On the response of physical systems governed by nonlinear ordinary differential equations to step input

Colombeau algebra and its applications in mechanics

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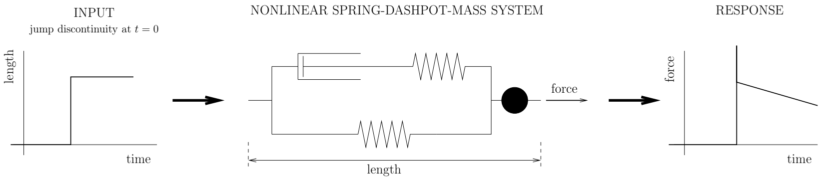
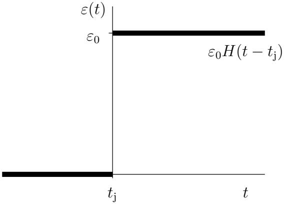
Problem

Physics



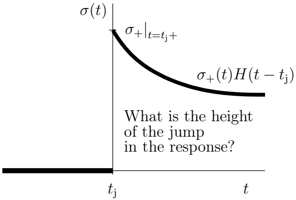
Mathematics



Input $\varepsilon(t)$

Nonlinear dynamical system.

$$f\left(\varepsilon, \sigma, \frac{\mathrm{d}\varepsilon}{\mathrm{d}t}, \frac{\mathrm{d}\sigma}{\mathrm{d}t}, \frac{\mathrm{d}^2\varepsilon}{\mathrm{d}t^2}, \frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2}\right) = 0$$



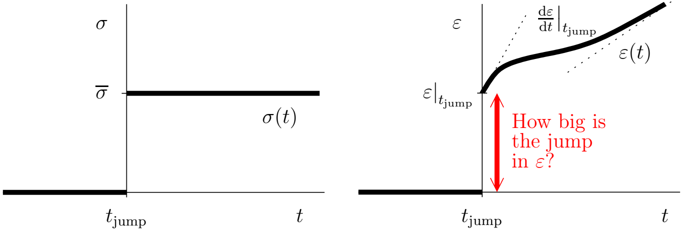
Response $\sigma(t)$

Jump discontinuity in a *linear* system – classical tools

Ordinary differential equation (stress σ versus deformation ε):

$$a_0\sigma + \frac{\mathrm{d}\sigma}{\mathrm{d}t} = b_0\varepsilon + c_0\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}$$

Input with jump discontinuity:



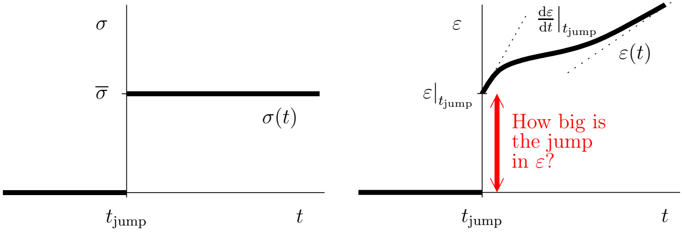
(a) Stress σ (input) (b) Deformation ε (response)

Jump discontinuity in a *nonlinear* system – classical tools

Ordinary differential equation (stress σ versus deformation ε):

$$a(\sigma,\varepsilon)\sigma + \frac{\mathrm{d}\sigma}{\mathrm{d}t} = b(\varepsilon,\sigma)\varepsilon + c(\varepsilon,\sigma)\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}$$

Input with jump discontinuity:



(a) Stress σ (input) (b) Deformation ε (response)

$$c(\varepsilon,\sigma)\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} =_{\mathrm{def}} \varepsilon^n \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} \quad \rightarrow \quad \varepsilon^n \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} \propto (H)^n \frac{\mathrm{d}H}{\mathrm{d}t} = (H)^n \,\delta$$

Fundamental problem

Heaviside function: $H = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$

How to handle the term $H^n \frac{\mathrm{d}H}{\mathrm{d}t}$? Naive calculation, $H^n = H$:

- $H^n \frac{\mathrm{d}H}{\mathrm{d}t} = H \frac{\mathrm{d}H}{\mathrm{d}t}$
- $H^n \frac{\mathrm{d}H}{\mathrm{d}t} = H \left[H^{n-1} \frac{\mathrm{d}H}{\mathrm{d}t} \right] = H \frac{1}{n} \frac{\mathrm{d}H^n}{\mathrm{d}t} = \frac{1}{n} H \frac{\mathrm{d}H}{\mathrm{d}t}$
- $H^n \frac{\mathrm{d}H}{\mathrm{d}t} = H^2 \left[H^{n-2} \frac{\mathrm{d}H}{\mathrm{d}t} \right] = H \frac{1}{n-1} \frac{\mathrm{d}H^{n-1}}{\mathrm{d}t} = \frac{1}{n-1} H \frac{\mathrm{d}H}{\mathrm{d}t}$

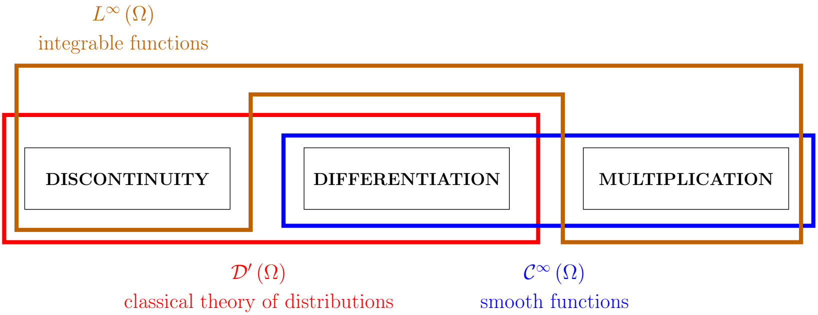
Really?

$$H \frac{\mathrm{d}H}{\mathrm{d}t} = \frac{1}{n} H \frac{\mathrm{d}H}{\mathrm{d}t} = \frac{1}{n-1} H \frac{\mathrm{d}H}{\mathrm{d}t}$$

Classical tools (theory of distributions) are useless in the nonlinear setting. Intuition could be misleading.

Fundamental problem

Is there a theory that would allow one to simultaneously handle discontinuity, differentiation and nonlinearity?



Everything is lost, it is impossible to introduce a structure that would allow "multiplication of distributions".

Laurent Schwartz. Sur l'impossibilité de la multiplication des distributions. *C. R. Acad. Sci. Paris*, 239:847–848, 1954

Nonlinear theory of distributions

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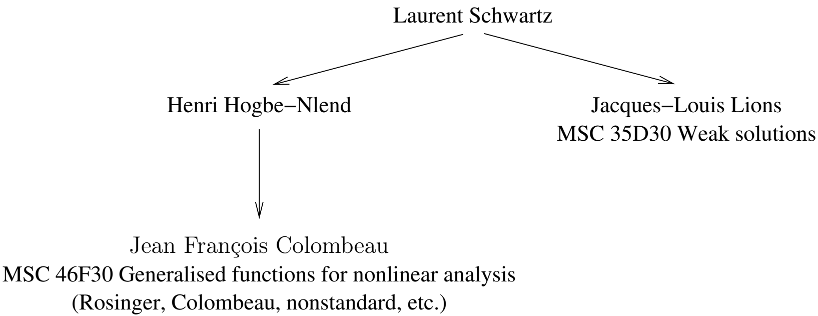
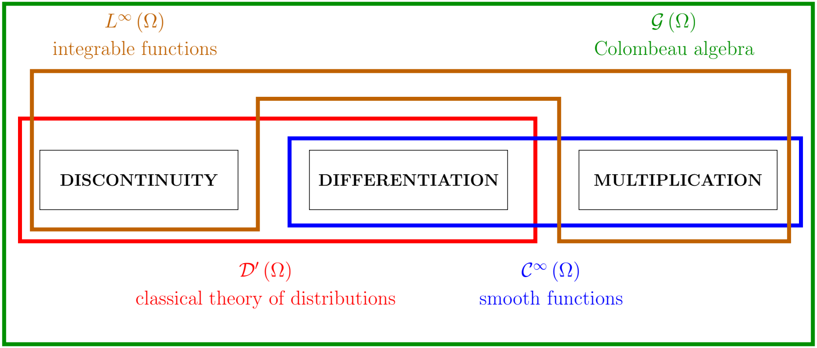


Mathematics Genealogy Project, http://genealogy.math.ndsu.nodak.edu/
2010 Mathematics Subject Classification, http://www.ams.org/msc/msc2010.html

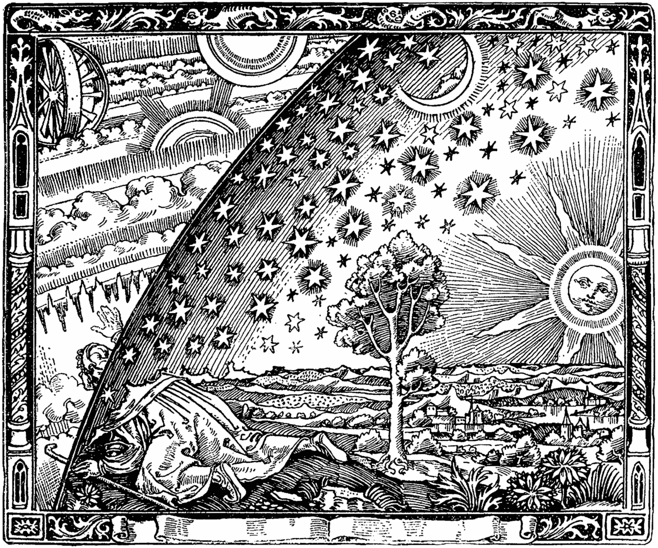
Fundamental problem – Colombeau algebra



Schwartz: "multiplication must be equal to the classical multiplication provided that we consider continuous functions"

Colombeau: "multiplication must be equal to the classical multiplication provided that we consider smooth functions"

How to escape from the world of smooth functions?



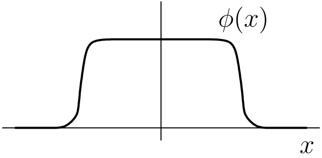
Colombeau algebra – analogy

nice objects	rational numbers
↓	$a \in \mathbb{Q}$
collections of nice objects	sequences of rational numbers
↓	$\{a_n\}_{n=1}^{+\infty}$
nice collections of nice objects	Cauchy sequences
↓	$\lvert a_n - a_m \rvert < \varepsilon$
two collections represent the same generalised object	zero difference
↓	$\{a_n\}_{n=1}^{+\infty}, \{b_n\}_{n=1}^{+\infty} : \lvert a_n - b_n \rvert < \varepsilon$
generalised objects and calculus	real numbers
	$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right)$
	$\mathrm{e} = \sum_{k=0}^{\infty} \frac{1}{k!}$
	$\mathrm{e}^{\mathrm{i}\pi} = -1$

Colombeau algebra – basic idea

The generalised functions are identified with collections of all smooth functions that are approximating the generalised function.

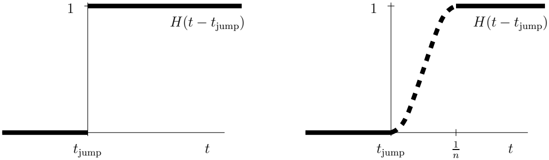
Smoothing kernel $\phi(x)$:



- $\phi \in \mathcal{D}(\mathbb{R})$
- $\int_{\mathbb{R}} \phi(\zeta)\,\mathrm{d}\zeta = 1$

Smoothing, $f \in L^1_{\mathrm{loc}}(\mathbb{R})$:

$$R_{\boldsymbol{f}}(\phi, t) =_{\mathrm{def}} \int_{\mathbb{R}} f(\zeta)\phi(\zeta - t)\,\mathrm{d}\zeta$$



Colombeau algebra – construction

nice objects	smooth functions
↓	$R_{\boldsymbol{f}}(\phi, t)$
collections of nice objects	representatives with diminishing mollification
↓	$R_{\boldsymbol{f}}(\phi_\varepsilon, t)$, $\phi_\varepsilon(t) =_{\text{def}} \frac{1}{\varepsilon}\phi\left(\frac{t}{\varepsilon}\right)$
nice collections of nice objects	reasonably singular representatives, $\mathcal{E}_M(\mathbb{R})$
↓	$\left\lvert \frac{\mathrm{d}^k}{\mathrm{d}t^k} R_{\boldsymbol{f}}(\phi_\varepsilon, t) \right\rvert \leq \frac{c}{\varepsilon^n}$
two collections represent the same generalised object	difference is in null space, $\mathcal{N}(\mathbb{R})$
↓	$\left\lvert \frac{\mathrm{d}^k}{\mathrm{d}t^k} \left(R_{\boldsymbol{f}}(\phi_\varepsilon, t) - R_{\boldsymbol{g}}(\phi_\varepsilon, t) \right) \right\rvert \leq c\varepsilon^{\beta(m)-l}$
generalised objects and calculus	generalised functions
	$\mathcal{G}(\mathbb{R}) =_{\text{def}} \mathcal{E}_M(\mathbb{R}) / \mathcal{N}(\mathbb{R})$
	$R_{\frac{\mathrm{d}^k \boldsymbol{g}}{\mathrm{d}t^k}}(\phi, t) =_{\text{def}} \frac{\mathrm{d}^k}{\mathrm{d}t^k} R_{\boldsymbol{g}}(\phi, t)$

Colombeau algebra – Heaviside function, Dirac distribution

Heaviside function:

$$R_{\boldsymbol{H}}(\phi, t) =_{\mathrm{def}} \int_{s=-t}^{+\infty} \phi(s)\,\mathrm{d}s$$

Dirac distribution:

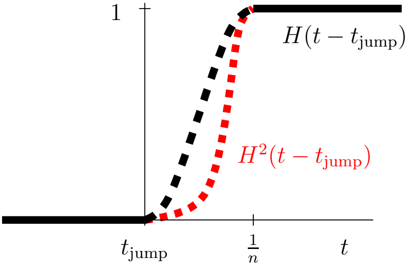
$$R_{\boldsymbol{\delta}}(\phi, t) =_{\mathrm{def}} \phi(-t)$$

Example:

$$\frac{\mathrm{d}\boldsymbol{H}}{\mathrm{d}t} = \boldsymbol{\delta}$$

Colomeau algebra – summary

- It is possible to multiply distributions.
- The price to pay is the incompatibility of the multiplication with the classical multiplication, $\boldsymbol{H}^2 \neq \boldsymbol{H}$.
- This drawback is fixed by coupled calculus.



Colombeau algebra – equivalence in the sense of association

- $=$ Strict equivalence, detailed behaviour in the vicinity of the jump is important:
- $\approx$ Equivalence in the sense of association, detailed behaviour in the vicinity of the jump is not important:

Colombeau algebra – equivalence in the sense of association

Definition (Equivalence in the sense of association)

Generalised functions $\boldsymbol{f}$ and $\boldsymbol{g}$ are equivalent in the sense of association, $\boldsymbol{f} \approx \boldsymbol{g}$, if and only if

$$\forall \psi \in \mathcal{D}(\mathbb{R}),\ \exists R_{\boldsymbol{f}} \in \mathcal{E}_M(\mathbb{R}),\ \exists R_{\boldsymbol{g}} \in \mathcal{E}_M(\mathbb{R}),\ \exists q \in \mathbb{N},\ \forall \phi \in \mathcal{A}_q :$$
$$\lim_{\varepsilon \to 0+} \int_{\mathbb{R}} \left[R_{\boldsymbol{f}}(\phi_\varepsilon, x) - R_{\boldsymbol{g}}(\phi_\varepsilon, x)\right] \psi(x)\, \mathrm{d}x = 0$$

Definition (Strict equivalence)

[. . .] for all $K \subset \mathbb{R}$, K compact, and for all $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ and $\beta \in \Gamma$ such that for all $m \in \mathbb{N}$, $m \geq l$ and for all $\phi \in \mathcal{A}_m$ there exists $\eta \in \mathbb{R}^+$ and $c \in \mathbb{R}^+$ such that

$$\forall t \in K,\ \forall \varepsilon \in (0, \eta) : \quad \left| \frac{\mathrm{d}^k}{\mathrm{d}t^k} \left(R_{\boldsymbol{f}}(\phi_\varepsilon, x) - R_{\boldsymbol{g}}(\phi_\varepsilon, x)\right) \right| \leq c\varepsilon^{\beta(m) - l}$$

Colombeau algebra – multiplication and the equivalence in the sense of association

Heaviside function and Dirac distribution as elements in Colombeau algebra $\mathcal{G}(\mathbb{R})$:

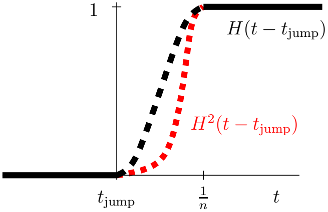
$$\boldsymbol{H}^2 \neq \boldsymbol{H}$$

$$\boldsymbol{H}^2 \approx \boldsymbol{H}$$

$$\boldsymbol{H}\boldsymbol{\delta} \not\approx \boldsymbol{H}^2\boldsymbol{\delta}$$

$$\boldsymbol{x}\boldsymbol{\delta} \neq \boldsymbol{0}$$

$$\boldsymbol{x}\boldsymbol{\delta} \approx 0$$



Colombeau algebra – example

System:

$$a(\varepsilon,\sigma)\sigma + \frac{\mathrm{d}\sigma}{\mathrm{d}t} = b(\varepsilon,\sigma)\varepsilon + c(\varepsilon,\sigma)\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}$$

Input:

$$\varepsilon = \begin{cases} 0 & t < 0 \\ \tilde{\varepsilon}(t) & t \geq 0 \end{cases}$$

Notion of solution:

$$a(\boldsymbol{\varepsilon},\boldsymbol{\sigma})\boldsymbol{\sigma} + \frac{\mathrm{d}\boldsymbol{\sigma}}{\mathrm{d}t} \approx b(\boldsymbol{\varepsilon},\boldsymbol{\sigma})\boldsymbol{\varepsilon} + c(\boldsymbol{\varepsilon},\boldsymbol{\sigma})\frac{\mathrm{d}\boldsymbol{\varepsilon}}{\mathrm{d}t}$$

Colombeau algebra – theorem

Theorem (Solution to $a(\varepsilon,\sigma)\sigma + \frac{\mathrm{d}\sigma}{\mathrm{d}t} = b(\varepsilon,\sigma)\varepsilon + c(\varepsilon,\sigma)\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}$)

System response described by an ordinary differential equation (stress σ versus deformation ε):

$$a(\varepsilon,\sigma)\sigma + \frac{\mathrm{d}\sigma}{\mathrm{d}t} = b(\varepsilon,\sigma)\varepsilon + c(\varepsilon,\sigma)\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}$$

Response σ of the system to the step input ε

$$\varepsilon = \begin{cases} 0, & t < 0, \\ \tilde{\varepsilon}(t), & t \geq 0, \end{cases}$$

where $\tilde{\varepsilon}$ is a smooth function, is given by the function

$$\sigma = \begin{cases} 0, & t < 0, \\ \tilde{\sigma}(t), & t \geq 0. \end{cases}$$

Colombeau algebra – theorem

Function $\tilde{\sigma}$ is for $t > 0$ a solution to the ordinary differential equation

$$a(\tilde{\varepsilon}, \tilde{\sigma})\tilde{\sigma} + \frac{\mathrm{d}\tilde{\sigma}}{\mathrm{d}t} = b(\tilde{\varepsilon}, \tilde{\sigma})\tilde{\varepsilon} + c(\tilde{\varepsilon}, \tilde{\sigma})\frac{\mathrm{d}\tilde{\varepsilon}}{\mathrm{d}t},$$
$$\left.\tilde{\sigma}\right|_{t=0+} = \sigma_0.$$

The initial condition σ_0 for the previous ordinary differential equation—that is the height of the jump in the response—is obtained by evaluating the solution ς of the ordinary differential equation

$$\frac{\mathrm{d}\varsigma}{\mathrm{d}\epsilon} = c(\epsilon, \varsigma),$$
$$\left.\varsigma\right|_{\epsilon=0} = 0,$$

at point $\epsilon = \varepsilon_0$, that is $\sigma_0 = \left.\varsigma\right|_{\epsilon=\varepsilon_0}$, where $\varepsilon_0 =_{\mathrm{def}} \left.\tilde{\varepsilon}\right|_{t=0+}$ denotes the height of the jump in the input.

Numerical results – sequetial approach

Equation:

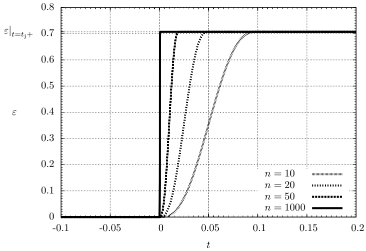
$$
\begin{aligned}
a(\varepsilon,\sigma)\sigma + \frac{\mathrm{d}\sigma}{\mathrm{d}t} &= b(\varepsilon,\sigma)\varepsilon + c(\varepsilon,\sigma)\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} \\
a(\varepsilon,\sigma) &=_{\mathrm{def}} \bar{a}\left(1+\varepsilon^2+\sigma^2\right) \\
b(\varepsilon,\sigma) &=_{\mathrm{def}} \bar{b}\left(1+\sigma^2\varepsilon^2\right) \\
c(\varepsilon,\sigma) &=_{\mathrm{def}} \bar{c}\left(1+\varepsilon^2\right)\mathrm{e}^{-\sigma} \\
\bar{a} &=_{\mathrm{def}} 1 \\
\bar{b} &=_{\mathrm{def}} 3 \\
\bar{c} &=_{\mathrm{def}} 5
\end{aligned}
$$

Input:

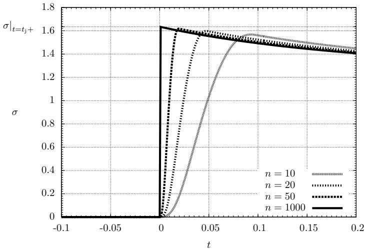
$$
\varepsilon = \begin{cases} 0 & t < 0 \\ \frac{\sqrt{2}}{2} & t \geq 0 \end{cases}
$$

Take a sequence of smooth inputs, and see what happens for $n \to +\infty$.

Numerical results – sequential approach

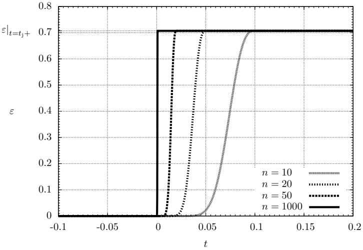


(a) Input $\varepsilon_n(t)$

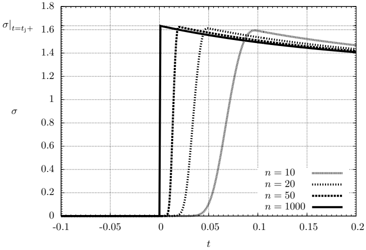


(b) Output $\sigma_n(t)$

Numerical results – sequential approach

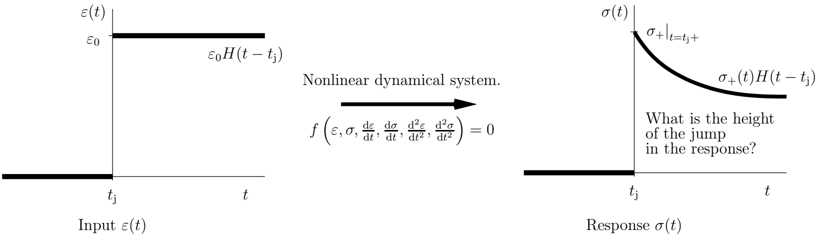


(a) Input $\hat{\varepsilon}_n(t)$



(b) Output $\hat{\sigma}_n(t)$

Response of nonlinear dynamical systems to step inputs

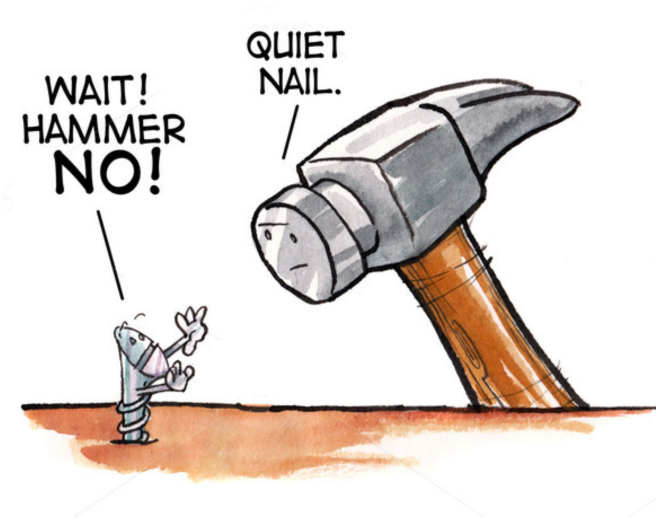


$$a(\sigma, \varepsilon)\sigma + \frac{\mathrm{d}\sigma}{\mathrm{d}t} = b(\varepsilon, \sigma)\varepsilon + c(\varepsilon, \sigma)\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}$$

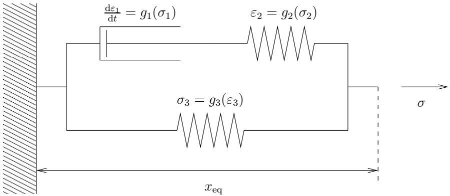
- Ordinary differential equation interpreted as an equation for generalised functions in Colombeau algebra.
- Equality understood *in the sense of association*.
- Explicit characterisation of the jump in terms of a, b and c.

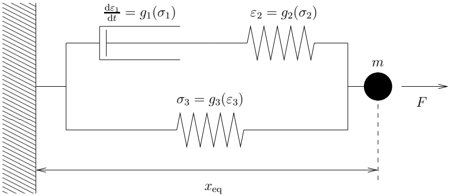
V. Průša and K. R. Rajagopal. On the response of physical systems governed by nonlinear ordinary differential equations to step input. *Int. J. Non-Linear Mech.*, 81:207–221, 2016

We have a hammer, what next?



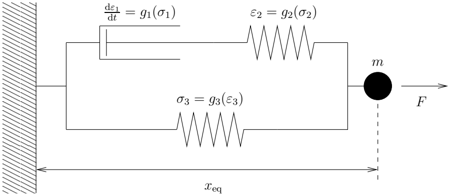
Response of nonlinear spring/dashpot systems to step deformation/load I





V. Průša, M. Řehoř, and K. Tůma. Colombeau algebra as a mathematical tool for investigating step load and step deformation of systems of nonlinear springs and dashpots. *Z. angew. Math. Phys.*, 68(1):1–13, 2017

Response of nonlinear spring/dashpot systems to step deformation/load II



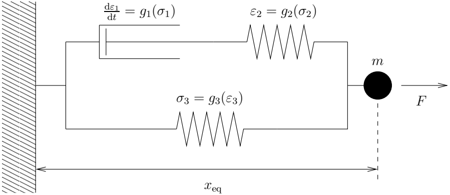
$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = F - \sigma$$

$$g_1(\sigma - g_3(\varepsilon)) + \frac{\mathrm{d}}{\mathrm{d}t} g_2(\sigma - g_3(\varepsilon)) = \frac{\mathrm{d}\varepsilon}{\mathrm{d}t}$$

$$\varepsilon =_{\mathrm{def}} \frac{x - x_{\mathrm{eq}}}{x_{\mathrm{eq}}}$$

V. Průša, M. Řehoř, and K. Tůma. Colombeau algebra as a mathematical tool for investigating step load and step deformation of systems of nonlinear springs and dashpots. *Z. angew. Math. Phys.*, 68(1):1–13, 2017

Response of nonlinear spring/dashpot systems to step deformation/load III



Input/response:

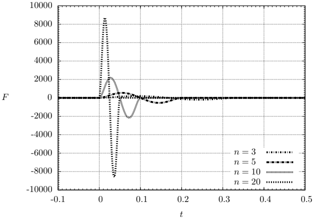
$$\boldsymbol{x} = x_{\mathrm{eq}} + (x_+ - x_{\mathrm{eq}})\boldsymbol{H}$$

Response/input:

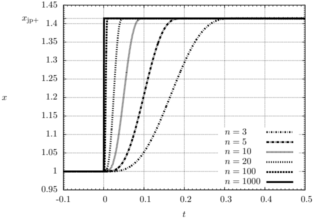
$$\boldsymbol{F} = \frac{\mathrm{d}}{\mathrm{d}t}\left(f_+\boldsymbol{H} + g_+\boldsymbol{\delta}\right)$$

V. Průša, M. Řehoř, and K. Tůma. Colombeau algebra as a mathematical tool for investigating step load and step deformation of systems of nonlinear springs and dashpots. *Z. angew. Math. Phys.*, 68(1):1–13, 2017

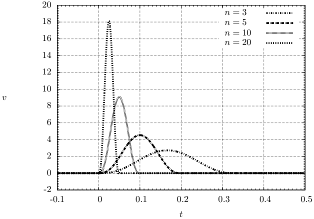
Response of nonlinear spring/dashpot systems to step deformation/load IV



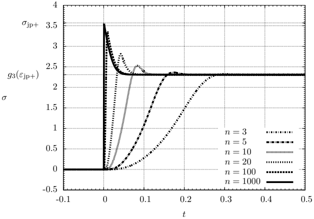
(c) Input, force F_n.



(d) Response, position x_n.

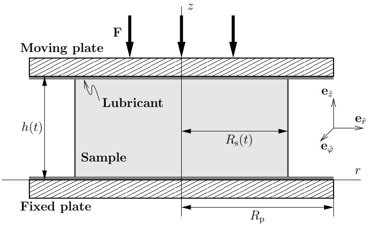


(e) Response, velocity v_n.



(f) Response, stress σ_n.

Response of a viscoelastic rate type fluid in a lubricated squeeze flow I



M. Řehoř, V. Průša, and K. Tůma. On the response of nonlinear viscoelastic materials in creep and stress relaxation experiments in the lubricated squeeze flow setting. *Phys. Fluids*, 28(10), 2016

Response of a viscoelastic rate type fluid in a lubricated squeeze flow II

Governing equations:

$$
\begin{aligned}
\operatorname{div} \boldsymbol{v} &= 0 \\
\varrho \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} &= \operatorname{div} \mathbb{T} \\
\mathbb{T} &= p\mathbb{I} + \mathbb{S} \\
\lambda \overset{\triangledown}{\mathbb{S}} + \mathbb{S} &= 2\mu\mathbb{D}
\end{aligned}
$$

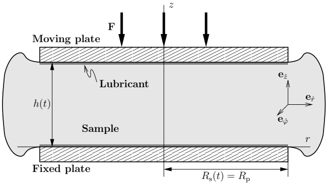
Input/response:

$$
F =_{\mathrm{def}} - \int_{\Sigma} \mathrm{T}_{\hat{z}\hat{z}}\big|_{z=h} \,\mathrm{d}\Sigma
$$

Response/input:

$$
h
$$

Response of a viscoelastic rate type fluid in a lubricated squeeze flow III



Deformation $\chi : \boldsymbol{X} \mapsto \boldsymbol{x} = \chi(\boldsymbol{X}, t)$ given by the formula

$$
\begin{aligned}
r &= \sqrt{\frac{h_0}{h}} R \\
\varphi &= \Phi \\
z &= \frac{h}{h_0} Z
\end{aligned}
$$

where $\boldsymbol{X} = [R, \Phi, Z]$ and $\boldsymbol{x} = [r, \varphi, z]$ are the coordinates in the reference/current configuration.

Response of a viscoelastic rate type fluid in a lubricated squeeze flow IV

After applying the ansatz for the deformation χ we arrive to:

$$
\begin{aligned}
F^*_{\mathrm{cm,ap}} + \frac{\mathrm{d}F^*_{\mathrm{cm,ap}}}{\mathrm{d}t^*} &= -\left[3\left(1+\sigma^*\right)\mathrm{e}^{-\varepsilon^*} + 2F^*_{\mathrm{cm,ap}}\right]\frac{\mathrm{d}\varepsilon^*}{\mathrm{d}t^*}, \\
\sigma^* + \frac{\mathrm{d}\sigma^*}{\mathrm{d}t^*} &= 2\left(1+\sigma^*\right)\frac{\mathrm{d}\varepsilon^*}{\mathrm{d}t^*}
\end{aligned}
$$

Height of the sample:

$$\varepsilon^* = \ln h^*$$

Input/response:

$$F^*_{\mathrm{cm,ap}}$$

Response/input:

$$h^*$$

M. Řehoř, V. Průša, and K. Tůma. On the response of nonlinear viscoelastic materials in creep and stress relaxation experiments in the lubricated squeeze flow setting. *Phys. Fluids*, 28(10), 2016

Conclusion

- Colombeau algebra is a nonlinear theory of distributions.
- Colombeau algebra provides a relatively easy to use tool for investigating the response of nonlinear systems to step inputs.
- Using Colombeau algebra we have investigated the response of a nonlinear spring-dashpot-mass system.
- Using Colombeau algebra we have investigated the response of a nonlinear viscoelastic rate type fluid in a setting relevant in rheology.

Thank you for your attention.