Weak Solutions in Fluid Dynamics

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Roztoky u Prahy
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What is a Weak Solution?

Consider the Poisson equation on $\mathbb{R}^n$,

$$\Delta u = f,$$

then u is called a weak solution if

$$\int_{\mathbb{R}^n} u \Delta \phi dx = \int_{\mathbb{R}^n} f \phi dx \qquad \text{for every } \phi \in C_c^2(\mathbb{R}^n).$$

Note: The weak formulation only requires u to be locally integrable!
Weak formulations are available for a vast class of PDEs, including most models from fluid dynamics.

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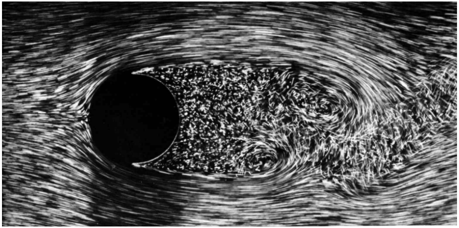
Why Study Weak Solutions?

- Often (e.g. elliptic theory) useful as an intermediate step to eventually obtain strong/classical solutions
- In fluid dynamics: Turbulent flows exhibit phenomena of irregularity, anomalous dissipation of energy, and non-deterministic behaviour, all of which are incompatible with differentiable solutions



(Picture from M. Van Dyke, *An Album of Fluid Motion*)

Mathematical Problems

- Existence, uniqueness, and regularity of weak solutions
- Conservation of energy/entropy vs. shock formation or anomalous dissipation
- Convergence of singular limits: e.g. viscosity $\to 0$, Mach number $\to 0$, hydrodynamic limit, numerical schemes
- Boundary layers, blow-up scenarios, long-time behaviour, stochastic forcing, etc. etc.

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The Incompressible Euler Equations

Consider the incompressible Euler equations

$$\begin{aligned} \partial_t v + (v \cdot \nabla) v + \nabla p &= 0, \\ \operatorname{div} v &= 0, \\ v(\cdot, 0) &= v^0. \end{aligned}$$

They model the motion of an inviscid incompressible fluid. Here, $v : \mathbb{R}^3 \times [0, T] \to \mathbb{R}^3$ is the velocity of the fluid and $p : \mathbb{R}^3 \times [0, T] \to \mathbb{R}$ is the pressure.

For smooth ("better than C^1") data, there is a unique smooth local-in-time solution.

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Non-Uniqueness

- Scheffer '93, Shnirelman '97: There exist non-trivial weak solutions (in the sense of distributions) of the 2D Euler equations with compact support in time!
- De Lellis–Székelyhidi '09: This is also true in any dimension, and the solutions can be L^∞.
- W. '11: Non-unique weak solutions can be constructed for any initial data in L^2, but they violate the energy inequality:

$$\frac{1}{2}\int_{\mathbb{R}^3}|v(x,t)|^2dx \not\leq \frac{1}{2}\int_{\mathbb{R}^3}|v^0(x)|^2dx.$$

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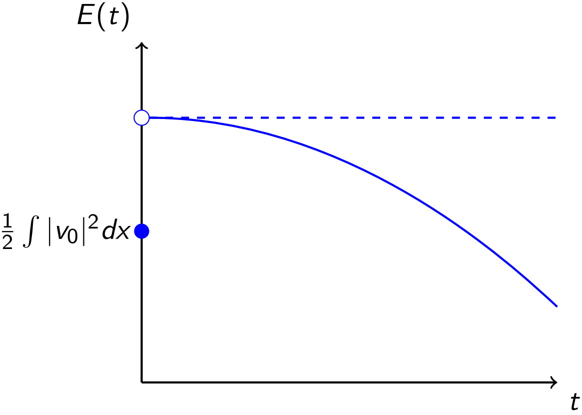
Energy Jump



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- De Lellis–Székelyhidi '10: Even assuming the weak energy inequality, solutions can be non-unique.
- Székelyhidi–W. '12: The set of initial data for which there exist non-unique solutions satisfying the energy inequality is dense in L^2.
- Daneri '14, Daneri–Székelyhidi '16: There are examples of Hölder continuous non-unique solutions satisfying the energy inequality.

All these examples rely on the *h*-principle, whose relevance to the Euler equations can be shown by convex integration (Gromov, Nash).

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Compressible Euler Equations

The isentropic compressible Euler system reads

$$\begin{aligned} \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \kappa \nabla(\rho^\gamma) &= 0 \\ \partial_t \rho + \operatorname{div}(\rho v) &= 0, \end{aligned}$$

where $v : \mathbb{R}^3 \times [0, T] \to \mathbb{R}^3$ is again the velocity and $\rho : \mathbb{R}^3 \times [0, T] \to \mathbb{R}^+$ is the density.

Note that now the pressure is not an unknown, but a constitutively given function of density! (Here we put $p(\rho) = \kappa\rho^\gamma$, $\kappa > 0$, $\gamma > 1$).

We consider only energy solutions, which satisfy in addition

$$\partial_t \left(\frac{\rho|v|^2}{2} + \frac{\kappa}{\gamma - 1}\rho^\gamma \right) + \operatorname{div}\left[\left(\frac{\rho|v|^2}{2} + \frac{\gamma}{\gamma - 1}\rho^\gamma \right) v \right] \leq 0.$$

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- Chiodaroli '14: Construction of such counterexamples with arbitrary smooth initial density
- Chiodaroli–De Lellis–Kreml '15: Infinitely many entropy solutions from Riemann data ($\gamma = 2$)
- Several similar results are available for other systems (Euler-Fourier, Euler-Korteweg-Poisson, nonlocal Euler, Savage-Hutter, Primitive Equations, ...):
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Weak-Strong Uniqueness

Consider again the incompressible Euler equations

$$\partial_t v + (v \cdot \nabla) v + \nabla p = 0, \qquad \operatorname{div} v = 0.$$

Theorem (Folklore)

Let u be a solution of Euler in the sense of distributions with initial data v_0, and suppose u satisfies

$$\int_{\mathbb{R}^d} |u(x,t)|^2 dx \leq \int_{\mathbb{R}^d} |v_0(x)|^2 dx \qquad \textit{for every } t > 0.$$

If v is a smooth solution with data v_0, then $u \equiv v$.

This is an instance of weak-strong uniqueness: *If there exists a sufficiently regular solution, then any weak solution with the same data coincides with it.*

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Relative Energy Method

The proof proceeds by a relative energy argument: Use the strong and the weak forms of the Euler equations and the assumption $|\nabla v| \leq C$ to estimate

$$E_{rel}(t) := \frac{1}{2} \int_{\mathbb{R}^3} |u - v|^2(t) dx$$

by its time integral $\int_0^t E_{rel}(s) ds$ and apply Gronwall's inequality to conclude $E_{rel} \equiv 0$.

Relative energy/entropy methods were introduced by C. Dafermos for hyperbolic conservation laws and have more recently been used widely in compressible fluid dynamics:

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Measure-Valued Solutions

DiPerna '85, DiPerna–Majda '87, Málek–Nečas–Rokyta–Růžička '96, ...

Motivation: Standard compactness arguments do not allow to pass to the weak limit in a sequence of approximate solutions (e.g. Navier-Stokes to Euler). Let (v_k) be a sequence of approximate solutions and identify v_k with the parametrised probability measure

$$(\delta_{v_k(x,t)})_{(x,t)\in\Omega\times\mathbb{R}^+}.$$

By weak compactness, a subsequence will converge weakly* in the sense of measures to a parametrised probability measure $\nu_{x,t}$.

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Then, $\nu_{x,t}$ will satisfy the Euler equations in the following sense:

$$\partial_t \int_{\mathbb{R}^d} \xi d\nu_{x,t}(\xi) + \operatorname{div} \int_{\mathbb{R}^d} \xi \otimes \xi d\nu_{x,t}(\xi) + \nabla p = 0$$

$$\operatorname{div} \int_{\mathbb{R}^d} \xi d\nu_{x,t}(\xi) = 0.$$

If $\int_{\mathbb{R}^d} \xi \nu_{x,0}(\xi) = v_0(x)$ and

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Weak-Strong Uniqueness

Theorem (Brenier–De Lellis–Székelyhidi '11)

Let $\nu_{x,t}$ be a measure-valued solution satisfying the weak energy inequality with initial data v_0. If v is a smooth solution with the same data, then

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Application to the viscosity limit

Corollary

If v is a smooth solution of Euler with data v_0, then any sequence of Leray-Hopf solutions of Navier-Stokes with initial data v_0 and $\nu \to 0$ converges to v strongly in $L^2_{t,x}$.

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Theorem (Gwiazda–Świerczewska-Gwiazda–W. '15)

Let $\gamma > 1$. Then measure-valued solutions of the isentropic Euler system that satisfy a weak energy inequality enjoy the weak-strong uniqueness property.

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Compressible Navier-Stokes Equations in 3D

For the isentropic compressible Navier-Stokes equations

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the existence of weak solutions is unknown in the case $1 < \gamma < 3/2$, as is the convergence of approximation schemes to weak solutions for general γ except in very particular cases.

On the other hand it is not obvious how to define measure-valued solutions: Nonlinearities appear both in v and ∇v, since the energy inequality reads

$$E(t) \leq E_0 - \int_0^t \int_{\mathbb{R}^3} \mathbb{S}(\nabla v) : \nabla v dxds.$$

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Feireisl–Gwiazda–Świerczewska-Gwiazda–W. '16: We

- give a reasonable definition of dissipative measure-valued solutions and prove their existence
- and show weak-strong uniqueness.

Application: Given smooth data, then approximate solutions converge to the unique smooth solution as long as the density remains bounded.
Further applications of this strategy:

- Feireisl–Lukáčová '16: Convergence of a finite element–finite volume scheme for compressible Navier-Stokes
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Admissible Measure-Valued Solutions

Feireisl–Gwiazda–Świerczewska-Gwiazda–W. '16: We

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Regularity and Energy Conservation

We have seen: There exist weak solutions of the incompressible Euler equations that do not conserve energy.

On the other hand, smooth solutions do conserve energy: Multiply the equations by v and integrate in x to obtain

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}^3}|v|^2dx+\int_{\mathbb{R}^3}v\cdot\operatorname{div}(v\otimes v)dx+\int_{\mathbb{R}^3}v\cdot\nabla p dx=0.$$

The last two integrals vanish due to an integration by parts and the condition $\operatorname{div} v=0$. For the integration by parts we need $v\in C^1$.

What is the threshold regularity that ensures energy conservation?

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Theorem

a) *If v is a weak solution of the incompressible Euler equations with $v \in C^\alpha$ for an $\alpha > \frac{1}{3}$, then the energy is conserved.*

b) *For every $\alpha < \frac{1}{3}$ there exists a weak solution $v \in C^\alpha$ that dissipates energy.*

- This was already conjectured by L. Onsager in 1949, based on Kolmogorov's 1941 theory of turbulence.
- Part a) was proved by Eyink '94 and Constantin–E–Titi '94. The latter showed the statement in Besov spaces $B^\alpha_{3,\infty}$ by a simple commutator estimate.
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Besov spaces

For $0 < \alpha < 1$ and $1 \leq p \leq \infty$, the Besov space $B^{\alpha}_{p,\infty}$ is defined as the set of functions $v \in L^p$ such that

$$\|u(\cdot - y) - u\|_{L^p} \leq C|y|^{\alpha}.$$

Observe that $B^{\alpha}_{\infty,\infty} = C^{\alpha}$.

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An Onsager-Type Result for Compressible Euler

Theorem (Feireisl–Gwiazda–Świerczewska-Gwiazda–W. '17)

Let (ρ, u) be a bounded weak solution of the compressible Euler equations such that

$$\rho \in B^{\alpha}_{3,\infty}, \qquad v \in B^{\beta}_{3,\infty}$$

with

$$\alpha \leq \beta, \qquad 2\alpha + \beta > 1, \qquad \alpha + 2\beta > 1$$

and either $\rho \geq c > 0$ or $\gamma \geq 2$.
Then the energy is conserved.

Remarks

- A special case is $\alpha = \beta > \frac{1}{3}$, which is similar to the incompressible case.
- Shock waves show that our result is sharp: A shock solution $(\rho, v) \in BV \cap L^\infty \subset B^{1/3}_{3,\infty}$ dissipates energy.
- This also shows that a shock cannot form in the density alone: If $\rho \in BV \cap L^\infty$ but v remains regular, then the energy is conserved, so that there can be no shock.
- Generalisations by Drivas–Eyink '17 and Gwiazda–Michálek–Świerczewska-Gwiazda '17

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Summary: Compressible vs. Incompressible Flows

- Turbulence is hard to define, but it may involve effects of irregularity, indeterminism, and energy dissipation. This motivates the study of weak solutions.
- Non-unique weak solutions are available for the incompressible and compressible Euler equations.
- Weak-strong uniqueness holds for a variety of incompressible and compressible models in very large classes of measure-valued solutions. This has important applications to singular limits.
- Onsager's Theorem gives a criterion for energy conservation in terms of regularity of the solution. For the incompressible Euler equations, the only known dissipative mechanism is an "anomalous", homogeneously turbulent one. For the compressible system, the classical dissipation mechanism is shock formation.

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