Nečas Center for Mathematical Modeling
MOdelling REvisited + MOdel REduction
ERC-CZ project LL1202 - MORE
MINISTRY OF EDUCATION, YOUTH AND SPORTS

Existence of large–data finite–energy global weak solutions to a compressible Oldroyd-B model

Yong Lu[1]

Chern Institute of Mathematics, Nankai University

Conference of the project ERC-CZ MORE, Roztoky, 2017.7.31–8.4

[1]in collaboration with John W. Barrett & Endre Süli

Newtonian fluids and Navier-Stokes equations

Consider the equations of balance of mass and momentum:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0,$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p - \mathrm{div}_x \mathbb{T} = \varrho \mathbf{f}.$$

ϱ: density; $\mathbf{u}$: velocity; p: pressure; $\mathbb{T}$: stress tensor; $\mathbf{f}$: external force.

Newtonian fluids: Newtonian stress tensor with shear and bulk viscosity coefficients: $\mu^S > 0$ and $\mu^B \geq 0$:

$$\mathbb{T} = \mathbb{S}(\nabla_x \mathbf{u}) = \mu^S \left(\frac{\nabla_x \mathbf{u} + \nabla_x^{\mathrm{T}} \mathbf{u}}{2} - \frac{1}{d}(\mathrm{div}_x \mathbf{u})\mathbb{I} \right) + \mu^B (\mathrm{div}_x \mathbf{u})\mathbb{I}.$$

In addition, for incompressible fluid flows, $\mathrm{div}_x \mathbf{u} = 0$:

$$\mathrm{div}_x \mathbb{T} = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) = \nu \Delta_x \mathbf{u}, \quad \nu = \mu^S/2.$$

Moreover, for homogeneous fluid flows $\varrho = 1$:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla_x \mathbf{u} + \nabla_x p - \nu \Delta_x \mathbf{u} = \mathbf{f}, \quad \mathrm{div}_x \mathbf{u} = 0.$$

Non-Newtonian fluids

For some fluids like honey, ketchup, etc., experiments show that the Cauchy stress tensor behaviors nonlinearly w.r.t. the velocity gradient. In such a case, the tress tensor is called non-Newtonian stress tensor and the fluids are called non-Newtonian fluids.

Examples:

1, Power law: $\mathbb{T} = \left(\nu_0 + \nu_1 |\mathbf{D}(\mathbf{u})|^2\right)^{\frac{p-2}{2}} \mathbf{D}(\mathbf{u}), \quad \mathbf{D}(\mathbf{u}) = \dfrac{\nabla_x \mathbf{u} + \nabla_x^{\mathrm{T}} \mathbf{u}}{2},$

2, Bingham law: $|\mathbb{T}| \leq \tau_* \Leftrightarrow \mathbf{D}(\mathbf{u}) = 0$ and $|\mathbb{T}| > \tau_* \Leftrightarrow \mathbb{T} = \dfrac{\tau_* \mathbf{D}(\mathbf{u})}{|\mathbf{D}(\mathbf{u})|} + \nu \mathbf{D}(\mathbf{u}$

3, Implicit law: $G(\mathbb{T}, \mathbf{D}(\mathbf{u})) = 0.$

Oldroyd-B model

Homogeneous fluid flows with $\mathbb{T} = \mathbb{S}(\nabla_x \mathbf{u}) + \tau$:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla_x \mathbf{u} + \nabla_x p - \nu \Delta_x \mathbf{u} = \mathrm{div}_x \tau + \mathbf{f}, \quad \mathrm{div}_x \mathbf{u} = 0,$$

$$\partial_t \tau + \mathbf{u} \cdot \nabla_x \tau - (\nabla_x \mathbf{u}\, \tau + \tau\, \nabla_x^{\mathrm{T}} \mathbf{u}) + \frac{1}{\mathrm{We}} \tau = \frac{\delta}{\mathrm{We}} \mathbf{D}(\mathbf{u}),$$

where the extra stress tensor τ is symmetric positive definite matrix.

Open problems:

- Global existence of strong solutions?
- Global existence of weak solutions?
- Domains in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Incompressible Oldroyd-B model: known results

Local-in-time strong solutions:

- Guillopé, Saut, 1990. Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, C^3 domain. Suppose $\mathbf{u}_0 \in H_0^1 \cap H^2$, $\mathrm{div}_x \mathbf{u}_0 = 0$, $\tau_0 \in H^2$. Then there exits $T > 0$ and a unique solution:

$$\mathbf{u} \in C([0,T]; H_0^2 \cap H^2) \cap L^2(0,T; H^3),$$
$$\partial_t \mathbf{u} \in C([0,T]; L^2) \cap L^2(0,T; H_0^1),$$
$$\tau \in C([0,T]; H^2), \quad p \in L^2(0,T; H_0^2).$$

- Fernández-Cara, Guillén, Ortega 1998. $L^q L^r$ framework.

Small data: global existence.

Incompressible Oldroyd-B model: known results

Large data, global existence of weak solutions:

N. Masmoudi, P.L. Lions, 2000, $d = 2, 3$.
$\nabla_x \mathbf{u}\, \tau + \tau\, \nabla_x^{\mathrm{T}} \mathbf{u}$ replaced by $\omega(\mathbf{u})\, \tau - \tau\, \omega(\mathbf{u})$, $\omega(\mathbf{u})$ is the anti-symmetric part of $\nabla_x \mathbf{u}$.

Newtonian limit: We $\to 0$: Oldroyd-B $\to$ Navier-Stokes.

- L. Molinet, R. Talhouk, 2008, in the framework of H^s strong solutions,
- D. Bresch, C. Prange 2013, in the framework of weak solutions of Masmoudi and Lions.

Incompressible Oldroyd-B model: known results

Large data, global existence of strong solutions

P. Constantin, M. Kliegl, 2012, $d = 2$, with diffusive stress: $\Delta_x \tau$:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla_x \mathbf{u} + \nabla_x p - \nu \Delta_x \mathbf{u} = \mathrm{div}_x \tau + \mathbf{f}, \quad \mathrm{div}_x \mathbf{u} = 0,$$

$$\partial_t \tau + \mathbf{u} \cdot \nabla_x \tau - (\nabla_x \mathbf{u}\, \tau + \tau\, \nabla_x^{\mathrm{T}} \mathbf{u}) + \frac{1}{\mathrm{We}} \tau = \frac{\delta}{\mathrm{We}} \mathbf{D}(\mathbf{u}) + \kappa \Delta_x \tau.$$

The micro-macro model for polymeric fluids

τ as a function of the polymer chains configurations:

$$\tau(t,x) = -\mathbb{I} + \int_D (q \otimes F(q))\,\psi(t,x,q)\,\mathrm{d}q,\ \ q \in D^K.$$

Potential and force

- Hookean model: $q \in D^K = \mathbb{R}^{Kd}$,

$$\Pi(q) = \frac{|q|^2}{2}, \quad F(q) = \nabla_q \Pi(q) = q.$$

- FENE (finitely extensible nonlinear elastic) model:
  $q \in D^K := B(0,\sqrt{b})^K \subset \mathbb{R}^{Kd}$,

$$\Pi(q) = -\frac{b}{2}\log(1-|q|^2/b), \quad F(q) = \nabla_q \Pi(q) = \frac{q}{1-|q|^2/b}.$$

The micro-macro model for polymeric fluids

Fokker–Planck equation

$$\partial_t \psi + \mathbf{u} \cdot \nabla_x \psi + \mathrm{div}_q \left(\left(\nabla_x \mathbf{u}\, q - F(q) \right) \psi \right) = \Delta_q \psi.$$

Relation to macroscopic models

Oldroyd-B model can be derived from the Navier-Stokes-Fokker-Planck equations under Hookean dumbbell model setting with $F(q) = q$.

The micro-macro model: known results

F.H. Lin, C. Liu, P. Zhang, 2007:

Near equilibrium, potential U satisfies some conditions.

For FENE:

- Strong solutions: W. E & P.W. Zhang, Masmoudi & P. Zhang & Z. Zhang ($d = 2$, FENE dumbbell)...
- Weak solutions: FENE dumbbell model: Masmoudi 2013.

Compressible Navier-Stokes-Fokker-Planck equations

Compressible Navier-Stokes equations with extra stress:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0,$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) - \mathrm{div}_x\, \mathbb{S}(\nabla_x \mathbf{u}) = \mathrm{div}_x \mathbb{T} + \varrho\, \mathbf{f}.$$

Extra stress tensor:

$$\mathbb{T}(\psi) := \mathbb{T}_1(\psi) - \delta \left(\int_D \psi \,\mathrm{d}\, q \right)^2 \mathbb{I}.$$

The first part, $\mathbb{T}_1(\psi)$, of $\mathbb{T}(\psi)$ is given by the *Kramers expression*

$$\mathbb{T}_1(\psi) := k \left[\left(\sum_{i=1}^{K} \mathbb{C}_i(\psi) \right) - L \left(\int_D \psi \;\mathrm{d}\, q \right) \mathbb{I} \right],$$

$$\mathbb{C}_i(\psi)(t,x) := \int_D \psi(t,x,q)\, U_i' \left(\frac{|q_i|^2}{2} \right) q_i \otimes q_i \,\mathrm{d}\, q, \quad i = 1, \ldots, K.$$

Compressible Navier-Stokes-Fokker-Planck equations

Fokker–Planck equation with diffusion:

$$\partial_t \psi + \mathrm{div}_x(\mathbf{u}\,\psi) + \sum_{i=1}^{K} \mathrm{div}_{q_i}\left((\nabla_x \mathbf{u})\, q_i\, \psi\right) = \varepsilon \Delta_x \psi$$

$$+ \frac{1}{4\lambda} \sum_{i=1}^{K} \sum_{j=1}^{K} A_{ij}\, \mathrm{div}_{q_i}\left(M \nabla_{q_j} \left(\frac{\psi}{M} \right) \right).$$

Polymer number density:

$$\eta := \int \psi \,\mathrm{d}\, q, \qquad \partial_t \eta + \mathrm{div}_x(\mathbf{u}\,\eta) = \varepsilon \Delta_x \eta.$$

Then,

$$\mathbb{T}(\psi) := k \left(\sum_{i=1}^{K} \mathbb{C}_i(\psi) \right) - \left(kL\eta + \delta\, \eta^2 \right) \mathbb{I}.$$

Some known results

Modelling: J. W. Barrett, E. Süli 2012.

Global-in-time existence of weak solutions, FENE setting: J. W. Barrett, E. Süli 2016, $\mathbb{R}^d$, $d = 2, 3$, $\gamma > d/2$.

Global-in-time existence of weak solutions when polymer density dependent viscosity coefficients, FENE setting: Feireisl-Lu-Süli 2016.

Compressible Oldroyd-B model

Recall: $\mathbb{T}(\psi) := k\left(\sum_{i=1}^{K}\mathbb{C}_i(\psi)\right) - \left(kL\eta + \delta\,\eta^2\right)\mathbb{I}$,

$$\partial_t\psi+\mathrm{div}_x(\mathbf{u}\,\psi)+\sum_{i=1}^{K}\mathrm{div}_{q_i}\left((\nabla_x\mathbf{u})\,q_i\,\psi\right) = \varepsilon\Delta_x\psi+\frac{1}{4\lambda}\sum_{i=1}^{K}\sum_{j=1}^{K}A_{ij}\,\mathrm{div}_{q_i}\left(\nabla_{q_j}\psi+\psi\,q_j\right).$$

Define:

$$\tau := \sum_{i=1}^{K}\mathbb{C}_i(\psi) = \int_D \psi(t,x,q)\,U_i'\left(\frac{|q_i|^2}{2}\right)q_i\otimes q_i\,\mathrm{d}\,q$$

$$= \int_D \psi(t,x,q)\,q_i\otimes q_i\,\mathrm{d}\,q, \quad \text{for Hookean model.}$$

Under Hookean dumbbell model, we derive a closed system in τ:

$$\partial_t\tau+\mathrm{Div}_x(\mathbf{u}\,\tau)-\left(\nabla_x\mathbf{u}\,\tau+\tau\,\nabla_x^{\mathrm{T}}\mathbf{u}\right) = \varepsilon\Delta_x\tau+\frac{k\,A_0}{2\lambda}\eta\,\mathbb{I}-\frac{A_0}{2\lambda}\tau,$$

$$(\mathrm{Div}_x(\mathbf{u}\,\tau))_{\kappa,\iota} := \mathrm{div}_x(\mathbf{u}\,\tau_{\kappa,\iota}), \quad 1\le\kappa,\iota\le d.$$

Compressible Oldroyd-B model

The system:

$$
\begin{aligned}
\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) &= 0,\\
\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) + \nabla_x q(\eta) - \mathrm{div}_x \, \mathbb{S}(\nabla_x \mathbf{u}) &= \mathrm{div}_x \tau + \varrho \, \mathbf{f},\\
\partial_t \eta + \mathrm{div}_x(\eta \mathbf{u}) &= \varepsilon \Delta_x \eta,\\
\partial_t \tau + \mathrm{Div}_x(\mathbf{u}\,\tau) - \left(\nabla_x \mathbf{u}\,\tau + \tau\,\nabla_x^{\mathrm{T}} \mathbf{u}\right) - \frac{k\,A_0}{2\lambda}\eta\,\mathbb{I} + \frac{A_0}{2\lambda}\tau &= \varepsilon \Delta_x \tau,
\end{aligned}
$$

where

$$
p(\varrho) = a\varrho^{\gamma}, \quad a > 0,\ \gamma > 1, \quad q(\eta) := kL\eta + \delta\,\eta^2,
$$

$$
\mathbb{S}(\nabla_x \mathbf{u}) = \mu^S \left(\frac{\nabla_x \mathbf{u} + \nabla_x^{\mathrm{T}} \mathbf{u}}{2} - \frac{1}{d}(\mathrm{div}_x \mathbf{u})\mathbb{I} \right) + \mu^B (\mathrm{div}_x \mathbf{u})\mathbb{I}.
$$

Boundary conditions:

$$
\mathbf{u} = \partial_{\mathbf{n}} \eta = \partial_{\mathbf{n}} \tau = \mathbf{0} \quad \text{on } (0, T] \times \partial\Omega.
$$

Global-in-time weak solutions

Initial data

$$\begin{aligned}
&\varrho(0,\cdot) = \varrho_0(\cdot) \text{ with } \varrho_0 \geq 0 \text{ a.e. in } \Omega, \quad \varrho_0 \in L^\gamma(\Omega),\\
&\mathbf{u}(0,\cdot) = \mathbf{u}_0(\cdot) \in L^r(\Omega;\mathbb{R}^d) \text{ for some } r \geq 2\gamma' \text{ such that } \varrho_0|\mathbf{u}_0|^2 \in L^1(\Omega),\\
&\eta(0,\cdot) = \eta_0(\cdot) \text{ with } \eta_0 \geq 0 \text{ a.e. in } \Omega, \quad \begin{cases} \eta_0 \in L^2(\Omega), & \text{if } \delta > 0,\\ \eta_0 \log \eta_0 \in L^1(\Omega), & \text{if } \delta = 0, \end{cases}\\
&\mathbb{T}(0,\cdot) = \mathbb{T}_0(\cdot) \text{ with } \mathbb{T}_0 = \mathbb{T}_0^{\mathrm{T}} \geq 0 \text{ a.e. in } \Omega, \quad \mathbb{T}_0 \in L^2(\Omega;\mathbb{R}^{d\times d}).
\end{aligned} \tag{1}$$

Theorem (Barrett, Süli, L. 2016)

Let $\gamma > 1$ and $\Omega \subset \mathbb{R}^2$ be a bounded $C^{2,\beta}$ domain with $\beta \in (0,1)$. Assume the parameters ε, k, λ, A_0 are all positive numbers and $\delta \geq 0$, $L \geq 0$ with $\delta + L > 0$. Then for any $T > 0$, there exists a finite-energy weak solution $(\varrho, \mathbf{u}, \eta, \mathbb{T})$ with initial data (1).

A priori estimates:

For compressible Navier-Stokes:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\,t}\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{a}{\gamma-1}\varrho^\gamma + \left(kL(\eta\log\eta+1)+\delta\,\eta^2\right)\right]\mathrm{d}x \\
&\quad + \int_\Omega \varepsilon\, kL\eta^{-1}|\nabla_x\eta|^2 + 2\,\varepsilon\,\delta\,|\nabla_x\eta|^2\,\mathrm{d}x \\
&\quad + \int_\Omega \mu^S\left|\frac{\nabla_x\mathbf{u}+\nabla_x^{\mathrm{T}}\mathbf{u}}{2} - \frac{1}{d}(\mathrm{div}_x\mathbf{u})\mathbb{I}\right|^2 + \mu^B|\mathrm{div}_x\mathbf{u}|^2\,\mathrm{d}x \\
&= -\int_\Omega \tau : \nabla_x\mathbf{u}\,\mathrm{d}x + \int_\Omega \varrho\,\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x.
\end{aligned} \tag{2}$$

For τ:

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\int_\Omega \mathrm{tr}\,\tau\,\mathrm{d}x + \frac{A_0}{2\lambda}\int_\Omega \mathrm{tr}\,\tau\,\mathrm{d}x = \frac{k\,A_0\,d}{2\lambda}\int_\Omega \eta\,\mathrm{d}x + 2\int_\Omega \tau : \nabla_x\mathbf{u}\,\mathrm{d}x. \tag{3}$$

A priori estimates:

Therefore, (2) + $\frac{1}{2}$(3) gives

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{a}{\gamma-1}\varrho^\gamma + \big(kL(\eta\log\eta + 1) + \delta\,\eta^2\big) + \frac{1}{2}\mathrm{tr}\,\tau\right]\mathrm{d}x$$
$$+ \int_\Omega \varepsilon\, kL\eta^{-1}|\nabla_x\eta|^2 + 2\,\varepsilon\,\delta\,|\nabla_x\eta|^2\,\mathrm{d}x$$
$$+ \int_\Omega \mu^S\left|\frac{\nabla_x\mathbf{u} + \nabla_x^{\mathrm{T}}\mathbf{u}}{2} - \frac{1}{d}(\mathrm{div}_x\mathbf{u})\mathbb{I}\right|^2 + \mu^B|\mathrm{div}_x\mathbf{u}|^2\,\mathrm{d}x + \frac{A_0}{4\lambda}\int_\Omega \mathrm{tr}\,\tau\,\mathrm{d}x$$
$$= \int_\Omega \varrho\,\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x + \frac{k\,A_0\,d}{4\lambda}\int_\Omega \eta\,\mathrm{d}x.$$

Problem: $\mathrm{tr}\,\tau \geq 0$, $\tau \geq 0$?

A regularized model

We introduce:

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) + \nabla_x \big(kL\eta + \delta\,\eta^2\big) - \mathrm{div}_x\, \mathbb{S}(\nabla_x \mathbf{u}) = \mathrm{div}_x \tau - \frac{\alpha}{2} \nabla_x\, (\mathrm{tr}\, \log \tau) + \varrho\, \mathbf{f}.$$

Functions in symmetric matrices: let g be a scalar function and τ be a symmetric matrix:

$$\tau = P^{-1} \mathrm{diag}\,\{\lambda_1, \dots, \lambda_d\} P, \ \ \lambda_\kappa \text{ eigenvalues of } \tau,$$

define

$$g(\tau) = P^{-1} \mathrm{diag}\,\{g(\lambda_1), \dots, g(\lambda_d)\} P.$$

Thus, if $\tau > 0$, we have the following identity:

$$\mathrm{tr}\, \log \tau = \log\, \det\ \tau.$$

New a priori estimates:

With the presence of logarithmic term:

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\,t}\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{a}{\gamma-1}\varrho^\gamma + \left(kL(\eta\log\eta + 1) + \delta\,\eta^2\right) + \frac{1}{2}\mathrm{tr}\,\tau\right]\mathrm{d}x \\
&\quad + \int_\Omega \varepsilon\, kL\eta^{-1}|\nabla_x\eta|^2 + 2\,\varepsilon\,\delta\,|\nabla_x\eta|^2\,\mathrm{d}x \\
&\quad + \int_\Omega \mu^S\left|\frac{\nabla_x\mathbf{u} + \nabla_x^{\mathrm{T}}\mathbf{u}}{2} - \frac{1}{d}(\mathrm{div}_x\mathbf{u})\mathbb{I}\right|^2 + \mu^B|\mathrm{div}_x\mathbf{u}|^2\,\mathrm{d}x + \frac{A_0}{4\lambda}\int_\Omega \mathrm{tr}\,\tau\,\mathrm{d}x \\
&= \int_\Omega \varrho\,\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x + \frac{k\,A_0\,d}{4\lambda}\int_\Omega \eta\,\mathrm{d}x + \frac{\alpha}{2}\int_\Omega (\mathrm{tr}\,\log\tau)\,(\mathrm{div}_x\mathbf{u})\,\mathrm{d}x.
\end{aligned}
\tag{4}
$$

To close the estimate, we need logarithmic type estimates.

A logarithmic bound:

Idea: $\int_\Omega$ (Equation in τ) : $\tau^{-1}\,\mathrm{d}x$.

Jacobi's formula: $\partial(\det A) = (\det A)\,\mathrm{tr}\,(\mathrm{A}^{-1}\partial \mathrm{A})$.

Hence, if $A = A^{\mathrm{T}} > 0$, $\quad \partial\,(\log\det A) = \mathrm{tr}\,(\mathrm{A}^{-1}\partial\mathrm{A}) = \partial \mathrm{A} : \mathrm{A}^{-1}$.

Thus,

- $\partial_t \tau : \tau^{-1} = \partial_t\,(\log\det\tau) = \partial_t(\mathrm{tr}\,\log\tau)$.
- $\mathrm{Div}_x(\mathbf{u}\,\tau) : \tau^{-1} = (\mathbf{u}\cdot\nabla_x)\,(\mathrm{tr}\,\log\tau) + d\,\mathrm{div}_x\mathbf{u}$.
- $-\left(\nabla_x\mathbf{u}\,\tau + \tau\,\nabla_x^{\mathrm{T}}\mathbf{u}\right) : \tau^{-1} = -2\,\mathrm{div}_x\mathbf{u}$.

Thus,

$$\partial_t\,(\mathrm{tr}\,\log\tau) + (\mathbf{u}\cdot\nabla_x)\,(\mathrm{tr}\,\log\tau) + (d-2)\,\mathrm{div}_x\mathbf{u}$$
$$= \varepsilon\,\Delta_x\tau : \tau^{-1} + \frac{k\,A_0}{2\lambda}\eta\,\mathrm{tr}\,\tau^{-1} - \frac{\mathrm{d}A_0}{2\lambda}.$$

A logarithmic bound:

Integral by parts gives:

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\int_\Omega (\mathrm{tr}\,\log\tau)\,\mathrm{d}x = \int_\Omega (\mathrm{div}_x \mathbf{u})\,(\mathrm{tr}\,\log\tau)\,\mathrm{d}x + \int_\Omega \varepsilon\,\Delta_x\tau : \tau^{-1}\,\mathrm{d}x$$
$$+ \int_\Omega \frac{k\,A_0}{2\lambda}\eta\,\mathrm{tr}\,\tau^{-1}\,\mathrm{d}x - \frac{\mathrm{d}\,\mathrm{A}_0}{2\lambda}|\Omega|. \tag{5}$$

For the diffusion term:

$$\int_\Omega \Delta_x\tau : \tau^{-1}\,\mathrm{d}x = \sum_{j=1}^{d}\int_\Omega \mathrm{tr}\left(\left((\partial_{x_j}\tau)(\tau^{-1})\right)^2\right)\mathrm{d}x$$
$$\geq \sum_{j=1}^{d}\int_\Omega \sum_{i=1}^{d}\left|\partial_{x_j}\log\lambda_i\right|^2\,\mathrm{d}x$$
$$\geq \frac{1}{d}\int_\Omega |\nabla_x(\mathrm{tr}\,\log\tau)|^2\,\mathrm{d}x.$$

A logarithmic bound:

(4) $- \frac{\alpha}{2}$(5) gives:

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\,t}\int_\Omega \Big[\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{a}{\gamma-1}\varrho^\gamma + \big(kL(\eta\log\eta + 1) + \delta\,\eta^2\big) \\
&\qquad + \frac{1}{2}\big(\mathrm{tr}\,\tau - \alpha\,(\mathrm{tr}\,\log\tau)\big)\Big]\,\mathrm{d}x \\
&\quad + \int_\Omega \varepsilon\, kL\eta^{-1}|\nabla_x\eta|^2 + 2\,\varepsilon\,\delta\,|\nabla_x\eta|^2\,\mathrm{d}x + \frac{\alpha\,\varepsilon}{2}\sum_{j=1}^{d}\int_\Omega \mathrm{tr}\,\Big(\big((\partial_{x_j}\tau)(\tau^{-1})\big)^2\Big)\,\mathrm{d}x \\
&\quad + \int_\Omega \mu^S\left|\frac{\nabla_x\mathbf{u} + \nabla_x^{\mathrm{T}}\mathbf{u}}{2} - \frac{1}{d}(\mathrm{div}_x\mathbf{u})\mathbb{I}\right|^2 + \mu^B|\mathrm{div}_x\mathbf{u}|^2\,\mathrm{d}x \\
&\quad + \frac{A_0}{4\lambda}\int_\Omega \mathrm{tr}\,\tau\,\mathrm{d}x + \frac{\alpha\,k\,A_0}{4\lambda}\int_\Omega \eta\,\mathrm{tr}\,\tau^{-1}\,\mathrm{d}x \\
&= \int_\Omega \varrho\,\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x + \frac{k\,A_0\,d}{4\lambda}\int_\Omega \eta\,\mathrm{d}x + \frac{\alpha\,d\,A_0}{4\lambda}|\Omega|. \qquad (6)
\end{aligned}
$$

A logarithmic bound

Observe:

$$\mathrm{tr}\,\tau - \alpha\,(\mathrm{tr}\ \log\tau) = \sum_{\kappa=1}^{\mathrm{d}} (\lambda_\kappa - \alpha \log \lambda_\kappa) \geq \sum_{\kappa=1}^{\mathrm{d}} (\alpha - \alpha \log \alpha) = \mathrm{d}\,(\alpha - \alpha \log \alpha)$$

Consider the following nonnegative energy functional:

$$E(t) := \int_\Omega \Big[\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{a}{\gamma - 1}\varrho^\gamma + \big(kL(\eta\log\eta + 1) + \delta\,\eta^2\big) + \frac{1}{2}\big(\mathrm{tr}\,\tau - \alpha\,(\mathrm{tr}\ \log\tau) + \mathrm{d}\,(\alpha\log\alpha - \alpha)\,\big)\Big]\mathrm{dx}.$$

Initial energy:

$$E_0 := \int_\Omega \Big[\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \frac{a}{\gamma - 1}\varrho_0^\gamma + \big(kL(\eta_0\log\eta_0 + 1) + \delta\,\eta_0^2\big) + \frac{1}{2}\big(\mathrm{tr}\,\tau_0 - \alpha\,(\mathrm{tr}\ \log\tau_0) + \mathrm{d}\,(\alpha\log\alpha - \alpha)\,\big)\Big]\mathrm{dx}.$$

A logarithmic a priori estimate:

Gronwall's inequality implies that

$$
\begin{aligned}
&E(t) + \int_0^t \int_\Omega \varepsilon\, kL\eta^{-1}|\nabla_x \eta|^2 + 2\,\varepsilon\,\delta\,|\nabla_x \eta|^2 \,\mathrm{d}x\,\mathrm{d}\,t' \\
&\quad + \frac{\alpha\,\varepsilon}{2d} \int_0^t \int_\Omega |\nabla_x (\mathrm{tr}\,\log\tau)|^2 \,\mathrm{d}x\,\mathrm{d}\,t' \\
&\quad + \int_0^t \int_\Omega \mu^S \left| \frac{\nabla_x \mathbf{u} + \nabla_x^{\mathrm{T}} \mathbf{u}}{2} - \frac{1}{d}(\mathrm{div}_x \mathbf{u})\mathbb{I} \right|^2 + \mu^B |\mathrm{div}_x \mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}\,t' \\
&\quad + \frac{A_0}{4\lambda} \int_0^t \int_\Omega \mathrm{tr}\,\tau \,\mathrm{d}x\,\mathrm{d}\,t' + \frac{\alpha\,k\,A_0}{4\lambda} \int_0^t \int_\Omega \eta\,\mathrm{tr}\,\tau^{-1} \,\mathrm{d}x\,\mathrm{d}\,\mathrm{t}' \\
&\leq (E_0 + C\,t)\,\mathrm{e}^{Ct}.
\end{aligned}
\tag{7}
$$

Recall Korn's inequality:

$$
\|\nabla_x \mathbf{u}\|_{L^2(\Omega;\mathbb{R}^{d\times d})} \leq C \left\| \frac{\nabla_x \mathbf{u} + \nabla_x^{\mathrm{T}} \mathbf{u}}{2} - \frac{1}{d}(\mathrm{div}_x \mathbf{u})\mathbb{I} \right\|_{L^2(\Omega;\mathbb{R}^{d\times d})} .
$$

A priori bounds:

From the *a priori* inequality (7):

$$\varrho \in L^\infty(0,T;L^\gamma(\Omega)),\ \mathbf{u} \in L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^d)),\ \varrho|\mathbf{u}|^2 \in L^\infty(0,T;L^1(\Omega)),$$
$$\eta \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W_0^{1,2}(\Omega)), \quad \eta\,\mathrm{tr}\,\tau^{-1} \in \mathrm{L}^1(0,\mathrm{T};\mathrm{L}^1(\Omega)),$$
$$\mathrm{tr}\,\tau - \alpha\,(\mathrm{tr}\,\log\tau) \in \mathrm{L}^\infty(0,\mathrm{T};\mathrm{L}^1(\Omega)), \quad \nabla_{\mathrm{x}}(\mathrm{tr}\,\log\tau) \in \mathrm{L}^2(0,\mathrm{T};\mathrm{L}^2(\Omega;\mathbb{R}^{\mathrm{d}})).$$

Problem:

Not enough regularity in τ.

A further bound in two space dimensions

From the equation in τ:

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\int_\Omega \frac{1}{2}|\tau|^2\,\mathrm{d}x + \varepsilon\int_\Omega |\nabla_x \tau|^2\,\mathrm{d}x + \frac{A_0}{2\lambda}\int_\Omega |\tau|^2\,\mathrm{d}x$$
$$\leq 4\int_\Omega |\nabla_x \mathbf{u}|\,|\tau|^2\,\mathrm{d}x + \frac{A_0}{4\lambda}\int_\Omega |\tau|^2\,\mathrm{d}x + \frac{2k^2\,A_0}{\lambda}\int_\Omega \eta^2\,\mathrm{d}x.$$

This implies

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\int_\Omega \frac{1}{2}|\tau|^2\,\mathrm{d}x + \varepsilon\int_\Omega |\nabla_x \tau|^2\,\mathrm{d}x + \frac{A_0}{4\lambda}\int_\Omega |\tau|^2\,\mathrm{d}x$$
$$\leq 4\|\nabla_x \mathbf{u}\|_{L^2(\Omega;\mathbb{R}^{2\times 2})}\,\|\tau\|^2_{L^4(\Omega;\mathbb{R}^{2\times 2})} + \frac{2k^2\,A_0}{\lambda}\int_\Omega \eta^2\,\mathrm{d}x.$$

Gagliardo–Nirenberg inequality $d = 2$:

$$\|\tau\|^2_{L^4(\Omega;\mathbb{R}^{2\times 2})} \leq C\,\|\tau\|_{L^2(\Omega;\mathbb{R}^{2\times 2})}\|\tau\|_{W^{1,2}(\Omega;\mathbb{R}^{2\times 2})}.$$

A further bound in two space dimensions

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\int_\Omega |\tau|^2\,\mathrm{d}x + \varepsilon\int_\Omega |\nabla_x \tau|^2\,\mathrm{d}x + \frac{A_0}{4\lambda}\int_\Omega |\tau|^2\,\mathrm{d}x \leq C\,\|\nabla_x \mathbf{u}\|^2_{L^2(\Omega)}\|\tau\|^2_{L^2(\Omega)} + \frac{4k^2\,A_0}{\lambda}\int_\Omega \eta^2\,\mathrm{d}x.$$

Gronwall's inequality implies that

$$\|\tau(t,\cdot)\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2})} \leq \|\tau_0\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2})}\,\mathrm{e}^{C\int_0^t \|\nabla_x \mathbf{u}(t',\cdot)\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2})}\,\mathrm{d}\,t'} + \frac{4k^2\,A_0}{\lambda}\int_0^t\int_\Omega \eta^2(t',x)\,\mathrm{d}x\,\mathrm{d}\,t'.$$

Moreover,

$$\int_\Omega |\tau|^2\,\mathrm{d}x + \varepsilon\int_0^t\int_\Omega |\nabla_x \tau|^2(t',x)\,\mathrm{d}x\,\mathrm{d}\,t' + \frac{A_0}{4\lambda}\int_0^t\int_\Omega |\tau|^2(t',x)\,\mathrm{d}x \leq C(t, E_0, \|\tau_0\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2})}).$$

A further bound in two space dimensions

Finally, we obtained enough regularity in τ:

$$\tau \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^{2\times 2})) \cap L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^{2\times 2})).$$

Above estimate is uniform as $\delta \to 0$:

It is sufficient to show $\|\eta\|_{L^2(0,T;L^2(\Omega))}$ is uniform as $\delta \to 0$. From (7),

$$\|\eta \log \eta\|_{L^\infty(0,T;L^1(\Omega))} + \|\nabla_x \eta^{\frac{1}{2}}\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^2))} \leq C \tag{8}$$

This gives $\|\eta\|_{L^2(0,T;W^{1,1}(\Omega))} \leq C$ by direct computation:

$$\int_\Omega |\nabla_x \eta| \,\mathrm{d}x = \int_\Omega |2\eta^{\frac{1}{2}} \nabla_x \eta^{\frac{1}{2}}| \,\mathrm{d}x \leq 2\|\eta\|_{L^1(\Omega)}^{\frac{1}{2}} \|\nabla_x \eta^{\frac{1}{2}}\|_{L^2(\Omega)},$$

As $d = 2$, the Sobolev embedding of $W^{1,1}(\Omega) \hookrightarrow L^2(\Omega)$ gives the desired result.

First level: artificial pressure approximation

Let $\sigma_1 > 0$ be small and $\Gamma \geq 4$.

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla_x p(\varrho) + \boxed{\sigma_1 \nabla_x \varrho^\Gamma} + \nabla_x\left(kL\eta + \delta\,\eta^2\right)$$
$$= \mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{u}) + \mathrm{div}_x\tau - \frac{\alpha}{2}\nabla_x\left(\mathrm{tr}\ \log\tau\right) + \varrho\,\mathbf{f},$$

Second level: dissipation approximation

Let $\sigma_2 > 0$ be small.

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = \boxed{\sigma_2 \Delta_x \varrho},$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) + \boxed{\sigma_1 \nabla_x \varrho^\Gamma} + \boxed{\sigma_2 \nabla_x \mathbf{u} \nabla_x \varrho}$$
$$+ \nabla_x \left(kL\eta + \delta\, \eta^2 \right) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \mathrm{div}_x \tau - \frac{\alpha}{2} \nabla_x \left(\mathrm{tr} \log \tau \right) + \varrho\, \mathbf{f},$$

Third level: Galerkin approximation

Consider eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ with $\lambda_n \to \infty$, $n \to \infty$, and an associated orthogonal eigenfunction basis in $L^2(\Omega; \mathbb{R}^2)$, denoted by $(\boldsymbol{\psi}_n)_{n\in\mathbb{N}}$, such that

$$-\Delta_x \boldsymbol{\psi}_n = \lambda_n \boldsymbol{\psi}_n \text{ in } \Omega; \quad \boldsymbol{\psi}_n = \mathbf{0} \text{ on } \partial\Omega.$$

Consider

$$\mathbf{u}_n \in C([0,T], X_n),\ \mathbf{u}_n(0) = \mathbf{u}_{0,n} = P_n \mathbf{u}_{0,\theta}; \quad \text{for any } \varphi \in X_n:$$

$$\int_\Omega \partial_t(\varrho_n \mathbf{u}_n) \cdot \varphi \, \mathrm{d}x$$

$$+ \int_\Omega \Big[\mathrm{div}_x(\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n) + \nabla_x p(\varrho_n) + \boxed{\sigma_1 \nabla_x \varrho_n^\Gamma} + \boxed{\sigma_2 \nabla_x \mathbf{u}_n \nabla_x \varrho_n}$$

$$+ \nabla_x \big(kL\eta_n + \delta\, \eta_n^2 \big) - \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}_n) \Big] \cdot \varphi \, \mathrm{d}x$$

$$= \int_\Omega \Big[\mathrm{div}_x \tau_n - \frac{\alpha}{2} \nabla_x \left(\mathrm{tr} \log \tau_\mathrm{n} \right) + \varrho_n \mathbf{f} \Big] \cdot \varphi \, \mathrm{d}x.$$

Third level: Galerkin approximation

Here $\varrho_n, \eta_n, \tau_n$ are determined by the parabolic equations:

$$\partial_t \varrho_n + \mathrm{div}_x(\varrho_n \mathbf{u}_n) = \boxed{\sigma_2 \Delta_x \varrho_n},$$

$$\partial_t \eta_n + \mathrm{div}_x(\eta_n \mathbf{u}_n) = \varepsilon \Delta_x \eta_n,$$

$$\partial_t \tau_n + \mathrm{Div}_x(\mathbf{u}_n \tau_n) - \left(\nabla_x \mathbf{u}_n \tau_n + \tau_n \nabla_x^{\mathrm{T}} \mathbf{u}_n \right) = \varepsilon \Delta_x \tau_n + \frac{k A_0}{2\lambda} \eta_n \mathbb{I} - \frac{A_0}{2\lambda} \tau_n.$$

Fourth level: regularization of the extra stress tensor

Let $\sigma_3 > 0$ small and $\chi_{\sigma_3}(s) := \max\{\sigma_3, s\}$. Consider:

$$\mathbf{u}_n \in C([0,T], X_n),\ \mathbf{u}_n(0) = \mathbf{u}_{0,n} = P_n \mathbf{u}_{0,\theta}; \quad \text{for any } \varphi \in X_n:$$

$$\int_\Omega \partial_t(\varrho_n \mathbf{u}_n) \cdot \varphi \,\mathrm{d}x$$

$$+ \int_\Omega \Big[\mathrm{div}_x(\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n) + \nabla_x p(\varrho_n) + \boxed{\sigma_1 \nabla_x \varrho_n^\Gamma} + \boxed{\sigma_2 \nabla_x \mathbf{u}_n \nabla_x \varrho_n}$$

$$+ \nabla_x \big(kL\eta_n + \delta\, \eta_n^2\big) - \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}_n) \Big] \cdot \varphi \,\mathrm{d}x$$

$$= \int_\Omega \Big[\boxed{\mathrm{div}_x \chi_{\sigma_3}(\tau_n)} - \frac{\alpha}{2} \nabla_x \left(\mathrm{tr}\, \log \tau_\mathrm{n}\right) + \varrho_n \mathbf{f} \Big] \cdot \varphi \,\mathrm{d}x.$$

Fourth level: regularization of the extra stress tensor

Here $\varrho_n, \eta_n, \tau_n$ are determined by the parabolic equations:

$$
\begin{aligned}
&\partial_t \varrho_n + \mathrm{div}_x(\varrho_n \mathbf{u}_n) = \boxed{\sigma_2 \Delta_x \varrho_n}, \\
&\partial_t \eta_n + \mathrm{div}_x(\eta_n \mathbf{u}_n) = \varepsilon \Delta_x \eta_n, \\
&\partial_t \tau_n + \mathrm{Div}_x(\mathbf{u}_n \boxed{\mathrm{div}_x \chi_{\sigma_3}(\tau_n)}) - \Big(\nabla_x \mathbf{u}_n \boxed{\mathrm{div}_x \chi_{\sigma_3}(\tau_n)} + \boxed{\mathrm{div}_x \chi_{\sigma_3}(\tau_n)} \nabla_x^{\mathrm{T}} \mathbf{u}_n \Big) \\
&\quad = \varepsilon \Delta_x \tau_n + \frac{k A_0}{2\lambda} \eta_n \mathbb{I} - \frac{A_0}{2\lambda} \boxed{\mathrm{div}_x \chi_{\sigma_3}(\tau_n)}.
\end{aligned}
$$

Final steps

1, Let $\alpha \to 0$ to finish the proof.

$\frac{\alpha}{2} \nabla_x (\mathrm{tr} \log \tau) \to 0$.

2, Let $\delta \to 0$ to cover the case $\delta = 0$.

$\nabla_x (kL\eta + \delta \eta^2)$ becomes $\nabla_x (kL\eta)$.

Local-in-time strong solutions

Data

Assume the external force $\mathbf{f} \in W^{1,2}((0,\infty)\times\Omega)$. Suppose

$$\varrho_0 \in W^{1,6}(\Omega),\ \eta_0 \in W^{2,2}_{\mathbf{n}}(\Omega),\ \mathbb{T}_0 \in W^{2,2}_{\mathbf{n}}(\Omega;\mathbb{R}^{d\times d}),\ \mathbf{u}_0 \in W^{1,2}_0 \cap W^{2,2}(\Omega;\mathbb{R}^d$$

where the notation $W^{2,2}_{\mathbf{n}}(\Omega) := \{f \in W^{2,2}(\Omega) :\ \partial_{\mathbf{n}} f = 0 \text{ on } \partial\Omega\}$.
Suppose there holds

$$-(\mu\Delta_x\mathbf{u}_0 + \nu\nabla_x \mathrm{div}_x\mathbf{u}_0) + \nabla_x p(\varrho_0) - \mathrm{div}_x\mathbb{T}_0 + \nabla_x(kL\eta_0 + \delta\,\eta_0^2) = \sqrt{\varrho_0}g$$

for some $g \in L^2(\Omega;\mathbb{R}^d)$.

Local-in-time strong solutions

Theorem (D. Fang-R. Zi 2013; Z. Zhang-L. 2017)

There exists a unique strong solution $(\varrho, \mathbf{u}, \eta, \mathbb{T})$ a maximal existence time $T_* \in (0, \infty]$ such that

$$\varrho \geq 0,\ \ \varrho \in C([0, T_*), W^{1,6}(\Omega)),$$
$$\mathbf{u} \in C([0, T_*), W_0^{1,2} \cap W^{2,2}(\Omega; \mathbb{R}^d)) \cap L^2_{\rm loc}([0, T_*); W^{2,r}(\Omega; \mathbb{R}^d)),$$
$$\eta \geq 0,\ \ \mathbb{T} = \mathbb{T}^{\rm T} \geq 0,\ \ (\eta, \mathbb{T}) \in C([0, T_*), W^{2,2}_{\mathbf{n}}) \cap L^2_{\rm loc}([0, T_*); W^{3,2})(\Omega; \mathbb{R} \times \mathbb{R}$$

where $r = 6$ when $d = 3$ and $r \in (1, \infty)$ is arbitrary when $d = 2$.
If $T_* < \infty$, the following quantity blow-up:

$$\limsup_{T \to T_*} \left(\|\varrho\|_{L^\infty((0,T)\times\Omega)} + \|\eta\|_{L^\infty((0,T)\times\Omega)} + \|\mathbb{T}\|_{L^2(0,T;L^\infty(\Omega;\mathbb{R}^{d\times d}))} \right) = \infty.$$

A refined blow-up criterion

Theorem (Z. Zhang-L. 2017)

Let $d = 2$ and $(\varrho, \mathbf{u}, \eta, \mathbb{T})$ be the strong solution with maximal existence time $T_* \in (0, \infty]$. If $T_* < \infty$, there holds

$$\limsup_{T \to T_*} \|\varrho\|_{L^\infty((0,T)\times\Omega)} = \infty.$$

Weak-strong uniqueness

Theorem (Z. Zhang-L. 2017)

Let $d = 2$. Let $(\varrho, \mathbf{u}, \eta, \mathbb{T})$ be a finite energy weak solution and $(\tilde{\varrho}, \tilde{\mathbf{u}}, \tilde{\eta}, \tilde{\mathbb{T}})$ be the strong solution with the same regular initial data. If in addition the initial data satisfy

$$\inf_{\Omega} \varrho_0 > 0, \quad \inf_{\Omega} \eta_0 > 0,$$

then there holds

$$(\varrho, \mathbf{u}, \eta, \mathbb{T}) = (\tilde{\varrho}, \tilde{\mathbf{u}}, \tilde{\eta}, \tilde{\mathbb{T}}) \quad \text{in} \quad [0, T_*) \times \Omega.$$

Conditional regularity

Theorem (Z. Zhang-L. 2017)

Let $d = 2$. Let $(\varrho, \mathbf{u}, \eta, \mathbb{T})$ be a finite energy weak solution with regular initial data satisfying

$$\inf_{\Omega} \varrho_0 > 0, \quad \inf_{\Omega} \eta_0 > 0,$$

If for some $T > 0$ there holds the upper bound

$$\sup_{(0,T)\times\Omega} \varrho < \infty,$$

then the weak solution $(\varrho, \mathbf{u}, \eta, \mathbb{T})$ is actually a strong one over time interval $[0, T]$.

Blow-up criterion– Step 1

$$\partial_t \rho - \varepsilon \Delta_x \rho = h \text{ in } (0,T) \times G; \ \rho(0,\cdot) = \rho_0 \text{ in } G; \ \partial_{\mathbf{n}} \rho = 0 \text{ in } (0,T) \times \partial G$$

Let $\rho_0 \in W_{\mathbf{n}}^{2-\frac{2}{p},q}$, $\quad h \in L^p(0,T;L^q(G))$. Then

$$\|\rho\|_{L^\infty(0,T;W^{2-\frac{2}{p},q}(G))} + \|\partial_t \rho\|_{L^p(0,T;L^q(G))} + \|\rho\|_{L^p(0,T;W^{2,q}(G))} \leq C.$$

Let $\rho_0 \in L^q(G)$, $h = \mathrm{div}\mathbf{g}$, $\mathbf{g} \in L^p(0,T;L^q(G;\mathbb{R}^d))$. Then

$$\|\rho\|_{L^\infty(0,T;L^q(G))} + \|\nabla_x \rho\|_{L^p(0,T;L^q(G;\mathbb{R}^d))} \leq C.$$

Blow-up criterion–Step 1

Further estimates for η and $\mathbb{T}$:

For any $r \in (1, \infty)$, there holds

$$\|\eta\|_{L^\infty(0,T_*;L^r(\Omega))} + \|\eta\|_{L^2(0,T_*;W^{1,r}(\Omega))} \leq C \tag{9}$$

and

$$\|\mathbb{T}\|_{L^\infty(0,T_*;L^r(\Omega))} + \|\mathbb{T}\|_{L^2(0,T_*;W^{1,r}(\Omega))} \leq C. \tag{10}$$

Blow-up criterion–Step 2

By contradiction, we assume $\|\varrho\|_{L^\infty((0,T_*)\times\Omega)} < \infty$.

Uniform estimates:

$$\|\varrho|\mathbf{u}|^\alpha\|_{L^\infty(0,T_*;L^1(\Omega))} \le C < \infty, \text{ for some } \alpha > 2. \tag{11}$$

$$\int_\Omega \text{momentum equation} \cdot |\mathbf{u}|^{\alpha-2}\mathbf{u}.$$

Blow-up criterion–Step 3

Define $\mathbf{v}_\varrho, \mathbf{v}_\eta, \mathbf{v}_\tau$ such that:

$$\begin{cases} -\mu\Delta_x \mathbf{v}_\varrho - \nu\nabla_x \mathrm{div}_x \mathbf{v}_\varrho = \nabla_x p(\varrho), & \text{in } \Omega, \\ \mathbf{v}_\varrho = 0, & \text{on } \partial\Omega, \end{cases}$$
$$\begin{cases} -\mu\Delta_x \mathbf{v}_\eta - \nu\nabla_x \mathrm{div}_x \mathbf{v}_\eta = \nabla_x (kL\eta + \delta\eta^2), & \text{in } \Omega, \\ \mathbf{v}_\eta = 0, & \text{on } \partial\Omega, \end{cases} \tag{12}$$
$$\begin{cases} -\mu\Delta_x \mathbf{v}_\tau - \nu\nabla_x \mathrm{div}_x \mathbf{v}_\tau = -\mathrm{div}_x \mathbb{T}, & \text{in } \Omega, \\ \mathbf{v}_\tau = 0, & \text{on } \partial\Omega. \end{cases}$$

Define $\mathbf{w} := \mathbf{u} - \mathbf{v}, \quad \mathbf{v} := (\mathbf{v}_\varrho + \mathbf{v}_\eta + \mathbf{v}_\tau)$. Then

$$\varrho\partial_t \mathbf{w} - \mu\Delta_x \mathbf{w} - \nu\nabla_x \mathrm{div}_x \mathbf{w} = -\varrho\mathbf{u}\cdot\nabla_x \mathbf{u} - \varrho\partial_t \mathbf{v}, \tag{13}$$

with no slip boundary condition

$$\mathbf{w} = 0 \quad \text{on } (0, T_*) \times \partial\Omega. \tag{14}$$

Blow-up criterion–Step 3

For some $T_1 \in (0, T_*)$,

$$\mathbf{w} \in L^\infty(0, T_*; W_0^{1,2}(\Omega; \mathbb{R}^2)) \cap L^2(0, T_*; W^{1,r}(\Omega; \mathbb{R}^2)) \cap L^2(T_1, T_*; W^{2,2}(\Omega; \mathbb{R}^2))$$

This implies

$$\mathbf{u} \in L^\infty(0, T_*; W_0^{1,2}) \cap L^2(0, T_*; W^{1,r}) \quad \text{for any } r \in (1, \infty).$$

Blow-up criterion–Step 4

Again by the regular results for parabolic problems, we finally obtain:

$$\|\eta\|_{L^\infty(0,T_*;L^\infty(\Omega))} \leq C < \infty$$

and

$$\|\mathbb{T}\|_{L^\infty(0,T_*;L^\infty(\Omega))} \leq C < \infty.$$

Relative entropy

Define:

$$H(s) := \frac{a}{\gamma - 1} s^{\gamma}, \quad G(s) := \left(kLs \log s + \delta\, s^2 \right).$$

Relative entropy functional

$$\mathcal{E}_1(\varrho, \mathbf{u}, \tilde{\varrho}, \tilde{\mathbf{u}})(t) := \int_{\Omega} \frac{1}{2} \varrho |\mathbf{u} - \tilde{\mathbf{u}}|^2 + \left(H(\varrho) - H(\tilde{\varrho}) - H'(\tilde{\varrho})(\varrho - \tilde{\varrho}) \right)(t, \cdot)\, \mathrm{d}x,$$

$$\mathcal{E}_2(\eta, \tilde{\eta})(t) := \int_{\Omega} \left(G(\eta) - G(\tilde{\eta}) - G'(\tilde{\eta})(\eta - \tilde{\eta}) \right)(t, \cdot)\, \mathrm{d}x.$$

Relative entropy

There exists $\delta > 0$, $c > 0$ depending only on a and γ such that for any $\varrho, \tilde{\varrho} \geq 0$,

$$H(\varrho) - H(\tilde{\varrho}) - H'(\tilde{\varrho})(\varrho - \tilde{\varrho}) \geq \begin{cases} c\tilde{\varrho}^{\gamma-2}(\varrho - \tilde{\varrho})^2, & \text{if } \delta\tilde{\varrho} \leq \varrho \leq \delta^{-1}\tilde{\varrho}, \\ c\max\{\varrho^\gamma, \tilde{\varrho}^\gamma\}, & \text{otherwise.} \end{cases} \tag{15}$$

For any $\eta, \tilde{\eta} \geq 0$, there holds

$$G(\eta) - G(\tilde{\eta}) - G'(\tilde{\eta})(\eta - \tilde{\eta}) \geq 2\delta(\eta - \tilde{\eta})^2 + \begin{cases} \dfrac{kL(\eta - \tilde{\eta})^2}{2\tilde{\eta}}, & \text{if } \eta \leq 2\tilde{\eta}, \\ \dfrac{kL\eta}{4}, & \text{if } \eta \geq 2\tilde{\eta}. \end{cases} \tag{16}$$

Relative entropy inequality

There holds

$$\mathcal{E}(t) + \int_0^t \int_\Omega \mu \, |\nabla_x(\mathbf{u} - \tilde{\mathbf{u}})|^2 + \nu |\mathrm{div}_x(\mathbf{u} - \tilde{\mathbf{u}})|^2 \, \mathrm{d}x \, \mathrm{d}\, t'$$
$$+ 2\varepsilon \int_0^t \int_\Omega 2kL |\nabla_x(\eta^{\frac{1}{2}} - \tilde{\eta}^{\frac{1}{2}})|^2 + \delta \, |\nabla_x(\eta - \tilde{\eta})|^2 \, \mathrm{d}x \, \mathrm{d}\, t'$$
$$\leq \mathcal{E}(0) + \int_0^t \mathcal{R}(t') \, \mathrm{d}\, t'.$$

Relative entropy inequality

The remainder $\mathcal{R}(t) = \sum_{j=1}^{5} \mathcal{R}_j(t)$ with

$$
\begin{aligned}
\mathcal{R}_1(t) := & \int_\Omega \varrho(\partial_t \tilde{\mathbf{u}} + \mathbf{u} \cdot \nabla_x \tilde{\mathbf{u}}) \cdot (\tilde{\mathbf{u}} - \mathbf{u}) \, \mathrm{d}x \\
& + \int_\Omega \mu \nabla_x \tilde{\mathbf{u}} : \nabla_x (\tilde{\mathbf{u}} - \mathbf{u}) + \nu \mathrm{div}_x \tilde{\mathbf{u}} \, \mathrm{div}_x (\tilde{\mathbf{u}} - \mathbf{u}) \, \mathrm{d}x + \int_\Omega \varrho \mathbf{f} \cdot (\mathbf{u} - \tilde{\mathbf{u}}) \, \mathrm{d}x \\
& + \int_\Omega (\tilde{\varrho} - \varrho) \partial_t H'(\tilde{\varrho}) + (\tilde{\varrho} \tilde{\mathbf{u}} - \varrho \mathbf{u}) \cdot \nabla_x H'(\tilde{\varrho}) \, \mathrm{d}x \\
& + \int_\Omega \mathrm{div}_x \tilde{\mathbf{u}} (p(\tilde{\varrho}) - p(\varrho)) \, \mathrm{d}x, \\
\mathcal{R}_2(t) := & \int_\Omega (\tilde{\eta} - \eta) \partial_t G'(\tilde{\eta}) + (\tilde{\eta} \tilde{\mathbf{u}} - \eta \mathbf{u}) \cdot \nabla_x G'(\tilde{\eta}) \, \mathrm{d}x \\
& + \int_\Omega \mathrm{div}_x \tilde{\mathbf{u}} (q(\tilde{\eta}) - q(\eta)) \, \mathrm{d}x,
\end{aligned}
$$

Relative entropy inequality

The remainder $\mathcal{R}(t) = \sum_{j=1}^{5} \mathcal{R}_j(t)$ with

$$\mathcal{R}_3(t) := -4\varepsilon k L \int_\Omega \nabla_x \tilde{\eta}^{\frac{1}{2}} \cdot \nabla_x (\eta^{\frac{1}{2}} - \tilde{\eta}^{\frac{1}{2}}) + \nabla_x \eta^{\frac{1}{2}} \cdot \nabla_x \tilde{\eta}^{\frac{1}{2}} (1 - \tilde{\eta}^{-\frac{1}{2}} \eta^{\frac{1}{2}}) \, \mathrm{d}x,$$

$$\mathcal{R}_4(t) := -2\varepsilon\delta \int_\Omega \nabla_x \tilde{\eta} \cdot \nabla_x (\eta - \tilde{\eta}) \, \mathrm{d}x,$$

$$\mathcal{R}_5(t) := \int_\Omega \mathbb{T} : \nabla_x (\tilde{\mathbf{u}} - \mathbf{u}) \, \mathrm{d}x.$$

Thank you for your attention!