). This fact can be used for deriving a version of the Gronwall inequality (1.4), in particular, the weak and strong solutions emanating from the same initial data coincide as long as the latter exists. This is the weak-strong uniqueness property shown in [17, Theorem 2.1]:

Theorem 4.2 *In addition to the hypotheses of Theorem 4.1 suppose that the initial data belong to the class:*

$$\varrho_0,\ \vartheta_0 \in W^{3,2}(\Omega),\ \mathbf{u}_0 \in W^{3,2}(\Omega; R^3). \tag{4.14}$$

Let $[\varrho, \vartheta, \mathbf{u}]$ be the weak solution of the Navier-Stokes-Fourier system, the existence of which is guaranteed by Theorem 4.1, and let $[\tilde\varrho, \tilde\vartheta, \tilde{\mathbf{u}}]$ be a strong solution of the same problem belonging to the class

$$\tilde\varrho,\ \tilde\vartheta \in C([0,T]; W^{3,2}(\Omega)),\ \tilde{\mathbf{u}} \in C([0,T]; W^{3,2}(\Omega; R^3)),$$

$$\tilde\vartheta \in L^2(0,T; W^{4,2}(\Omega)),\ \partial_t\tilde\vartheta \in L^2(0,T; W^{2,2}(\Omega)),$$

$$\tilde{\mathbf{u}} \in L^2(0,T; W^{4,2}(\Omega; R^3)),\ \partial_t\tilde{\mathbf{u}} \in L^2(0,T; W^{2,2}(\Omega; R^3)),$$

and emanating from the same initial data.

Then $\varrho = \tilde\varrho$, $\vartheta = \tilde\vartheta$, and $\mathbf{u} = \tilde{\mathbf{u}}$ in $[0,T]$.

Note that local-in-time strong solutions in the afore-mentioned class were constructed by Valli [36], [37], Valli and Zajackowski [38]. Since the proof uses only the relative entropy inequality, the same result is valid in the class of dissipative solutions.

Finally, we report a conditional regularity result in the spirit of Beale, Kato, and Majda [1], see [18, Theorem 2.1]:

Theorem 4.3 *In addition to the hypotheses of Theorem 4.1 suppose that the initial data belong to the regularity class (4.14) and satisfy the compatibility conditions:*

$$\nabla_x\vartheta_0 \cdot \mathbf{n}|_{\partial\Omega} = \mathbf{u}_0|_{\partial\Omega} = 0,\ \nabla_x p(\varrho_0, \vartheta_0)|_{\partial\Omega} = \mathrm{div}_x \mathbb{S}(\vartheta_0, \nabla_x\mathbf{u}_0) + \varrho_0 f|_{\partial\Omega}. \tag{4.15}$$

Let $[\varrho, \vartheta, \mathbf{u}]$ be a weak (dissipative) solution of the Navier-Stokes-Fourier system satisfying

$$\text{ess} \sup_{(t,x)\in(0,T)\times\Omega} |\nabla_x \mathbf{u}(t,x)| < \infty.$$

Then $[\varrho, \vartheta, \mathbf{u}]$ is a classical solution in the open space-time cylinder $(0,T)\times\Omega$.

The reader will have noticed that the compatibility conditions (4.15) reflex the no-slip boundary condition for the velocity. The same result, with an obvious modification, applies to a general Navier slip boundary condition.

5 Singular limits

Singular limits are closely related to scale analysis of differential equations - an efficient tool used both theoretically and in numerical experiments to reduce the undesirable and mostly unnecessary complexity of the underlying physical system. The Navier-Stokes-Fourier system, in the entropy formulation, can be written in the dimensionless form:

$$\mathrm{Sr}\ \partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{5.1}$$

$$\mathrm{Sr}\ \partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u}\otimes \mathbf{u}) + \frac{1}{\mathrm{Ma}^2}\nabla_x p = \frac{1}{\mathrm{Re}}\mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) + \frac{1}{\mathrm{Fr}^2}\varrho \nabla_x F, \tag{5.2}$$

$$\mathrm{Sr}\ \partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \frac{1}{\mathrm{Pe}}\mathrm{div}_x\Big(\frac{\mathbf{q}(\vartheta, \nabla_x\vartheta)}{\vartheta}\Big) = \sigma, \tag{5.3}$$

$$\mathrm{Sr}\ \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Big(\frac{\mathrm{Ma}^2}{2}\varrho|\mathbf{u}|^2 + \varrho e - \frac{\mathrm{Ma}^2}{\mathrm{Fr}^2}\varrho F\Big)\mathrm{d}x = 0, \tag{5.4}$$

with the scaled entropy production rate

$$\sigma \geq \frac{1}{\vartheta}\Big(\frac{\mathrm{Ma}^2}{\mathrm{Re}}\mathbb{S} : \nabla_x \mathbf{u} - \frac{1}{\mathrm{Pe}}\frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\Big), \tag{5.5}$$

where we have taken the potential driving force $\mathbf{f} = \nabla_x F(x)$.

The dimensionless *characteristic numbers* appearing in the preceding system are defined as follows, see Klein et al. [23]:

Symbol	Definition	Name
Sr	$L_{\mathrm{ref}}/(T_{\mathrm{ref}}U_{\mathrm{ref}})$	Strouhal number
Ma	$U_{\mathrm{ref}}/\sqrt{p_{\mathrm{ref}}/\varrho_{\mathrm{ref}}}$	Mach number
Re	$\varrho_{\mathrm{ref}}U_{\mathrm{ref}}L_{\mathrm{ref}}/\mu_{\mathrm{ref}}$	Reynolds number
Fr	$U_{\mathrm{ref}}/\sqrt{L_{\mathrm{ref}}f_{\mathrm{ref}}}$	Froude number
Pe	$p_{\mathrm{ref}}L_{\mathrm{ref}}U_{\mathrm{ref}}/(\vartheta_{\mathrm{ref}}\kappa_{\mathrm{ref}})$	Péclet number

Here L_{ref} stands for the characteristic length, T_{ref} is the characteristic time, and U_{ref} is the characteristic velocity.

5.1 Inviscid incompressible limits

In many real world applications, in particular in meteorology, the fluid motion is rather slow, and, at the same time, the transport coefficients are small. This the situation corresponding to the choice:

$$\mathrm{Sr} = 1,\ \mathrm{Ma} = \varepsilon,\ \mathrm{Re} = \varepsilon^{-a},\ \mathrm{Pe} = \varepsilon^{-b},\ a, b > 0,$$

where $\varepsilon \to 0$ is a small parameter. Moreover, for the sake of simplicity, we set $F = 0$.

The initial data are *ill-prepared*, specifically,

$$\varrho(0, \cdot) = \varrho_{0,\varepsilon} = \overline{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon},\ \vartheta(0, \cdot) = \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon},\ \mathbf{u}(0, \cdot) = \mathbf{u}_{0,\varepsilon}, \tag{5.6}$$

where $\overline{\varrho}$, $\overline{\vartheta}$ are positive constants, and the perturbations $\varrho^{(1)}_{0,\varepsilon}$, $\vartheta^{(1)}_{0,\varepsilon}$ are allowed to be large.

For $[\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon]$ a family of solutions to the scaled Navier-Stokes-Fourier system, we may anticipate that

$$\varrho_\varepsilon \to \overline{\varrho},\ \vartheta_\varepsilon \to \overline{\vartheta},\ \mathbf{u}_\varepsilon \to \mathbf{v},\ \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to T, \tag{5.7}$$

where the limit velocity $\mathbf{v}$ and the temperature deviation T satisfy

$$\mathrm{div}_x \mathbf{v} = 0, \tag{5.8}$$

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = 0, \tag{5.9}$$

$$\partial_t T + \mathbf{v} \cdot \nabla_x T = 0, \tag{5.10}$$

cf. [16]. The system (5.8), (5.9) is nothing other than the incompressible Euler system known to possess a local in time strong solution for any regular initial data. The equation (5.10) represents pure transport of the temperature deviation.

5.2 Mathematical analysis

A rigorous justification of the limit (5.7), carried over in [16], is rather technical and demonstrates the strength of the method of relative entropies. Results of this type for a simpler compressible Navier-Stokes system (without temperature) were obtained by Masmoudi [26], [27].

The leading idea of the analysis is rather simple, namely, take the trio

$$\mathbf{U} = \nabla_x \Phi_\varepsilon + \mathbf{v},\ r = \overline{\varrho} + \varepsilon R_\varepsilon,\ \Theta = \overline{\vartheta} + \varepsilon T_\varepsilon$$

as test functions in the relative entropy inequality (4.3). The function $\mathbf{v}$ is the solution of the Euler system (5.8), (5.9), while R_ε, T_ε, and Φ_ε solve the *acoustic-transport system*:

$$\varepsilon \partial_t (\alpha R_\varepsilon + \beta T_\varepsilon) + \omega \Delta \Phi_\varepsilon = 0, \tag{5.11}$$

$$\varepsilon \partial_t \nabla_x \Phi_\varepsilon + \nabla_x (\alpha R_\varepsilon + \beta T_\varepsilon) = 0, \tag{5.12}$$

$$\partial_t (\delta T_\varepsilon - \beta R_\varepsilon) + \mathbf{U}_\varepsilon \cdot \nabla_x (\delta T_\varepsilon - \beta R_\varepsilon) + (\delta T_\varepsilon - \beta R_\varepsilon) \mathrm{div}_x \mathbf{U}_\varepsilon = 0, \tag{5.13}$$

with the constants

$$\alpha = \frac{1}{\overline{\varrho}} \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho}, \ \beta = \frac{1}{\overline{\varrho}} \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta}, \ \delta = \overline{\varrho} \frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta}, \ \omega = \overline{\varrho} \left(\alpha + \frac{\beta^2}{\delta} \right).$$

For $Z_\varepsilon = \alpha R_\varepsilon + \beta T_\varepsilon$, the system (5.11), (5.12) can be written in the form of

ACOUSTIC EQUATION

$$\varepsilon \partial_t Z_\varepsilon + \omega \Delta \Phi_\varepsilon = 0, \ \varepsilon \partial_t \Phi_\varepsilon + Z_\varepsilon = 0. \tag{5.14}$$

The system (5.14) governs the propagation of acoustic waves supposed to "disappear" in the incompressible limit. The principal idea of the analysis is therefore to show that

$$\Phi_\varepsilon \to 0, \ Z_\varepsilon \to 0 \text{ in some sense,} \tag{5.15}$$

and to recover the limit equation (5.10) from (5.13). In order to show (5.15), we use the dispersive (Strichartz type) estimates discussed in the next section.

5.3 Propagation of acoustic waves

We consider a fluid flow confined to a general (unbounded) domain $\Omega \subset R^3$, where the velocity $\mathbf{u}_\varepsilon$ satisfies the complete slip boundary conditions (3.7), (3.10). Accordingly, the *acoustic potential* Φ_ε appearing in (5.14) satisfies the homogeneous *Neumann boundary condition*

$$\nabla_x \Phi_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{5.16}$$

Note that the complete slip boundary conditions are also necessary in order to avoid the up to now unsurmountable difficulties connected with the presence of a boundary layer in the inviscid limit, see e.g. Kato [21].

5.3.1 Frequency localized Strichartz estimates

A short inspection of the solution formula associated to the acoustic problem (5.15), (5.16) reveals that solutions may be expressed by means of the wave propagator

$$h \mapsto \exp\left(\pm \mathrm{i} \frac{t}{\varepsilon} \sqrt{-\Delta_N} \right) [h],$$

where Δ_N denotes the L^2-realization of the Neumann Laplacean on Ω. Our goal will be to show

$$\int_{-\infty}^{\infty} \left\| G(-\Delta_N) \exp\left(\pm \mathrm{i} \sqrt{-\Delta_N} t \right) [h] \right\|^p_{L^q(\Omega)} \leq c(G) \|h\|^p_{H^{1,2}(\Omega)}, \ \frac{1}{2} = \frac{1}{p} + \frac{3}{q}, \ q < \infty, \tag{5.17}$$

where $G \in C_c^\infty(0, \infty)$, and where $H^{1,2}$ denotes the homogeneous Sobolev space. The estimate (5.17) can be viewed as frequency localized Strichartz estimates, cf. [34]. They provide the necessary piece of information in order to show the (local) decay of acoustic waves claimed in (5.15), cf. [16]. In the remaining part of this section, we show (5.17) by means of the arguments of developed by Burq [4], Smith and Sogge [33]. To this end, we suppose that $\Omega = R^3 \setminus K$ is a regular *exterior* domain, K a compact set in R^3 with a smooth boundary.

5.3.2 Dispersive estimates for the free Laplacean

We recall the standard *Strichartz estimates* for the free Laplacean Δ in R^3,

$$\int_{-\infty}^{\infty} \left\| \exp\left(\pm\mathrm{i}\sqrt{-\Delta}t\right)[h] \right\|^p_{L^q(R^3)} \ \mathrm{d}t \leq \|h\|^p_{H^{1,2}(R^3)},\ \frac{1}{2} = \frac{1}{p} + \frac{3}{q},\ q < \infty, \tag{5.18}$$

see Keel and Tao [22], Strichartz [34].

In addition, the free Laplacean satisfies the local energy decay in the form

$$\int_{-\infty}^{\infty} \left\| \varphi \exp\left(\pm\mathrm{i}\sqrt{-\Delta}t\right)[h] \right\|^2_{H^{\alpha,2}(R^3)} \ \mathrm{d}t \leq c(\varphi)\|h\|^2_{H^{\alpha,2}(R^3)},\ \alpha \leq \frac{3}{2}, \tag{5.19}$$

see Smith and Sogge [33, Lemma 2.2].

5.3.3 Frequency localized estimates

To show (5.17), we decompose the function

$$U(t,\cdot) = G(-\Delta_N)\exp\left(\pm\mathrm{i}\sqrt{-\Delta_N}t\right)[h] = \exp\left(\pm\mathrm{i}\sqrt{-\Delta_N}t\right)G(-\Delta_N)[h]$$

as

$$U = v + w,\ v = \chi U,\ w = (1-\chi)U,$$

where

$$\chi \in C^\infty_c(R^3),\ 0 \leq \chi \leq 1,\ \chi(x) = 1 \text{ for } |x| \leq R.$$

Here R is chosen so large that the complement K of Ω is contained in the ball of the radius R.

Thus we write

$$w = w^1 + w^2,$$

where w^1 solves the homogeneous wave equation

$$\partial^2_{t,t} w^1 - \Delta w^1 = 0 \text{ in } R^3,$$

supplemented with the initial conditions

$$w^1(0) = (1-\chi)G(-\Delta_N)[h],\ \partial_t w^1(0) = \pm\mathrm{i}(1-\chi)\sqrt{-\Delta_N}G(-\Delta_N)[h],$$

while

$$\partial^2_{t,t} w^2 - \Delta w^2 = F \text{ in } R^3,$$

$$w^2(0) = \partial_t w^2(0) = 0,$$

with

$$F = -\nabla_x \chi \nabla_x U - U\Delta\chi.$$

As a direct consequence of the standard Strichartz estimates (5.18), we obtain

$$\int_{-\infty}^{\infty} \left\| w^1 \right\|^p_{L^q(R^3)} \ \mathrm{d}t \leq c(G)\|h\|^p_{H^{1,2}(R^3)},\ \frac{1}{2} = \frac{1}{p} + \frac{3}{q},\ q < \infty. \tag{5.20}$$

As the next step, we use Duhamel's formula to deduce

$$w^2(\tau,\cdot) = \frac{1}{2\sqrt{-\Delta}}\left[\exp\left(\mathrm{i}\sqrt{-\Delta}\tau\right)\int_0^\tau \exp\left(-\mathrm{i}\sqrt{-\Delta}s\right)[\eta^2 F(s)] \ \mathrm{d}s\right]$$

$$-\frac{1}{2\sqrt{-\Delta}}\left[\exp\left(-\mathrm{i}\sqrt{-\Delta}\tau\right)\int_0^\tau \exp\left(\mathrm{i}\sqrt{-\Delta}s\right)[\eta^2 F(s)]\ \mathrm{d}s\right],$$

with

$$\eta \in C_c^\infty(R^3),\ 0 \leq \eta \leq 1,\ \eta = 1 \text{ on } \mathrm{supp}[F].$$

Now, similarly to Burq [4], we use the following result of Christ and Kiselev [6]:

Lemma 5.1 *Let X and Y be Banach spaces and assume that $K(t,s)$ is a continuous function taking its values in the space of bounded linear operators from X to Y. Set*

$$\mathcal{T}[f](t) = \int_a^b K(t,s)f(s)\ \mathrm{d}s,\ \mathcal{W}[f](t) = \int_a^t K(t,s)f(s)\ \mathrm{d}s,$$

where

$$0 \leq a \leq b \leq \infty.$$

Suppose that

$$\|\mathcal{T}[f]\|_{L^p(a,b;Y)} \leq c_1 \|f\|_{L^r(a,b;X)}$$

for certain

$$1 \leq r < p \leq \infty.$$

Then

$$\|\mathcal{W}[f]\|_{L^p(a,b;Y)} \leq c_2 \|f\|_{L^r(a,b;X)},$$

where c_2 depends only on c_1, p, and r.

We apply Lemma 5.1 to

$$X = L^2(R^3),\ Y = L^q(R^3),\ q < \infty,\ \frac{1}{2} = \frac{1}{p} + \frac{3}{q},\ r = 2,$$

and

$$f = F,\ K(t,s)[F] = \frac{1}{\sqrt{-\Delta}}\exp\left(\pm\mathrm{i}\sqrt{-\Delta}(t-s)\right)[\eta^2 F].$$

Writing

$$\int_0^\infty K(t,s)F(s)\ \mathrm{d}s = \exp\left(\pm\mathrm{i}\sqrt{-\Delta}t\right)\frac{1}{\sqrt{-\Delta}}\int_0^\infty \exp\left(\mp\mathrm{i}\sqrt{-\Delta}s\right)[\chi^2 F(s)]\ \mathrm{d}s,$$

we have to show, in accordance with the Strichartz estimates (5.18), that

$$\left\|\int_0^\infty \exp\left(\pm\mathrm{i}\sqrt{-\Delta}s\right)[\eta^2 F(s)]\ \mathrm{d}s\right\|_{L^2(R^3)} \leq c\|F\|_{L^2(0,\infty;L^2(R^3))}. \tag{5.21}$$

On the other hand, however,

$$\left\|\int_0^\infty \exp\left(\pm\mathrm{i}\sqrt{-\Delta}s\right)[\chi^2 F(s)]\ \mathrm{d}s\right\|_{L^2(R^3)}$$

$$= \sup_{\|v\|_{L^2(R^3)}\leq 1}\int_0^\infty \left\langle \exp\left(\pm\mathrm{i}\sqrt{-\Delta}s\right)[\chi^2 F(s)];v\right\rangle\ \mathrm{d}s$$

$$= \sup_{\|v\|_{L^2(R^3)}\leq 1}\int_0^\infty \left\langle \chi F(s);\chi\exp\left(-\mathrm{i}\sqrt{-\Delta}s\right)[v]\right\rangle\ \mathrm{d}s;$$

whence the desired conclusion (5.21) follows from the local energy decay estimates (5.19). As the norm of F is bounded, we may infer that

$$\int_{-\infty}^{\infty} \|w^2\|^p_{L^q(R^3)} \ \mathrm{d}t \leq c(G)\|h\|^p_{H^{1,2}(R^3)},\ \frac{1}{2} = \frac{1}{p} + \frac{3}{q},\ q < \infty. \tag{5.22}$$

Finally, since $v = \chi U$ is compactly supported, we deduce form the standard elliptic regularity for $-\Delta_N$ that

$$\int_0^{\infty} \|v\|^2_{L^q(\Omega)} \ \mathrm{d}t \leq c(G)\|h\|^2_{H^{1,2}(\Omega)}; \tag{5.23}$$

while, by virtue of the standard energy estimates,

$$\sup_{t>0} \|v(t,\cdot)\|_{L^q(\Omega)} \leq c(G)\|h\|_{H^{1,2}(\Omega)}. \tag{5.24}$$

where $q < \infty$ is the same as in (5.18). Interpolating (5.23), (5.24), we get the desired conclusion (5.17).

To conclude this section, we note that similar estimates on *exterior* domain can be obtained by the method of Isozaki [19]. On the other hand, the present method seems more versatile and applicable to a larger class of unbounded domains, for instance to a perturbed half-space or wave operators with non-constant coefficients arising in the stratified limits, cf. [14].

5.4 Singular limit - main result

In order to formulate our main result, several remarks are in order. In agreement with the previous section, we consider the fluid confined to an unbounded domain $\Omega \subset R^3$ with a compact and regular boundary $\partial\Omega$, on which the velocity field $\mathbf{u}_\varepsilon$ satisfies the complete slip boundary conditions (3.7), (3.10). Moreover, the initial data are taken in the form (5.6), where

$$\varrho^{(1)}_{0,\varepsilon} \to \varrho^{(1)}_0 \text{ in } L^2(\Omega),\ \vartheta^{(1)}_{0,\varepsilon} \to \vartheta^{(1)}_0 \text{ in } L^2(\Omega),\ \|\varrho^{(1)}_{0,\varepsilon}\|_{L^\infty(\Omega)},\ \|\varrho^{(1)}_{0,\varepsilon}\|_{L^\infty(\Omega)} \leq c, \tag{5.25}$$

and

$$\mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \text{ in } L^2(\Omega; R^3). \tag{5.26}$$

Since the spatial domain is un bounded, the far field conditions must be prescribed. In agreement with (5.25), (5.26), we take

$$\varrho_\varepsilon \to \overline{\varrho},\ \vartheta_\varepsilon \to \overline{\vartheta},\ \mathbf{u}_\varepsilon \to 0 \text{ as } |x| \to \infty. \tag{5.27}$$

Accordingly, the natural function spaces the solution is sought in read

$$\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \in L^\infty(0,T; L^{5/3} + L^2(\Omega)),\ \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \in L^\infty(0,T; L^4 + L^2(\Omega)), \tag{5.28}$$

and, if we fix $\Lambda = 1$ in the hypotheses (4.11 - 4.13),

$$\vartheta_\varepsilon \in L^2(0,T; W^{1,2}(\Omega)),\ \mathbf{u}_\varepsilon \in L^2(0,T; W^{1,2}(\Omega; R^3)). \tag{5.29}$$

Finally, we denote

$$\mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0],\ \text{where } \mathbf{H} \text{ denotes the standard Helmholtz projection,}$$

and suppose that

$$\mathbf{v}_0 \in W^{k,2}(\Omega; R^3), \ k > \frac{5}{2}.$$

Our result concerning the inviscid, incompressible limit of the Navier-Stokes-Fourier system will be formulated directly in terms of the dissipative solutions, meaning the functions $[\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon]$ satisfying the relative entropy inequality (4.3). Since the domain Ω is unbounded, we have to modify the space of test functions accordingly, namely

$$r > 0, \ \Theta > 0, \ \text{and } \mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ r - \overline{\varrho}, \Theta - \overline{\vartheta}, \ \mathbf{U} \text{ in } C_c^\infty([0,T] \times \overline{\Omega}).$$

Combining the dispersive estimates obtained in Section 5.3 with the method of [16] we obtain the following generalization of [16, Theorem 3.1]:

Theorem 5.2 *Let $\Omega \subset R^3$ be an unbounded domain with a compact boundary of class $C^{2+\nu}$. Suppose that the thermodynamic functions p, e, and s and the transport coefficients μ, η, κ satisfy the hypotheses (4.4 - 4.13), with $\Lambda = 1$. Let*

$$b > 0, \ \frac{10}{3} > a > 0.$$

Furthermore, suppose that the initial data (5.6) are chosen in such a way that

$$\{\varrho_{0,\varepsilon}^{(1)}\}_{\varepsilon>0}, \ \{\vartheta_{0,\varepsilon}^{(1)}\}_{\varepsilon>0} \ \text{are bounded in } L^2 \cap L^\infty(\Omega), \ \varrho_{0,\varepsilon}^{(1)} \to \varrho_0^{(1)}, \ \vartheta_{0,\varepsilon}^{(1)} \to \vartheta_0^{(1)} \ \text{in } L^2(\Omega),$$

and

$$\{\mathbf{u}_{0,\varepsilon}\}_{\varepsilon>0} \ \text{is bounded in } L^2(\Omega; R^3), \ \mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \ \text{in } L^2(\Omega; R^3),$$

where

$$\varrho_0^{(1)}, \ \vartheta_0^{(1)} \in W^{1,2} \cap W^{1,\infty}(\Omega), \ \mathbf{H}[\mathbf{u}_0] = \mathbf{v}_0 \in W^{k,2}(\Omega; R^3) \ \text{for a certain } k > \frac{5}{2}.$$

Let $T_{\max} \in (0, \infty]$ denote the maximal life-span of the regular solution $\mathbf{v}$ to the Euler system (5.8), (5.9) satisfying $\mathbf{v}(0, \cdot) = \mathbf{v}_0$. Finally, let $\{\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon\}$ be a dissipative solution of the scaled Navier-Stokes-Fourier system in $(0,T) \times \Omega$, $T < T_{\max}$, with

$$\mathrm{Sr} = 1, \ \mathrm{Ma} = \varepsilon, \ \mathrm{Re} = \varepsilon^{-a}, \ \mathrm{Pe} = \varepsilon^{-b}.$$

Then

$$\operatorname{ess} \sup_{t \in (0,T)} \| \ \varrho_\varepsilon(t, \cdot) - \overline{\varrho} \ \|_{L^2 + L^{5/3}(\Omega)} \leq \varepsilon c,$$

$$\sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon \to \sqrt{\overline{\varrho}} \ \mathbf{v} \ \text{in } L^\infty_{\rm loc}((0,T]; L^2_{\rm loc}(\Omega; R^3)) \ \text{and weakly-(*) in } L^\infty(0,T; L^2(\Omega; R^3)),$$

and

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to T \ \text{in } L^\infty_{\rm loc}((0,T]; L^q_{\rm loc}(\Omega; R^3)), \ 1 \leq q < 2, \ \text{and weakly-(*) in } L^\infty(0,T; L^2(\Omega)),$$

where $\mathbf{v}$, T is the unique solution of the Euler-Boussinesq system (5.8 - 5.10), with the initial data

$$\mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0], \ T_0 = \overline{\varrho} \frac{\partial s(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \vartheta_0^{(1)} - \frac{1}{\overline{\varrho}} \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \varrho_0^{(1)}.$$

Finally, we note that *existence* of the dissipative solutions for the Navier-Stokes-Fourier system in general (unbounded) domains was shown by Jesslé, Jin, and Novotný [20].

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