Traveling waves in one-dimensional non-linear models of strain-limiting viscoelasticity

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Conference MORE
Implicitly constituted materials: Modeling, Analysis and Computing

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The equation

We are interested in the equation

$$T_{xx} + \nu T_{xxt} = g(T)_{tt},$$

where $T(x, t)$ is the Cauchy stress at point x and time t, $g(\cdot)$ is a nonlinear function, and $\nu > 0$ is a constant.

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Current interests

Example 1

An explicitly constituted material:

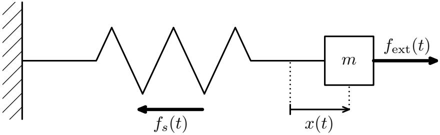


Figure: Applying an external force puts the system in motion

We can write the constitutive specification for the spring as

$$f_s = g(x) \underset{\text{(linear spring)}}{\Longrightarrow} f_s = kx,\ k \text{ spring constant.}$$

One then writes the balance of linear momentum and use this relation to get an ODE in terms of the displacement as

$$m\ddot{x} + f_s = f_{\text{ext}}.$$

Example 2

An implicitly constituted material:

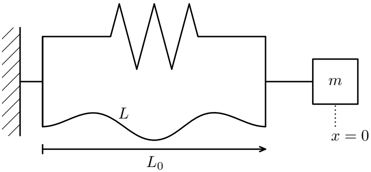


Figure: A mass-spring-wire system in its equilibrium

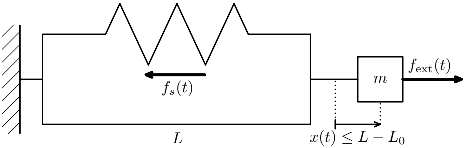
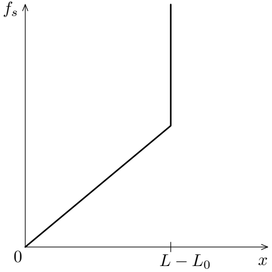


Figure: Applying an external force puts the system in motion

Example 2

- The wire of maximal length L cannot break whatever force is applied to it.
- The extension of the spring is limited to L.
- Once the maximal length L is obtained, no change in the position occurs.

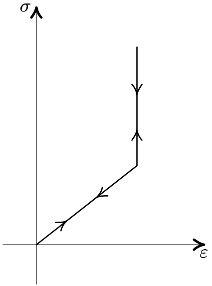


In this case it is much more sensible to prescribe an implicit relation between the force and the displacement as

$$g(f_s, x) = 0.$$

Focus of the talk

The focus of my talk is a particular subclass of elastic bodies which are defined through implicit constitutive relations where the linearized strain is a nonlinear function of the stress and exhibits a limiting strain irrespective of the stress to which the material is subject.



Notations

- $\Omega \subset \mathbb{R}^d$ for $d \in \mathbb{N}$ with Lipschitz boundary
- $\mathbf{u} : \Omega \times [0, T] \to \mathbb{R}^d$ is the deformation of the body
- $\mathbf{F} = \nabla \mathbf{u} \in \mathbb{R}^{d \times d}$ is the deformation gradient
- $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ is the left Cauchy-Green stretch tensor
- $\mathbf{L} = \nabla \mathbf{v}$ is the velocity gradient, $\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T)$ is the symmetric part of $\mathbf{L}$
- $\epsilon(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ is the linearized strain

Derivation of the model

We are interested in class of implicit models defined through

$$G(\mathbf{T}, \mathbf{B}) = 0.$$

Since the body is isotropic, it has to satisfy the condition $G(\mathbf{Q}\mathbf{T}\mathbf{Q}^T, \mathbf{Q}\mathbf{B}\mathbf{Q}^T) = \mathbf{Q}G(\mathbf{T}, \mathbf{B})\mathbf{Q}^T, \forall \mathbf{Q} \in \mathrm{SO}(3)$, which leads to

$$\begin{aligned} G(\mathbf{T}, \mathbf{B}) = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{T} + \alpha_2 \mathbf{B} + \alpha_3 \mathbf{T}^2 + \alpha_4 \mathbf{B}^2 + \alpha_5 (\mathbf{T}\mathbf{B} + \mathbf{B}\mathbf{T}) \\ + \alpha_6 (\mathbf{T}^2 \mathbf{B} + \mathbf{B}\mathbf{T}^2) + \alpha_7 (\mathbf{T}^2 \mathbf{B}^2 + \mathbf{B}^2 \mathbf{T}^2) = 0, \end{aligned}$$

where α_i depend on the invariants

$$\mathrm{tr}\mathbf{T}, \mathrm{tr}\mathbf{B}, \mathrm{tr}\mathbf{T}^2, \mathrm{tr}\mathbf{B}^2, \mathrm{tr}\mathbf{T}^3, \mathrm{tr}(\mathbf{T}\mathbf{B}), \mathrm{tr}(\mathbf{T}^2\mathbf{B}), \mathrm{tr}(\mathbf{T}\mathbf{B}^2), \mathrm{tr}(\mathbf{T}^2\mathbf{B}^2).$$

Derivation of the model

An implicit subclass is $\mathbf{B} = \beta_0 \mathbf{I} + \beta_1 \mathbf{T} + \beta_2 \mathbf{T}^2$, where β_i depend on $\mathrm{tr}\mathbf{T}, \mathrm{tr}\mathbf{T}^2, \mathrm{tr}\mathbf{T}^3$.
Linearization gives

$$2\epsilon = (\beta_0 - 1)\mathbf{I} + \beta_1 \mathbf{T} + \beta_2 \mathbf{T}^2,$$

which is a nonlinear relationship between the linearized strain and the stress.
We are interested in the viscoelastic version

$$\gamma \mathbf{B} + \nu \mathbf{D} = \beta_0 \mathbf{I} + \beta_1 \mathbf{T} + \beta_2 \mathbf{T}^2,$$

where γ and ν are nonnegative constants.
Rajagopal & Saccomandi (2014) introduced this model which explains responses of viscoelastic bodies such as Titanium and Gum metal alloys.

Derivation of the model

Linearizing the strain we get

$$\epsilon + \nu\epsilon_t = \beta_0 \mathbf{I} + \beta_1 \mathbf{T} + \beta_2 \mathbf{T}^2,$$

where $\epsilon_t = \partial\epsilon/\partial t$ is the linearized counterpart of $\mathbf{D}$ and β_i depend on $\mathrm{tr}\mathbf{T}, \mathrm{tr}\mathbf{T}^2, \mathrm{tr}\mathbf{T}^3$.
We want to consider the one-dimensional problem with more general right-hand sides;

$$\epsilon + \nu\epsilon_t = g(T),$$

which gives the linearized strain $\epsilon = u_x$ and the strain rate ϵ_t as a nonlinear function of the Cauchy stress T.

Derivation of the model

In the absence of external body forces the equation of motion leads to

$$u_{tt} = T_x \quad \Rightarrow \quad u_{ttx} = T_{xx} \quad \Rightarrow \quad \epsilon_{tt} = T_{xx}.$$

On the other hand, $\epsilon + \nu\epsilon_t = g(T)$ gives

$$\epsilon_{tt} + \nu\epsilon_{ttt} = [g(T)]_{tt}.$$

Combining these two relations, we obtain the PDE we want to study:

$$T_{xx} + \nu\, T_{xxt} = [g(T)]_{tt}.$$

Previously studied nonlinearities

Model A: 1D version of model introduced by Kannan, Rajagopal & Saccomandi (2014)

$$g(T) = \beta T + \alpha \left(1 + \frac{\gamma}{2} T^2\right)^n T,$$

where $\alpha \geq 0, \beta \leq 0, \gamma \geq 0$ and n are constants.

Note that when $n = 0$, or $\gamma = 0$, one recovers the standard constitutive equation for a linearized material.

Previously studied nonlinearities

Model B: Simplified version of a model introduced by Rajagopal (2011)

$$g(T) = \frac{T}{(1 + |T|^r)^{1/r}},$$

where $r > 0$ is a constant.

Note that when $\beta = 0, n = -1/2, \alpha = 1$ and $\gamma = 2$, Model A becomes Model B with $r = 2$.

This model is studied in elastic setting by many authors in different contexts, see e.g., Bulíček, Málek, Rajagopal & Süli (2014), Bulíček, Málek, Rajagopal & Walton (2015), Bulíček, Málek & Süli (2015).

Previously studied nonlinearities

Model C: 1D version of a nonlinearity introduced by Rajagopal (2010, 2011)

$$g(T) = \alpha \left\{ \left[1 - \exp\left(\frac{-\beta T}{1 + \delta |T|} \right) \right] + \frac{\gamma T}{1 + |T|} \right\},$$

where $\alpha, \beta, \gamma, \delta$ are constants.

Note that when $\beta = 0$ and $\alpha = \gamma = 1$, Model C reduces to Model B with $r = 1$.

Previously studied nonlinearities

Model D: 1D version of another model introduced by Rajagopal (2010, 2011)

$$g(T) = \alpha \left(1 - \frac{1}{1 + \frac{T}{1+\delta|T|}} \right) + \beta \left(1 + \frac{1}{1 + \gamma T^2} \right) T,$$

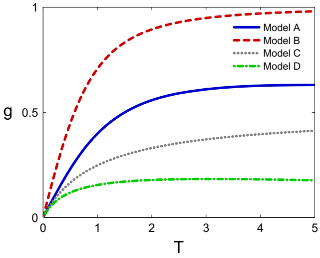
where $\alpha, \beta, \gamma, \delta$ are constants.

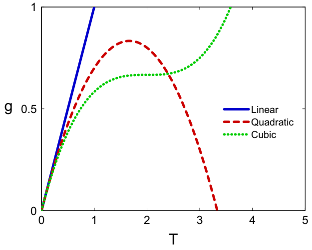
Note that when $\alpha = 0$, with appropriate choice of the remaining parameters, one can derive Model A from this model.

Previously studied nonlinearities

Remarks:

- Models C and D have a drawback when T is compressive and sufficiently large since they violate the assumption of small strain due to their initial terms when stress is negative and large.
- In a moderate stress regime all these models look as follows:





Traveling wave solutions

We look for traveling wave solutions of the PDE we derived with $g(T)$ as in Models A, B, C and D, as well as quadratic and cubic nonlinearities:

$$T_{xx} + \nu\, T_{xxt} = [g(T)]_{tt} \text{ where } T = T(\xi) \text{ with } \xi = x - c\,t.$$

(c is the constant wave propagation speed)
The equation becomes

$$T^{''} - \nu\, c\, T^{'''} = c^2\, [g(T)]^{''}.$$

Traveling wave solutions

Setting

$$\lim_{\xi \to -\infty} T(\xi) = T_\infty^-, \ \lim_{\xi \to +\infty} T(\xi) = T_\infty^+,$$

integrating and using $T^{'}(\xi), T^{''}(\xi) \to 0$ as $\xi \to \mp\infty$ we get $T^{'} = f(T)$ with

$$f(T) = \frac{1}{\nu} \left\{ \left(T - \frac{T_\infty^- + T_\infty^+}{2} \right) - c^2 \left(g(T) - \frac{g(T_\infty^-) + g(T_\infty^+)}{2} \right) \right\}.$$

Two obvious equilibrium points are $T = T_\infty^-$ and $T = T_\infty^+$. Integrating, one get an implicit solution of the form

$$\xi - \xi_0 = \int_{T_0}^{T} \frac{ds}{f(s)}$$

with $T(\xi_0) = T_0$.

Traveling wave solutions

We also find

$$c^2 = \frac{T_\infty^- - T_\infty^+}{g(T_\infty^-) - g(T_\infty^+)}.$$

Therefore we have two possible cases:

(i) $T_\infty^- > T_\infty^+$ and $g(T_\infty^-) > g(T_\infty^+)$.

(ii) $T_\infty^- < T_\infty^+$ and $g(T_\infty^-) < g(T_\infty^+)$.

Without loss of generality, we look at case (i) and take $T_\infty^- = 1$ and $T_\infty^+ = 0$ to get

$$T' = \frac{1}{\nu c g(1)}(g(1)T - g(T)).$$

We will take $T(0) = 1/2$.

Traveling wave solutions

Remarks:

- There is no heteroclinic traveling wave solution when we consider an elastic solid, i.e. $\nu = 0$.
- There is no heteroclinic traveling wave solution for the linear viscoelastic model where $g(T) = g'(0)T$ with $g'(0) \neq 0$.

Traveling wave solutions

Quadratic case: We let $g(T) = g'(0)T + \frac{1}{2}g''(0)T^2$.
In this case we obtain the explicit solution

$$T(\xi) = (1 + e^{a_2 \xi})^{-1}$$

where $a_2 = -\frac{g''(0)c}{2\nu}$.

- We need $a_2 > 0$, hence traveling wave solution exists if $g''(0) < 0$ and $c > 0$ (right-going wave) or $g''(0) > 0$ and $c < 0$ (left-going wave).
- No solution if $a_2 < 0$, or $g''(0)$ and c have the same sign.
- Traveling wave solution becomes smaller as c increases.
- No shock waves.

Traveling wave solutions

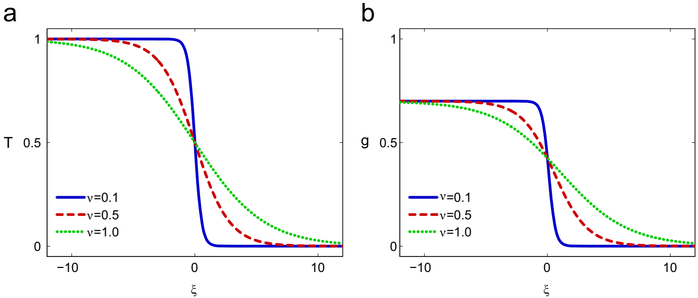


Figure: Variation of (a) T and (b) $g(T)$ of the quadratically nonlinear model

Traveling wave solutions

Cubic case: We let $g(T) = g'(0)T + \frac{1}{2}g''(0)T^2 + \frac{1}{6}g'''(0)T^3$. In this case we obtain the implicit solution

$$\frac{T^{1+b}}{(1-T)^b(T+b)} = \frac{1}{1+2b}e^{b(1+b)a\xi},$$

where $a = -\frac{g'''(0)c}{6\nu}$ and $b = 1 + 3\frac{g''(0)}{g'''(0)}$.

- When b (or equivalently $g'''(0)$) increases, traveling wave solution becomes smoother.
- When $g''(0) = 0$ we obtain the explicit solution

$$T(\xi) = \frac{e^{a\xi}}{(3+e^{2a\xi})^{1/2}}.$$

- Traveling wave solution exists if $a < 0$ or equivalently if $g'''(0)$ and c have the same sign.

Traveling wave solutions

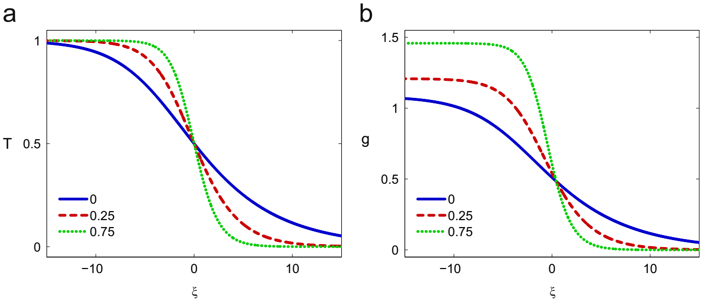


Figure: Variation of (a) T and (b) $g(T)$ of the cubically nonlinear model

Traveling wave solutions

Case of Model A: When $n = 1$ we obtain the explicit solution we obtained in the cubic case with a replaced by

$$\frac{\alpha\gamma}{[\alpha(1+\gamma)+\beta]\nu c}.$$

Traveling wave solutions

Case of Model B: When $r = 2$ we obtain the solution implicitly as

$$H(T) = H(1/2)e^{\xi/\nu c},$$

where $H(s) = \frac{(1-s^2)^2}{s(3+s^2+2^{3/2}(1+s^2))}\left(\frac{(1+s^2)^{1/2}+1}{s}\right)^{2^{1/2}}$.

- Note that $H(\mp 1) = 0$, $H(1/2) > 0$ and $H(s) \to \infty$ as $s \to 0^+$.
- If $c > 0$ we have $T \to 0^+$ as $\xi \to +\infty$, and $T \to \mp 1$ as $\xi \to -\infty$, which are incompatible with the conditions we chose.
- Traveling wave solution exists if $a < 0$ or equivalently if $g'''(0)$ and c have the same sign.

Traveling wave solutions

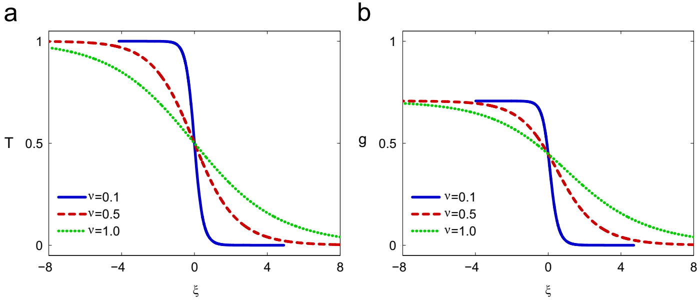


Figure: Variation of (a) T and (b) $g(T)$ of Model B

Traveling wave solutions

- For Models C and D, we obtain highly nonlinear equations for which analytical solutions are not available.
- We find kink-type traveling wave solutions numerically.
- The profiles for stress T are in good agreement with those derived from analytical solutions belonging to previous models.
- However, the profiles for the strain are significantly different.

Traveling wave solutions

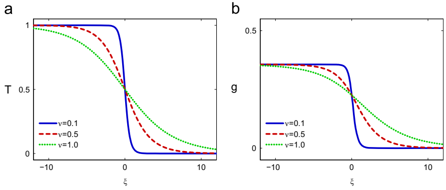


Figure: Variation of (a) T and (b) $g(T)$ of Model C

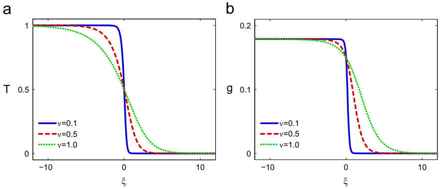


Figure: Variation of (a) T and (b) $g(T)$ of Model D

Further analysis of the PDE

We study the initial-boundary value problem for the PDE

$$T_{xx} + \nu\, T_{xxt} = [g(T)]_{tt},$$

with the following initial and boundary conditions:

$$\begin{aligned} &T(x,0) = T_0(x), \quad T_t(x,0) = T_1(x). \\ &T(0,t) = T(1,t) = 0. \end{aligned}$$

- This model is different from classical viscoelastic models since the inertia term is nonlinear.
- The unknown is the stress T instead of the deformation u unlike classical models.

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