On the existence of integrable solutions to nonlinear elliptic systems and variational problems with linear growth

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The talk is based on the following results

- M. Bulíček, J. Málek, K. R. Rajagopal and J. R. Walton: **Existence of solutions for the anti-plane stress for a new class of “strain-limiting” elastic bodies**, Calc. Var. Partial Differential Equations, 2015
- M. Bulíček, J. Málek and E. Süli: **Analysis and approximation of a strain-limiting nonlinear elastic model**, Mathematics and Mechanics of Solids, 2014
- M. Bulíček, J. Málek, K. R. Rajagopal and E. Süli: **On elastic solids with limiting small strain: modelling and analysis**, EMS Surveys in Mathematical Sciences, 2014.
- L. Beck, M. Bulíček, J. Málek and E. Süli: **On the existence of integrable solutions to nonlinear elliptic systems and variational problems with linear growth**, ARMA 2017
- L. Beck, M. Bulíček, E. Maringová: **On regularity up to the boundary for variational problems with linear growth**, submitted

Linearized nonlinear elasticity

We consider the elastic deformation of the body $\Omega \subset \mathbb{R}^d$ with $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\overline{\Gamma_D \cup \Gamma_N} = \partial\Omega$ described by

$$\begin{aligned} -\operatorname{div} \mathbf{T} &= \boldsymbol{f} && \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{u}_0 && \text{on } \Gamma_D, \\ \mathbf{T}\boldsymbol{n} &= \boldsymbol{g} && \text{on } \Gamma_N. \end{aligned} \tag{El}$$

where $\boldsymbol{u}$ is displacement, $\mathbf{T}$ the Cauchy stress, $\boldsymbol{f}$ the external body forces, $\boldsymbol{g}$ the external surface forces and ε is the linearized strain tensor, i.e.,

$$\varepsilon = \varepsilon(\boldsymbol{u}) := \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)$$

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- The key assumption in linearized elasticity

$$\boxed{|\boldsymbol{\varepsilon}| \ll 1}. \tag{A}$$

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$\Longrightarrow$ contradicts the assumption of the model (A) $\Longrightarrow$ not valid model at least in the neighborhood of x_0.

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- Consider implicit models which a priori guarantees $|\varepsilon| \le K$:

$$\boxed{\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^*(\mathbf{T}) := \lambda_1(|\operatorname{tr}\mathbf{T}|)(\operatorname{tr}\mathbf{T})\mathbf{I} + \lambda_2(|\mathbf{T}|)\mathbf{T} + \lambda_3(|\mathbf{T}^d|)\mathbf{T}^d}, \tag{L-S}$$

where

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From the equation, we may hope that

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- The reasonable assumptions (∞-Laplacian-like problem):

$$\left.\lambda_{1,2,3}(s) \ge \frac{\alpha}{s+1}.\quad\right\} \Longrightarrow \int_\Omega |\mathbf{T}| \le C.$$

Limiting strain model & monotonicity

- Apriori estimates for $\mathbf{T}$ in L^1
- For the convergence at least some monotonicity needed, the minimal assumption:

$$0 \le \frac{d}{ds}(\lambda_{1,2,3}(s)s). \tag{M}$$

- If we would have a sequence fulfilling

$$\int_{\Omega_0} |\mathbf{T}^n|^{1+\delta} \le C(\Omega_0) \qquad \text{for all } \Omega_0 \subset\subset \Omega,$$
$$\implies \mathbf{T}^n \rightharpoonup \mathbf{T} \qquad \text{weakly in } L^1_{loc}.$$

  then using (M) we can identify the limit.
- Assume kind of uniform monotonicity, i.e., for some $\alpha, a, K > 0$

$$\frac{\alpha}{(K+s)^{a+1}} \le \frac{d}{dt}(\lambda_i(s)s) \tag{UM}$$

  for example

$$\boxed{\lambda_i(s) := \frac{1}{(1+s^a)^{\frac{1}{a}}}} \qquad \text{for simplicity} \qquad \boxed{\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^*(\mathbf{T}) := \frac{\mathbf{T}}{(1+|\mathbf{T}|^a)^{\frac{1}{a}}}}.$$

Simplified setting - potential structure

We look for $(\boldsymbol{u}, \mathbf{T})$ such that $\boldsymbol{u} = \boldsymbol{u}_0$ on Γ_D and $\mathbf{T}\boldsymbol{n} = \boldsymbol{g}$ on Γ_N such that in Ω there holds

$$\left.\begin{aligned} -\operatorname{div} \mathbf{T} &= \boldsymbol{f}, \\ \varepsilon(\boldsymbol{u}) &= \varepsilon^*(\mathbf{T}). \end{aligned}\right\} \quad \Leftrightarrow \quad \left\{ -\operatorname{div} \mathbf{T}^*(\varepsilon(\boldsymbol{u})) = \boldsymbol{f}. \right.$$

with

$$\boxed{\varepsilon^*(\mathbf{T}) := \frac{\mathbf{T}}{(1+|\mathbf{T}|^a)^{\frac{1}{a}}} \qquad \text{and} \qquad T^*(\mathbf{W}) := (\varepsilon^*)^{-1}(\mathbf{W}) := \frac{\mathbf{W}}{(1-|\mathbf{W}|^a)^{\frac{1}{a}}}}$$

for all $\mathbf{T} \in \mathbb{R}^{d\times d}_{sym}$ and $\mathbf{W} \in \mathbb{R}^{d\times d}_{sym}$ such that $|\mathbf{W}| < 1$.

Simplified setting - potential structure

First, we introduce the space of functions having bounded the symmetric gradient

$$E := \{\boldsymbol{u} \in W^{1,1}(\Omega)^d;\ \boldsymbol{\varepsilon}(\boldsymbol{u}) \in L^\infty(\Omega)^{d\times d}\}.$$

and assume at least $\boldsymbol{u}_0 \in E$, $\boldsymbol{f} \in L^2(\Omega)^d$ and $\boldsymbol{g} \in L^1(\Gamma_N)^d$.

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- the set of admissible displacement

$$\mathcal{V} := \{\boldsymbol{u} \in W^{1,1}(\Omega):\ \boldsymbol{u} - \boldsymbol{u}_0 \in W^{1,1}_{\Gamma_D}(\Omega)^d,\ \boldsymbol{u} \in E\}$$

- the set of admissible stresses

$$\mathcal{S} := \left\{\mathbf{T} \in L^1(\Omega)^{d\times d}_{sym}:\ \forall \boldsymbol{v} \in E \cap W^{1,1}_{\Gamma_D}\ \int_\Omega \mathbf{T}\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} + \int_{\Gamma_N} \boldsymbol{g}\cdot\boldsymbol{v}\right\}$$

Weak solution: Find $(\boldsymbol{u}, \mathbf{T}) \in \mathcal{V}\times\mathcal{S}$ such that $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \boldsymbol{\varepsilon}^*(\mathbf{T})$ a.e. in Ω.

Potential structure - primary formulation

Find potential $F : \mathbb{R}^{d\times d}_{sym} \to \mathbb{R}_+$ such that $F(0) = 0$ and

$$\frac{\partial F(\mathbf{W})}{\partial \mathbf{W}} = \mathbf{T}^*(\mathbf{W}) \quad \text{if } |\mathbf{W}| < 1,$$
$$F(\mathbf{W}) = \infty \quad \text{if } |\mathbf{W}| > 1.$$

Primary (variational) formulation: Find $\boldsymbol{u} \in \mathcal{V}$ such that for all $\boldsymbol{v} \in \mathcal{V}$

$$\boxed{\int_\Omega F(\varepsilon(\boldsymbol{u})) - f\cdot \boldsymbol{u} - \int_{\Gamma_N} \boldsymbol{g}\cdot\boldsymbol{u} \le \int_\Omega F(\varepsilon(\boldsymbol{v})) - f\cdot\boldsymbol{v} - \int_{\Gamma_N} \boldsymbol{g}\cdot\boldsymbol{v}}$$

Lemma

Let $\|\boldsymbol{\varepsilon}(\boldsymbol{u}_0)\|_\infty < 1$ (the safety strain condition). Then there exists a unique $\boldsymbol{u}$ solving the primary formulation. Moreover there exists $\mathbf{T} \in L^1(\Omega)^{d\times d}$ such that $\varepsilon(\boldsymbol{u}) = \varepsilon^*(\mathbf{T})$ and for all $\boldsymbol{v} \in \mathcal{V}$ such that $\mathbf{T}^*(\boldsymbol{\varepsilon}(\boldsymbol{v})) \in L^1$ there holds

$$\int_\Omega \mathbf{T}\cdot\varepsilon(\boldsymbol{u}-\boldsymbol{v}) \le \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{u}-\boldsymbol{v}) + \int_{\Gamma_N} \boldsymbol{g}\cdot(\boldsymbol{u}-\boldsymbol{v})$$

In addition, if there is a weak solution then it also solves the primary formulation. Similarly, if $\boldsymbol{u}$ satisfies the safety strain condition, then $(\boldsymbol{u},\mathbf{T})$ is a weak solution.

Potential structure - dual formulation

Find potential $F^*:\mathbb{R}^{d\times d}_{sym}\to\mathbb{R}_+$ such that $F(0)=0$ and (note here that $F(\mathbf{W})\sim|\mathbf{W}|$ at infinity

$$\frac{\partial F^*(\mathbf{W})}{\partial \mathbf{W}} = \boldsymbol{\varepsilon}^*(\mathbf{W}).$$

Dual (variational) formulation: Find $\mathbf{T}\in\mathcal{S}$ such that for all $\mathbf{W}\in\mathcal{S}$

$$\boxed{\int_\Omega F^*(\mathbf{T})-\mathbf{T}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_0)\le\int_\Omega F(\mathbf{W})-\mathbf{W}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_0)}$$

Lemma

The existence of weak solution is equivalent to the existence of the minimizer to the dual problem. Moreover, if $\|\varepsilon(\boldsymbol{u}_0)\|_\infty<1$ (the safety strain condition) then there exists a finite infimum of the dual formulation which maybe attained by $\overline{\mathbf{T}}\in\mathcal{M}(\overline{\Omega})^{d\times d}_{sym}$.

Potential structure - relaxed dual formulation

- the relaxed set of admissible stresses

$$\mathcal{S}^m := \left\{ \mathbf{T} \in \mathcal{M}(\overline{\Omega})^{d\times d}_{sym} : \ \forall \boldsymbol{v} \in \mathcal{C}^1_{\Gamma_D}(\Omega)^d \ \int_\Omega \mathbf{T} \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} + \int_{\Gamma_N} \boldsymbol{g} \cdot \boldsymbol{v} \right\}$$

Dual (variational) relaxed formulation: For $\boldsymbol{u}_0 \in \mathcal{C}^1(\Omega)^d$, find $\mathbf{T} \in \mathcal{S}^m$ such that for all $\mathbf{W} \in \mathcal{S}^m$

$$\boxed{\int_\Omega F^*(\mathbf{T}^r) + (\mathbf{W}^r - \mathbf{T}^r)\cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_0) + |\mathbf{T}^s|(\overline{\Omega}) + \langle \mathbf{W}^s - \mathbf{T}^s, \boldsymbol{\varepsilon}(\boldsymbol{u}_0)\rangle \le \int_\Omega F^*(\mathbf{W}^r) + |\mathbf{W}^s|(\overline{\Omega})}$$

where $\mathbf{T} = \mathbf{T}^r + \mathbf{T}^s$ and $\mathbf{T}^r$ is a regular part (i.e., absolutely continuous w.r.t. Lebesgue measure) and $\mathbf{T}^s$ is a singular part (i.e., supported on the set of zero Lebesgue measure).

Lemma

Let $\|\boldsymbol{\varepsilon}(\boldsymbol{u}_0)\|_\infty < 1$. Then there exists a minimizer to relaxed dual formulation. Moreover, the regular part $\mathbf{T}^r$ is unique and satisfies $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \boldsymbol{\varepsilon}^*(\mathbf{T}^r)$, where $\boldsymbol{u}$ is (unique) minimizer to primary formulation. In addition, if $\mathbf{T}^s_1$ and $\mathbf{T}^s_2$ are two singular parts then for all $\boldsymbol{v} \in \mathcal{C}^1_{\Gamma_D}(\Omega)^d$

$$|\mathbf{T}^s_1|(\overline{\Omega}) - \langle \mathbf{T}^s_1, \boldsymbol{\varepsilon}(\boldsymbol{u}_0)\rangle = |\mathbf{T}^s_2|(\overline{\Omega}) - \langle \mathbf{T}^s_2, \boldsymbol{\varepsilon}(\boldsymbol{u}_0)\rangle \ \textit{and} \ \langle \mathbf{T}^s_1 - \mathbf{T}^s_2, \nabla \boldsymbol{v}\rangle = 0$$

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- Where is the singular measure supported? Is it really there? How do you explain that the regular part did not solve the balance equation? Is there some crack/damage possible region? Is there any influence of the shape Ω or the parameter a? etc. etc.

Limiting strain model - anti-plane stress

We consider the following special geometry

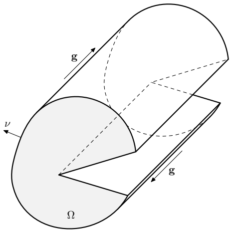

Figure: Anti-plane stress geometry.

and we look for the solution in the following from:

$$\boldsymbol{u} = \boldsymbol{u}(x_1, x_2) = (0, 0, u(x_1, x_2)), \quad \boldsymbol{g}(x) = (0, 0, g(x_1, x_2)),$$

and

$$\mathbf{T}(x) = \begin{pmatrix} 0 & 0 & T_{13}(x_1, x_2) \\ 0 & 0 & T_{23}(x_1, x_2) \\ T_{13}(x_1, x_2) & T_{23}(x_1, x_2) & 0 \end{pmatrix}. \tag{1}$$

Equivalent reformulation-simply connected domain

- Find $U:\Omega\to\mathbb{R}$ - the Airy stress function such that

$$T_{13}=\frac{1}{\sqrt{2}}U_{x_2}\quad\text{and}\quad T_{23}=-\frac{1}{\sqrt{2}}U_{x_1}.$$

$\Longrightarrow\ \operatorname{div}\mathbf{T}=\mathbf{0}$ is fulfilled.

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- U must satisfy $(\varepsilon(\boldsymbol{u}) = \frac{\mathbf{T}}{(1+|\mathbf{T}|^a)^{\frac{1}{a}}})$

$$\operatorname{div}\left(\frac{\nabla U}{(1+|\nabla U|^a)^{\frac{1}{a}}}\right) = 0 \qquad \text{in } \Omega,$$

$$U_{x_2} \boldsymbol{n}_1 - U_{x_1} \boldsymbol{n}_2 = \sqrt{2} g \qquad \text{on } \partial\Omega.$$

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- Dirichlet problem, indeed assume that $\partial\Omega$ is parameterized by $\gamma(s) = (\gamma_1(s), \gamma_2(s))$. Then

$$U(\gamma(s_0)) = a_0 + \sqrt{2}\int_0^{s_0} g(\boldsymbol{\gamma}(s))\sqrt{(\gamma_1'(s))^2 + (\gamma_2'(s))^2}ds =: U_0(x).$$

Consequences for U

- We look for $U \in W^{1,1}(\Omega)$

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- In general does not exists - relaxed formulation: fixed $\Omega \subset\subset \Omega_0$ and find $U \in BV(\Omega_0)$ such that $U = U_0$ in $\Omega_0 \setminus \overline{\Omega}$ and

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$$\int_\Omega F^*((\nabla U)^r) + |\nabla U^s|(\overline{\Omega}) \le \int_\Omega F^*((\nabla V)^r) + |\nabla V^s|(\overline{\Omega}).$$
- We have the same result as before:(But consider $a = 2$ then we know that $(\nabla U)^s$ is supported only on $\partial\Omega$ and we have "half"-relaxed formulation: Find $u \in W^{1,1}(\Omega)$ such that
$$\int_\Omega \sqrt{1+|\nabla U|^2} + \int_{\partial\Omega} |U - U_0| \le \int_\Omega \sqrt{1+|\nabla V|^2} + \int_{\partial\Omega} |V - U_0|.$$

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- $a < 2$ we can localize and prove $\nabla U \in L^{\infty}_{loc}$
- $a \in (1, 2)$ the weak solution may not exists eg. for $\Omega = B_2 \setminus B_1$
- on the flat part of the boundary, one can extend the solution outside

Consequences for solution in general case/geometry

- Bildhauer & Fuchs (2001–): General theory for $a \in (0, 2]$ there exists $u \in W^{1,1}(\Omega)$

$$\int_\Omega F^*(\nabla U) + \int_{\partial\Omega} |U - U_0| \leq \int_\Omega F^*(\nabla V) + \int_{\partial\Omega} |V - U_0|.$$

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i.e., smoothness locally in Ω, the trace may not be attained; for convex domains everything is nice up to the boundary
- We cannot solve the problem in general for the Neumann data - counterexamples
- Maybe we can avoid to be **T** measure in the interior of Ω at last for some a's

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$$\int_\Omega F^*(\nabla U) + \int_{\partial\Omega} |U - U_0| \leq \int_\Omega F^*(\nabla V) + \int_{\partial\Omega} |V - U_0|.$$

  i.e., smoothness locally in Ω, the trace may not be attained; for convex domains everything is nice up to the boundary
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- Maybe the Dirichlet problem is easier to handle - we do not need the estimates up to the boundary
- But in all cases we need to face the problem with symmetric gradient contrary to the full gradient as in Bildhauer & Fuchs
- Is really the assumption $a \le 2$ essential? Counterexamples only for non-smooth data

Limiting strain - anti-plane stress geometry

Theorem (anti-plane stress)

Let U_0 be arbitrary. Then there exists unique weak solution U provided that one of the following holds.

- Ω *is uniformly convex,* $a > 0$ *is arbitrary and* U_0 *smooth.*
- $a \in (0,2)$ *and* $\partial\Omega = \bigcup_{i=1}^{N} \Gamma_i$ *such that either* Γ_i *is uniformly convex and* U_0 *is smooth on* Γ_i *or* Γ_i *is flat and* U_0 *is constant there.*
- $a \in (0,1]$, Ω *arbitrary piece-wise* $\mathcal{C}^{1,1}$ *and* U_0 *piece-wise in* $\mathcal{C}^{1,1}$. *Moreover, if* U_0 *and* Ω *smooth then* U *is* $\mathcal{C}^{1,\alpha}(\overline{\Omega})$.

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Theorem (anti-plane stress II)

Let $a \in (0, 2]$, U_0 and $\Omega \subset \mathbb{R}^d$ be arbitrary. Then there exists unique weak solution $U \in W^{1,1}(\Omega)$ in the following sense

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Defining $\mathbf{T}_{13} := U_{x_2}$ and $\mathbf{T}_{23} := -U_{x_1}$ we have $\operatorname{div} \mathbf{T} = 0$ but $\mathbf{T}\boldsymbol{n} = \boldsymbol{g}$ is not attained but we have "best approximation".

General result

Theorem (Beck, Bulíček, Maringová)

Let $F \in \mathcal{C}^2(0,\infty)$ be increasing strictly convex fulfilling

$$\lim_{s\to\infty} \frac{F(s)}{s} = \lim_{s\to\infty} F'(s) = K > 0.$$

Then the following is equivalent

- *For any $\Omega \in \mathcal{C}^{1,1}$ and any $u_0 \in \mathcal{C}^{1,1}(\overline{\Omega})$ there exists unique $u \in W^{1,\infty}(\Omega)$ fulfilling*

$$\boxed{\int_\Omega F(|\nabla u|) \le \int_\Omega F(|\nabla u_0 + \nabla\varphi|) \qquad \textit{for all } \varphi \in W_0^{1,1}(\Omega).}$$

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The second condition is equivalent to the fact that

$$\lim_{s\to K_-} F^*(s) = \infty.$$

Result for particular model and general geometry

Consider $\boldsymbol{\varepsilon}^*(\mathbf{T}) = \mathbf{T}/(1+|\mathbf{T}|^a)^{\frac{1}{a}}$:

Theorem (General result for $a > 0$)

Let $a > 0$ and $\boldsymbol{u}_0$ satisfy the safety strain condition. Then there exists a unique triple $(\boldsymbol{u}, \mathbf{T}, \tilde{\boldsymbol{g}}) \in \mathcal{V} \times L^1(\Omega)^{d\times d}_{sym} \times (\mathcal{C}^1_0(\Gamma_N))^*$ such that for all $\boldsymbol{v} \in \mathcal{C}^1_{\Gamma_D}(\overline{\Omega})$

$$\varepsilon(\boldsymbol{u}) = \varepsilon^*(\mathbf{T})$$

$$\int_\Omega \mathbf{T}\cdot\varepsilon(\boldsymbol{u}-\boldsymbol{w}) \le \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{u}-\boldsymbol{w}) + \int_{\Gamma_N} \boldsymbol{g}\cdot(\boldsymbol{u}-\boldsymbol{w})$$

$$\boldsymbol{u} = \boldsymbol{u}_0 \ \ on \ \Gamma_D,$$

where $\boldsymbol{w} \in \mathcal{V}$ is arbitrary such that there exists $\tilde{\mathbf{T}} \in L^1$ fulfilling $\varepsilon(\boldsymbol{w}) = \varepsilon^*(\tilde{\mathbf{T}})$.

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$$\boxed{\int_\Omega \mathbf{T}\cdot\nabla\boldsymbol{v} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} + \langle \boldsymbol{g}-\tilde{\boldsymbol{g}}, \boldsymbol{v}\rangle_{\Gamma_N}}$$

Assumptions for general model

Assumptions on ε^*: Denote $\mathbf{A}(\mathbf{T}) := \frac{\partial \boldsymbol{\varepsilon}^*(\mathbf{T})}{\partial \mathbf{T}}$.

- ε^* is coercive, i.e.,
$$\boldsymbol{\varepsilon}^*(\mathbf{T}) \cdot \mathbf{T} \geq C_1|\mathbf{T}| - C_2$$
- $\boldsymbol{\varepsilon}^*$ is h-elliptic, i.e., there exists nonincreasing function h such that for all $\mathbf{W} \neq 0$
$$0 < h(|\mathbf{T}|)|\mathbf{W}|^2 \leq (\mathbf{W}, \mathbf{W})_{\mathbf{A}(\mathbf{T})} \leq \frac{|\mathbf{W}|^2}{1 + |\mathbf{T}|},$$
where
$$(\mathbf{W}, \mathbf{W})_{\mathbf{A}(\mathbf{T})} := \sum \mathbf{A}_{\mu j}^{\nu i}(\mathbf{T})\mathbf{W}^{\nu i}\mathbf{W}^{\mu j}, \qquad \mathbf{A}_{\mu j}^{\nu i}(\mathbf{T}) := \frac{\partial(\boldsymbol{\varepsilon}^*)^{\nu i}(\mathbf{T})}{\partial \mathbf{T}^{\mu j}}.$$
- $\mathbf{A}$ is asymptotically symmetric, i.e.,
$$\frac{|\mathbf{A}^s(\mathbf{T}) - \mathbf{A}(\mathbf{T})|^2}{h(|\mathbf{T}|)} \leq \frac{C_2}{1 + |\mathbf{T}|}.$$
- either h does not decrease faster than $|\mathbf{T}|^{-1-2/d}$ or $\boldsymbol{\varepsilon}^*$ is asymptotically radial, i.e., there exists a function g such that $g(|\mathbf{T}|) \leq C(1 + |\mathbf{T}|)$ fulfilling
$$\frac{|g(|\mathbf{T}|)\boldsymbol{\varepsilon}^*(\mathbf{T}) - \mathbf{T}|^2}{h(|\mathbf{T}|)} \leq C_2(1 + |\mathbf{T}|^3).$$

Assumptions for general models

Assumptions on data:

- $\boldsymbol{f} \in L^2$
- $\boldsymbol{g} \in L^1$
- $\boldsymbol{u}_0$ satisfies safety strain condition, i.e., there exists a compact set $K \subset \varepsilon^*(\mathbb{R}^{d\times d}_{sym})$ such that for almost all $x \in \Omega$

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_0(x)) \in K$$

.

Result for limiting strain models

Theorem (General result)

There exists a unique triple $(\boldsymbol{u}, \mathbf{T}, \tilde{\boldsymbol{g}}) \in W^{1,1}(\Omega)^d \times L^1(\Omega)^{d\times d}_{sym} \times (\mathcal{C}^1_0(\Gamma_d))^*$ *such that* $\boldsymbol{u} - \boldsymbol{u}_0 \in W^{1,1}_{\Gamma_D}(\Omega'\mathbb{R}^d)$ *and for all* $\boldsymbol{v} \in \mathcal{C}^1_{\Gamma_D}(\overline{\Omega})$

$$\int_\Omega \mathbf{T} \cdot \varepsilon(\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} + \langle \boldsymbol{g} - \tilde{\boldsymbol{g}}, \boldsymbol{v} \rangle_{\Gamma_N}$$

$$\varepsilon(\boldsymbol{u}) = \varepsilon^*(\mathbf{T}) \in L^\infty(\Omega; \mathbb{R}^{d\times d})$$

Moreover, for all $\boldsymbol{w} \in W^{1,\infty}(\Omega)$ *being equal to* $\boldsymbol{u}_0$ *on* Γ_D *such that there exists* $\tilde{\mathbf{T}} \in L^1(\Omega)^{d\times d}_{sym}$ *fulfilling* $\varepsilon(\boldsymbol{w}) = \varepsilon^*(\tilde{\mathbf{T}})$ *we have*

$$\int_\Omega \mathbf{T} \cdot \varepsilon(\boldsymbol{u} - \boldsymbol{w}) \le \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{u} - \boldsymbol{w}) + \int_{\Gamma_N} \boldsymbol{g} \cdot (\boldsymbol{u} - \boldsymbol{w})$$

Conclusion II

- The first result for the symmetric gradient, where the structure of the nonlinearity plays the crucial role
- The same result obviously holds also for the full gradient case
- For any $\mathcal{C}^1$ strictly monotone operator being asymptotically symmetric and having asymptotically radial structure we avoided the presence of the singular part in the interior!
- At least in 2D and a simply connected domains, we can convert this setting to the minimal surface-like problems and get the same result. Improvement of the known results in a significant way!
- The method does not use the improved integrability result (which even may not be true)!
- The same theory for minimal surface-like problems and general geometries. Sharp identification of the cases when the theory can be built up to the boundary without any restriction on the shape of the domain.

Scheme of the proof

We find a mollified problem for which we have a solution and then go to the limit. The approximation is of the form

$$\varepsilon_n^*(\mathbf{T}) := \varepsilon^*(\mathbf{T}) + n^{-1}\frac{\mathbf{T}}{(1+|\mathbf{T}|)^{1-\frac{1}{n}}}.$$

- The first a priori estimate

$$\int_\Omega |\mathbf{T}^n| \le C, \qquad \|\varepsilon(\boldsymbol{u}^n)\|_n \le C.$$

-
$$\begin{aligned} \mathbf{T}^n \rightharpoonup^* \overline{\mathbf{T}} \qquad & \text{in } \mathcal{M}(\overline{\Omega})^{d\times d}_{sym}, \\ \varepsilon(\boldsymbol{u}^n) \rightharpoonup \varepsilon(\boldsymbol{u}) \qquad & \text{in } L^q(\Omega)^{d\times d}_{sym}, \text{ for all } q < \infty. \end{aligned}$$

  and $\overline{\mathbf{T}}$ solves the equation but we do not know that $\varepsilon(\boldsymbol{u}) = \varepsilon^*(\overline{\mathbf{T}})$

Scheme

- First we show that

$$\mathbf{T}^n \to \mathbf{T} \qquad \text{a.e. in } \Omega,$$

where $\mathbf{T} \in L^1(\Omega)^{d\times d}_{sym}$ but we do not know that $\mathbf{T} = \overline{\mathbf{T}}$.

- Then due to the continuity of ε^* we have

$$\varepsilon(\boldsymbol{u}) = \varepsilon^*(\mathbf{T}) \text{ a.e. in } \Omega.$$

- Fatou lemma and monotonicity justifies the limit passage in

$$\int_\Omega \mathbf{T} \cdot \varepsilon(\boldsymbol{u} - \boldsymbol{w}) \le \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{u} - \boldsymbol{w}) + \int_{\Gamma_N} \boldsymbol{g} \cdot (\boldsymbol{u} - \boldsymbol{w})$$

- the final step is to show that

$$\boxed{-\operatorname{div} \mathbf{T} = \boldsymbol{f}}$$