# Three-dimensional numerical simulations of viscoelastic models in real situations

Karel Tůma

in cooperation with J.M. Krishnan, J. Málek, V. Průša, J. Stein

Charles University, Faculty of Mathematics and Physics, Mathematical Institute

# Asphalt concrete

- composite material
- consists of mineral aggregate bound with asphalt binder and compacted
- 2% of air voids $\Rightarrow$ almost incompressible
- exhibits viscoelastic behavior

# Bovine eye

- transparent, colorless, gelatinous
- 98% of water, NaCl, hyaluronan
- maintains the shape of the eye, keeps a clear path to the retina
- viscoelastic behavior (parameters by Sharif-Kashani et al. 2011)

# Incompressible viscoelastic fluid like models

Microstructure too complicated $\Rightarrow$ minimalistic macroscopic model

Incompressible rate-type fluid models

$$\operatorname{div} \mathbf{v} = 0,$$
$$\rho \left( \frac{\partial \mathbf{v}}{\partial t} + [\nabla \mathbf{v}]\mathbf{v} \right) = \operatorname{div} \mathbf{T}, \quad \mathbf{T} = \mathbf{T}^{\mathrm{T}},$$

where the Cauchy stress tensor $\mathbf{T} = -p\mathbf{I} + \mathbf{S}$, and $\mathbf{S}$ satisfies an evolutionary equation

$$\frac{\partial \mathbf{S}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{S} + \mathbf{f}(\mathbf{v}, \nabla \mathbf{v}, \mathbf{S}, \nabla \mathbf{S}) = \mathbf{0},$$

which belongs to the class of implicitly constituted models.

*Burgers model*

$$\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D} + \mathbf{S}, \quad \mathbf{D} = \frac{1}{2}\left(\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}}\right),$$

$$\overset{\triangledown\triangledown}{\mathbf{S}} + \left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)\overset{\triangledown}{\mathbf{S}} + \frac{1}{\tau_1\tau_2}\mathbf{S} = 2\left(\frac{G_1}{\tau_2} + \frac{G_2}{\tau_1}\right)\mathbf{D} + 2(G_1 + G_2)\overset{\triangledown}{\mathbf{D}},$$

$$\overset{\triangledown}{\mathbf{A}} = \dot{\mathbf{A}} - (\nabla\mathbf{v})\mathbf{A} - \mathbf{A}(\nabla\mathbf{v})^{\mathrm{T}}$$

## Asphalt concrete

Fits the stress relaxation and creep test performed by Monismoth, Secor (1962).

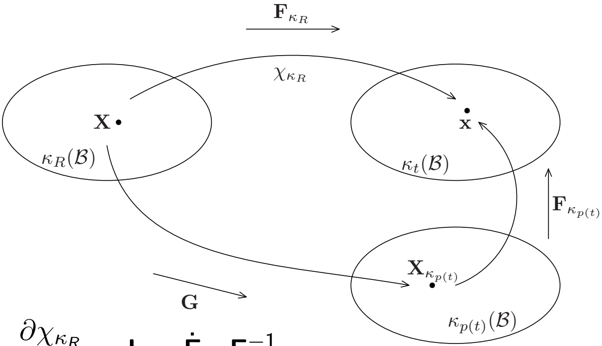
## Vitreous body

Fits the creep test performed by Sharif-Kashani (2011).

- Is it thermodynamically consistent?
- What is the physical meaning of the model?
- How to implement the second order equation?
- What initial conditions to use?

# Thermodynamic approach

Using natural configuration the deformation is split into purely elastic and dissipative part.



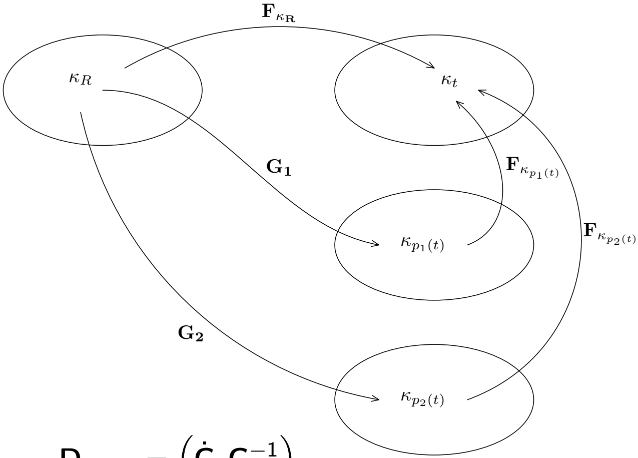

$$\mathbf{F}_{\kappa_R} = \frac{\partial \chi_{\kappa_R}}{\partial \mathbf{X}}, \quad \mathbf{L} = \dot{\mathbf{F}}_{\kappa_R}\mathbf{F}_{\kappa_R}^{-1}$$

$$\mathbf{B}_{\kappa_{p(t)}} = \mathbf{F}_{\kappa_{p(t)}}\mathbf{F}_{\kappa_{p(t)}}^{\mathrm{T}}, \quad \mathbf{L}_{\kappa_{p(t)}} = \dot{\mathbf{G}}\mathbf{G}^{-1}, \quad \mathbf{D}_{\kappa_{p(t)}} = \frac{\mathbf{L}_{\kappa_{p(t)}} + \mathbf{L}_{\kappa_{p(t)}}^{\mathrm{T}}}{2}$$

$$\dot{\mathbf{B}}_{\kappa_{p(t)}} = \mathbf{L}\mathbf{B}_{\kappa_{p(t)}} + \mathbf{B}_{\kappa_{p(t)}}\mathbf{L}^{\mathrm{T}} - 2\mathbf{F}_{\kappa_{p(t)}}\mathbf{D}_{\kappa_{p(t)}}\mathbf{F}_{\kappa_{p(t)}}^{\mathrm{T}}$$

# Two relaxation mechanisms

Experiments show that ashpalt has at least two different relaxation mechanisms.




$$\mathsf{B}_{\kappa_{p_1(t)}} = \mathsf{F}_{\kappa_{p_1(t)}} \mathsf{F}^{\mathrm{T}}_{\kappa_{p_1(t)}}, \ \mathsf{D}_{\kappa_{p_1(t)}} = \left( \dot{\mathsf{G}}_1 \mathsf{G}_1^{-1} \right)_{\mathrm{sym}}$$

$$\mathsf{B}_{\kappa_{p_2(t)}} = \mathsf{F}_{\kappa_{p_2(t)}} \mathsf{F}^{\mathrm{T}}_{\kappa_{p_2(t)}}, \ \mathsf{D}_{\kappa_{p_2(t)}} = \left( \dot{\mathsf{G}}_2 \mathsf{G}_2^{-1} \right)_{\mathrm{sym}}$$

Two constitutive relations for scalars are prescribed: Helmholtz free energy $\psi$, and rate of entropy production $\xi$.

**Helmholtz free energy $\psi$ – compressible neo-Hookean**

$$\psi = \frac{G_1}{2\rho}\left(\operatorname{tr}\mathbf{B}_{\kappa_{p_1(t)}} - 3 - \ln\det\mathbf{B}_{\kappa_{p_1(t)}}\right) + \frac{G_2}{2\rho}\left(\operatorname{tr}\mathbf{B}_{\kappa_{p_2(t)}} - 3 - \ln\det\mathbf{B}_{\kappa_{p_2(t)}}\right)$$

**Rate of entropy production $\xi$**

$$0 \leq \tilde{\xi} = 2\mu|\mathbf{D}|^2 + 2G_1\tau_1|\mathbf{F}_{\kappa_{p_1(t)}}\mathbf{D}_{\kappa_{p_1(t)}}|^2 + 2G_2\tau_2|\mathbf{F}_{\kappa_{p_2(t)}}\mathbf{D}_{\kappa_{p_2(t)}}|^2$$

**Derivation of isothermal model**

**Step 1.** Take the $\frac{\mathrm{d}}{\mathrm{d}t}$ derivative of $\psi(\mathbf{B}_{\kappa_{p_1(t)}}, \mathbf{B}_{\kappa_{p_2(t)}})$.

**Step 2.** Use the reduced TD identity $\xi = \mathbf{T}\cdot\mathbf{D} - \rho\dot{\psi}$.

**Step 3.** Compare $\xi = \tilde{\xi}$.

$$\boxed{\begin{aligned} \mathbf{T} &= -p\mathbf{I} + 2\mu\mathbf{D} + G_1(\mathbf{B}_{\kappa_{p_1(t)}} - \mathbf{I}) + G_2(\mathbf{B}_{\kappa_{p_2(t)}} - \mathbf{I}), \\ &\overset{\triangledown}{\mathbf{B}}_{\kappa_{p_1(t)}} + \frac{1}{\tau_1}(\mathbf{B}_{\kappa_{p_1(t)}} - \mathbf{I}) = \mathbf{0}, \\ &\overset{\triangledown}{\mathbf{B}}_{\kappa_{p_2(t)}} + \frac{1}{\tau_2}(\mathbf{B}_{\kappa_{p_2(t)}} - \mathbf{I}) = \mathbf{0}. \end{aligned}}$$

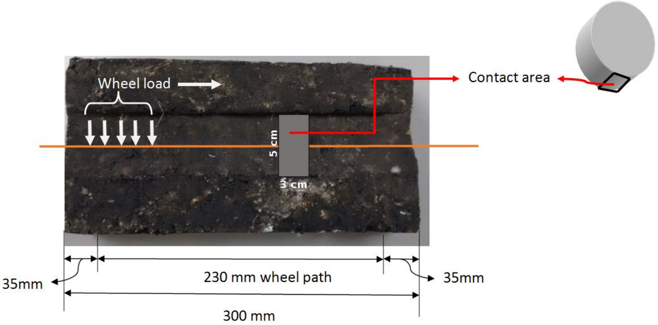
It can be shown that it is equivalent to a classical Burgers model

$$\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D} + \mathbf{S},$$

$$\overset{\triangledown\triangledown}{\mathbf{S}} + \left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)\overset{\triangledown}{\mathbf{S}} + \frac{1}{\tau_1\tau_2}\mathbf{S} = 2\left(\frac{G_1}{\tau_2} + \frac{G_2}{\tau_1}\right)\mathbf{D} + 2(G_1 + G_2)\overset{\triangledown}{\mathbf{D}}.$$

# Application 1: Wheel tracker test of the asphalt concrete

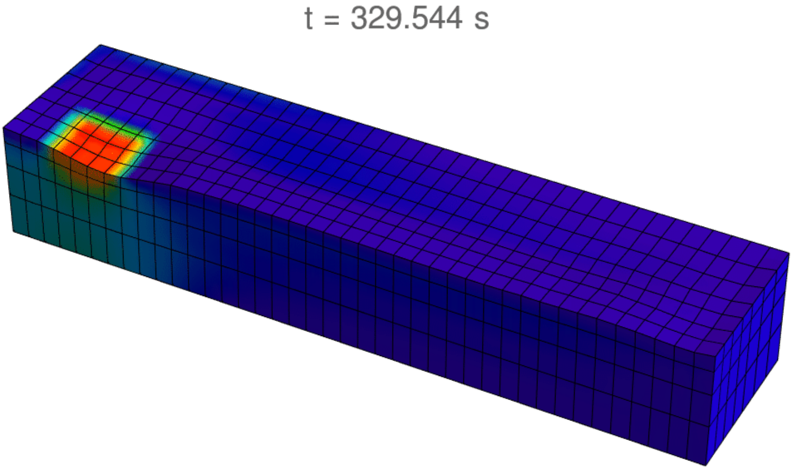
- "torture test" to check the abilities of the material
- done by the group of J.M. Krishnan (IITM)
- speed 1 km/h and 10 km/h, $\approx 10^4$ cycles performed
- brick dimensions $30 \times 13.8 \times 5$ cm
- capable of measuring the deformation $u_z$



# 3D simulation in deforming domains, FEM implementation

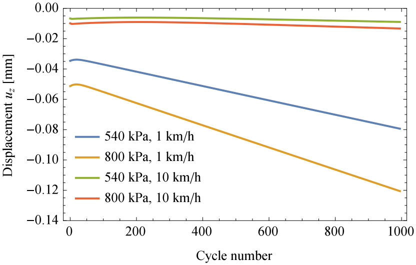
- arbitrary Langrangian-Eulerian method to treat deforming domains
- monolithic approach
- weak ALE formulation implemented in AceFEM/AceGen system (J. Korelc, Ljublan), automatic differentiation provides exact tangent matrix
- non-linearities treated with the standard Newton method
- implicit Euler, 200 time steps per one rutting cycle
- hexahedra mesh 1 120 elements
- to decrease the size of the problem and complexity of the global matrix (gives 157 DOFs per element):
    - $\mathbf{v} \to$ H2 elements
    - $p, \mathbf{B}_1, \mathbf{B}_2 \to \mathrm{P1}^{\mathrm{disc}}$ elements
    - $\mathbf{u} \to$ H1 elements
- 88 052 DOFs (Dirichlet BC excluded)

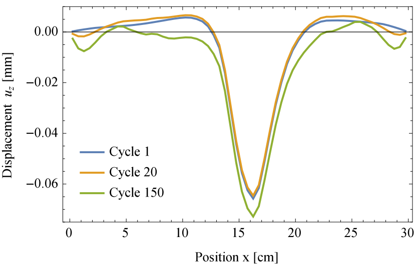
# Wheel tracker test simulation




Cycle 200, 800 kPa, speed 1 km/h
pressure distribution, deformation scaled 100×

# Dependence of the deformation on the cycle




- the wheel in the middle going to the left

**left**: deformation at the top in the middle for all cases

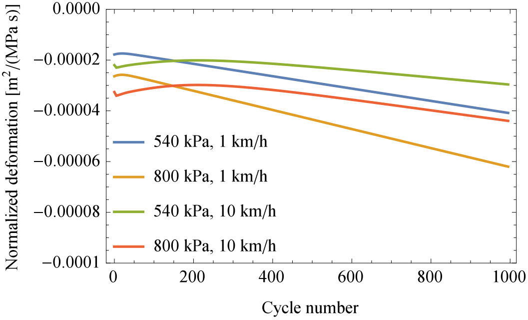
**right**: deformation at the top surface for $p_{\mathrm{app}} = 800\,\mathrm{kPa}$, $v = 1\,\mathrm{km/h}$

# Normalized deformation

Expectation: the deformation $u_z$ is proportional to the applied stress $p_{\mathrm{app}}$ and reciprocal to the speed $v$

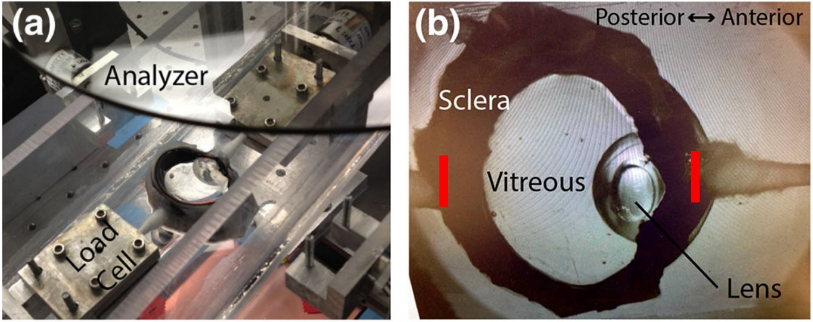
Define normalized displacement

$$u_z^N := \frac{v}{p_{app}} u_z$$



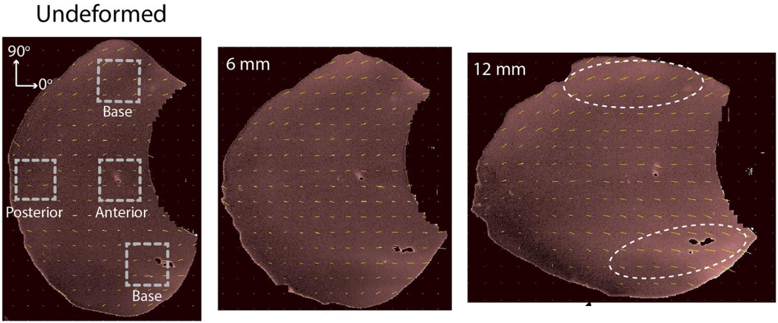

higher $p_{\mathrm{app}}$ + lower $v \Rightarrow$ higher relative deformation

# Application 2: Stretching of the bovine eye

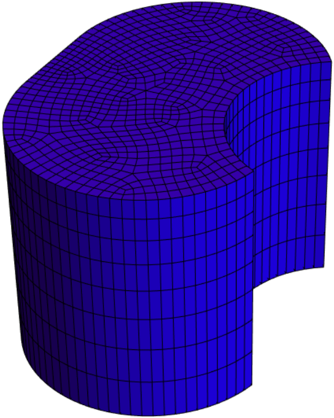



- performed by Shah et al. (2016)
- 2 cm thick plate cut out
- put into loading machine and glued on the sides
- let it relax and then $4\times$ prolongated in 3 mm increments
- tracking of markers on the top surface

# Experiment (Shah et al. 2016)



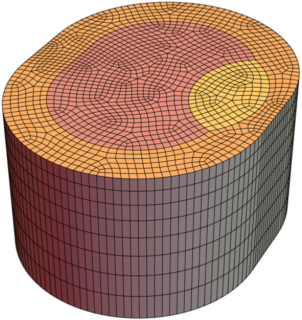
- a need to compute 3D problem
- first attempt: the deformation of vitreous only, deformation too different from what they did
- vitreous fluid can not hold by itself $\Rightarrow$ need to compute a more complex problem



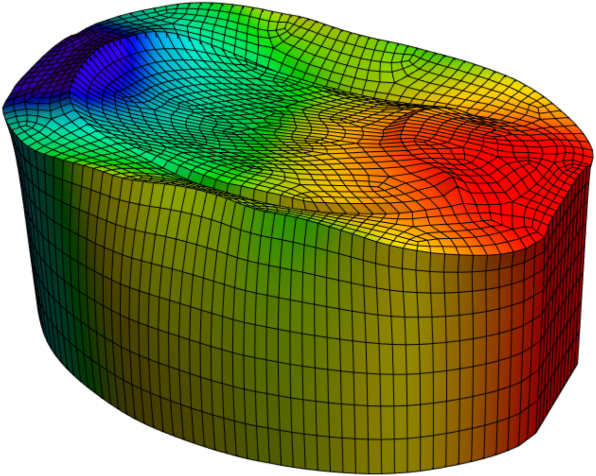
- full model for vitreous, sclera and lens
- compressible elastic neo-Hookean solid with a strain energy density, inertia included

$$W(\mathbf{u}) = \frac{1}{2}\mu(J^{-2/3}\,\mathrm{tr}\,\mathbf{C} - 3) + \frac{1}{2}\kappa(\ln J)^2, \quad J = \det \mathbf{F}, \quad \mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$$

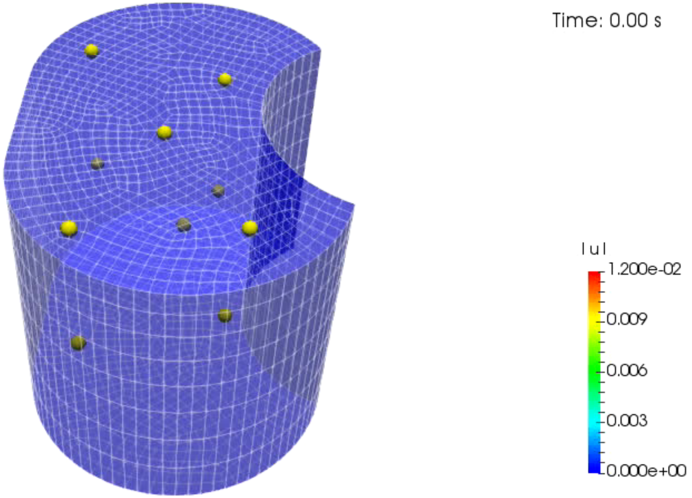
- $\kappa = 1000\mu$ makes the material almost incompressible
- sclera: $\mu = 330\,\mathrm{kPa}$ (Grytz et al. 2014)
- lens: $\mu = 10\,\mathrm{kPa}$ (Fisher 1971)
- interaction: continuous displacement and stress



# Bovine eye deformation (Burgers model)

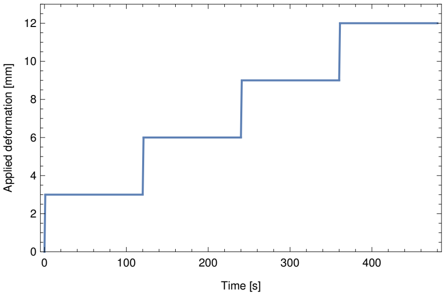


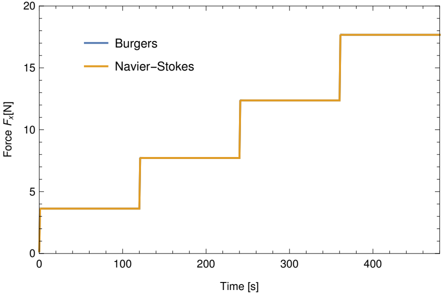
# Bovine eye deformation (Burgers model)



# Dependence of force on time

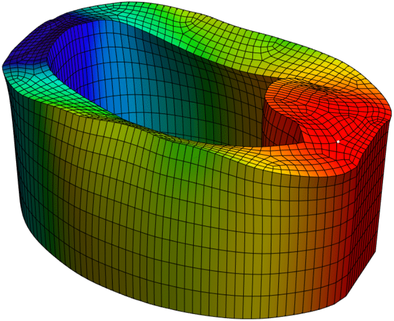
Navier-Stokes vs. Burgers

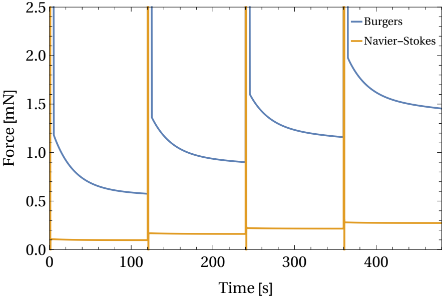


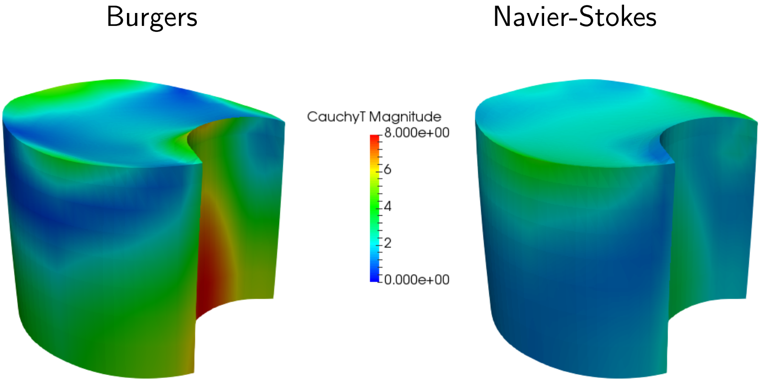
# Dependence of force on time

Navier-Stokes vs. Burgers





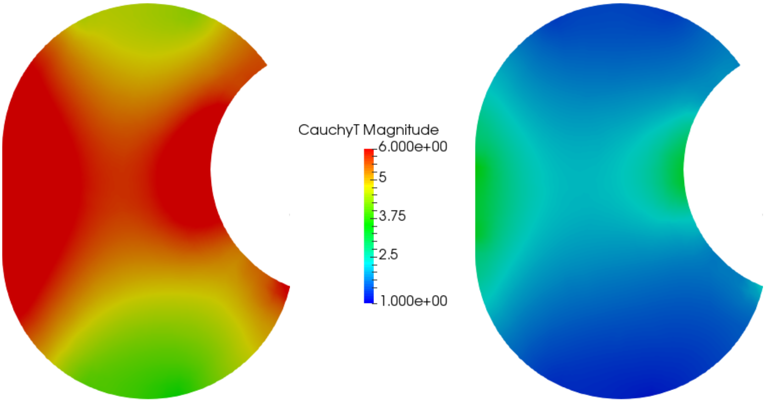
# Navier-Stokes vs. Burgers, stresses




$|\mathbf{T}|$

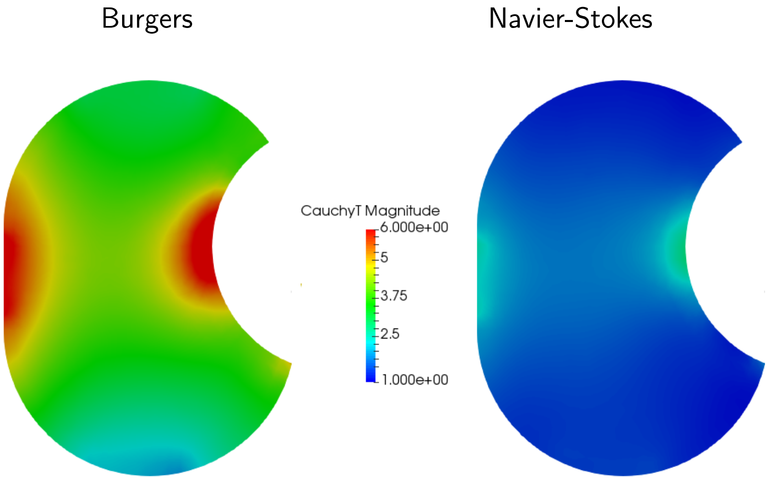
# Navier-Stokes vs. Burgers, stresses

Burgers | Navier-Stokes




$$z = 0$$

# Navier-Stokes vs. Burgers, stresses




$z = h/2$

# Conclusion

**Summary of the results:**

- performed 3D simulation of the deformation of the bovine eye
- described by Burgers model:
  - can be obtained using thermodynamic approach
  - satisfy the second law of thermodynamics
  - one should view them as the fluids where the elastic response corresponds to that of a compressible neo-Hookean solid

## Wheel tracker test of the asphalt concrete

- phase-shift – applied stress is ahead the maximum depression
- higher pressures and lower speeds deform the material relatively more

## Stretching of the bovine eye

- interaction between viscoelastic vitreous, elastic sclera and lens (orders of magnitude different)
- difference in stresses between Navier-Stokes and Burgers