Random ~~walks~~ talks with a busy guy on permanent vacation

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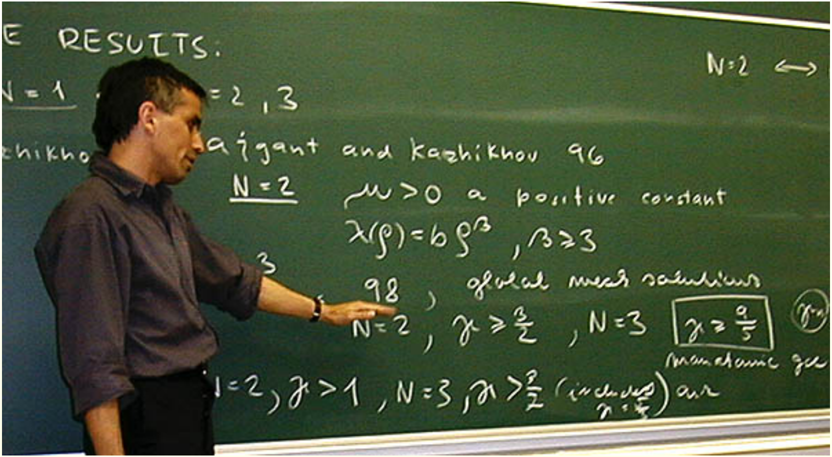
Busy guy on permanent vacation



Plan of the talk

1. Etymology of the title
2. A short cv of Eduard Feireisl
3. Why we like him ($\equiv$ Why he is great)
4. Bits and pieces from mathematics of Eduard Feireisl
   - Feireisl-Lions existence theory for CNSE
   - Stability and Weak strong uniqueness for CNSE

Etymology of the title

We shall proceed in two steps.

1. Why adjective "busy" ?

2. Appropriateness of noun "vacation" ?

Etymology of the title

Step 1 : Adjective "busy"

1. Random talks with a guy on permanent vacation

2. Random talks with a ~~hardworking~~ guy on permanent vacation

3. Random talks with a busy guy on permanent vacation

Etymology of the title

Step 2 : Noun "vacation"

1. **First argument**
   Vacation is according to Cambridge dictionary ". . . a time when someone does not go to his usual work or to his usual school but is free to do what he, she wants (including mathematics) such as travel (including to seminars and conferences) or relax (including doing mathematics) . . ."
2. **Second argument**

A short cv of Eduard Feireisl

- Born in December, 60 years ago
- High school at "Nové Strašeci"
- Master degree, Charles University in Prague (1982)
- Phd., Mathematical Institute CSAV (1986)
- DrSc. (99), Docent habilitation (09), Full Professor (11)
- Permanent leading senior researcher in MU CAV
- Long term visiting positions in Oxford (England), Madrid (Spain), Ecole Polytechnique, Paris-Sud, Besancon, Nancy(France), Athens(Ohio-USA), Munich, Vienna, Aquilla, Budapest, Warsaw
- Member of Editorial boards of 18 mostly highly impacted journals
- Member (sometimes chairing) Scientific Committees of many international conferences (including ECM, 2012)
- At least 17 invited plenary lectures at International conferences
- Unenumarable number of talks at seminars, conferences and scientific schools
- Awards : Prizes of Acad. Sci. (04, 09), Premium Academae (07-13), The Neuron Award for Contribution to Science (15)
- ERC Advanced Grant (13-18)
- 260 publications (4 monographs), around 3500 citations.

Why we like him . . .

- high-talent, hard work, modesty
  ". . . belongs to a few mathematicians who are able to translate with unimaginable and unsupportable lightness the problems of real word applications into elegant and treatable mathematical formulations . . ."
- resistance, endurance, pertinence
- authentic mathematician
- generosity

Compressible Navier-Stokes equations (CNSE)

We consider in $[0,T) \times \Omega$, $\Omega \subset R^3$ (a bounded Lipschitz domain) the following system of equations

Continuity equation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0 \tag{1}$$

Momentum equation

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) \tag{2}$$

Boundary conditions

$$\mathbf{u}\Big|_{(0,T)\times\partial\Omega} = 0 \tag{3}$$

Initial conditions

$$\varrho(0,x) = \varrho_0(x), \quad \varrho \mathbf{u}(0,x) = \varrho_0 \mathbf{u}_0(x). \tag{4}$$

Stress tensor

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu\left(\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} - \frac{2}{3}\mathrm{div}\mathbf{u}\mathbb{I}\right) + \eta \mathrm{div}\mathbf{u}\mathbb{I}, \ \mu > 0; \ \eta \geq 0 \tag{5}$$

Pressure

$$p \in C^1[0, \infty) \quad p(0) = 0. \tag{6}$$

Helmholtz function H

$$\varrho H'(\varrho) - H(\varrho) = p(\varrho), \quad H(\varrho) = \varrho \int_1^\varrho \frac{p(s)}{s^2} \mathrm{d}s$$

Relative energy function E

$$E(\varrho, r) = H(\varrho) - H'(r)(\varrho - r) - H(r)$$

Weak solutions to CNSE

Functional spaces

$\varrho(t,x) \geq 0$ for a.a. $(t,x) \in (0,T) \times \Omega$, $\varrho \in L^\infty(0,T;L^1(\Omega))$,
$\varrho\mathbf{u} \in L^\infty(0,T;L^1(\Omega;R^3))$, $\varrho\mathbf{u}^2 \in L^\infty(0,T;L^1(\Omega))$,
$\mathbf{u} \in L^2(0,T;W_0^{1,2}(\Omega;R^3))$, $p(\varrho) \in L^\infty(0,T;L^1(\Omega))$.

Continuity equation

$\varrho \in C_{\text{weak}}([0,T];L^1(\Omega))$ and equation (1) is replaced by the family of integral identities

$$\int_\Omega \varrho\varphi \,\mathrm{d}x\Big|_0^\tau = \int_0^\tau \int_\Omega \Big(\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t \qquad (7)$$

for all $\tau \in [0,T]$ and for any $\varphi \in C^1([0,T]\times\overline{\Omega})$;

Momentum equation

$\varrho\mathbf{u} \in C_{\rm weak}([0,T];L^1(\Omega;R^3))$ and momentum equation (2) is satisfied in the sense of distributions, specifically,

$$\int_\Omega \varrho\mathbf{u}\cdot\varphi\,{\rm d}x\Big|_0^\tau = \int_0^\tau\int_\Omega \Big(\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\varphi\Big)\,{\rm d}x{\rm d}t \tag{8}$$

$$+\int_0^\tau\int_\Omega \Big(p(\varrho){\rm div}_x\varphi - \mathbb{S}(\vartheta,\nabla_x\mathbf{u}) : \nabla_x\varphi\Big)\,{\rm d}x\,{\rm d}t$$

for all $\tau\in[0,T]$ and for any $\varphi\in C^1_c([0,T]\times\Omega;R^3)$;

Energy inequality

$$\int_\Omega \Big(\frac{1}{2}\varrho\mathbf{u}^2 + E(\varrho,\overline{\varrho})\Big)\,{\rm d}x\Big|_0^\tau + \int_0^\tau\int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}\,{\rm d}x{\rm d}t \le 0, \tag{9}$$

for a.a. $\tau\in(0,T)$, where $\overline{\varrho}>0$.

Feireisl-Lions theory of existence of weak solutions

Pressure for existence of weak solutions

$$p'(\varrho) \geq a_1 \varrho^{\gamma-1} - b, \quad \varrho > 0, \tag{10}$$

$$p(0) = 0, \quad p(\varrho) \leq a_2 \varrho^{\gamma} + b, \quad \varrho \geq 0,$$

with some $\gamma > 3/2$, $a_1 > 0$, $a_2, b \in R$.

Finite energy initial data

$$0 \neq \varrho_0 \geq 0, \int_\Omega \frac{1}{2} \varrho_0 \mathbf{u}_0^2 + E(\varrho_0 \,|\, \overline{\varrho}) \,\mathrm{d}x < \infty. \tag{11}$$

Existence of Weak solutions : Lions,98 ($\gamma \geq \frac{9}{5}$), Feireisl, Petzeltova, N., 02 ($\gamma > \frac{3}{2}$), Feireisl (02) non-monotone pressure as above

Under assumptions on the initial data (11), and pressure $p \in C^1[0, \infty)$, (10), the compressible Navier-Stokes system (1–5) admits at least one weak solution.

Lions approach

- Equations

$$\partial_t \varrho_n + \mathrm{div}_x(\varrho_n \mathbf{u}_n) = 0 \tag{12}$$

$$\partial_t(\varrho_n \mathbf{u}_n) + \mathrm{div}_x(\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n) + \nabla_x p(\varrho_n) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}_n) \tag{13}$$

$$\partial_t b(\varrho_n) + \mathrm{div}_x(b(\varrho_n)\mathbf{u}_n) + (\varrho_n b'(\varrho_n) - b(\varrho_n))\mathrm{div}\mathbf{u}_n = 0 \tag{14}$$

- Energy inequality plus one additional estimate yield estimates that guarantee weak convergence of $(\varrho_n, \mathbf{u}_n, p(\varrho_n))$ to $(\varrho, \mathbf{u}, \overline{p(\varrho)})$.

- With this information one can pass to the limit in the above equations and get :

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0 \qquad (15)$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x \overline{p(\varrho)} = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) \qquad (16)$$

$$\partial_t \overline{\varrho \ln \varrho} + \mathrm{div}_x(\overline{\varrho \ln \varrho} \mathbf{u}) + \overline{\varrho \mathrm{div} \mathbf{u}} = 0 \qquad (17)$$

- The first key point in this approach is that the approximated and momentum equation verify Effective viscous flux identity :

$$0 \leq \overline{\varrho p(\varrho)} - \varrho \overline{p(\varrho)} \approx \overline{\varrho \mathrm{div} \mathbf{u}} - \varrho \mathrm{div} \mathbf{u} \geq 0 \qquad (18)$$

- The second key point in the approach is DiPerna-Lions transport theory : If $(\varrho, \mathbf{u})$ verifies (15) then it verifies also (15) in the renormalized sense provided $\varrho \in L^2(Q_T)$. In particular :

$$\partial_t \varrho \ln \varrho + \mathrm{div}_x(\varrho \ln \varrho \mathbf{u}) + \varrho \mathrm{div} \mathbf{u} = 0 \qquad (19)$$

- Consequently (17) and (19) yield

$$\int_0^T \int_\Omega \overline{\varrho \ln \varrho} - \varrho \ln \varrho \, \mathrm{d}x\mathrm{d}t \leq 0$$

which means a.e. in Q_T convergence of ϱ_n and finishes the story.

Feireisl's approach

- Oscillations defect measure for a weakly convergent sequence ϱ_n :

$$\mathrm{osc}_p[\varrho_n \rightharpoonup \varrho](Q_T) \equiv \sup_{k\geq 1}\left(\limsup_{n\to 0}\int_{Q_T}\left|T_k(\varrho_n) - T_k(\varrho)\right|^p \mathrm{d}x\mathrm{d}t\right), \quad (20)$$

where truncation $T_k(\varrho) = \min(\varrho, k)$.
- First key observation is that

$$\mathrm{osc}_{\gamma+1}[\varrho_n \rightharpoonup \varrho](Q_T) \text{ is bounded.}$$

- Effective viscous flux identity reads :

$$\int_0^T\int_\Omega\Big(\overline{\varrho^\gamma T_k(\varrho)} - \overline{\varrho^\gamma}\ \overline{T_k(\varrho)}\Big)\mathrm{d}x\mathrm{d}t \approx \int_0^T\int_\Omega\Big(\overline{T_k(\varrho)\mathrm{div}\mathbf{u}} - \overline{T_k(\varrho)}\ \mathrm{div}\mathbf{u}\Big)\mathrm{d}x\mathrm{d}t$$

Its r.h.s. is

$$\leq c\Big[\mathrm{osc}_{\gamma+1}[\varrho_n \rightharpoonup \varrho](Q_T)\Big]^{\frac{1}{2\gamma}}$$

Its left hand side is

$$\geq \limsup_{n\to 0}\int_0^T\int_\Omega\Big(\varrho_n^\gamma - \varrho^\gamma\Big)\Big(T_k(\varrho_n) - T_k(\varrho)\Big)\ \mathrm{d}x\mathrm{d}t +$$

$$\int_0^T\int_\Omega\Big(\varrho^\gamma - \overline{\varrho^\gamma}\Big)\Big(\overline{T_k(\varrho)} - T_k(\varrho)\Big)\ \mathrm{d}x\mathrm{d}t$$

$$\geq \limsup_{n\to 0}\int_0^T\int_\Omega\Big|T_k(\varrho_n) - T_k(\varrho)\Big|^{\gamma+1}\ \mathrm{d}x\mathrm{d}t := \mathrm{osc}_{\gamma+1}[\varrho_n \rightharpoonup \varrho](Q_T).$$

- Second key observation : If the oscillation defect measure with $p > 2$ is bounded and if the sequence $(\varrho_n, \mathbf{u}_n)$ verifies continuity equation in the renormalized sense, then its weak limit verifies continuity equation also in the renormalized sense. (Condition $\varrho \in L^2(Q_T)$ is not needed !).
- With the above statement at hand one concludes exactly as in the Lions approach that

$$\int_0^T \int_\Omega \overline{\varrho \ln \varrho} - \varrho \ln \varrho \, \mathrm{d}x\mathrm{d}t \leq 0$$

which means a.e. in Q_T convergence of ϱ_n and finishes the story.

And what about uniqueness of weak solutions ?

- This is millenium problem (to difficult)
- What about weak strong uniqueness ? (Are strong solutions -provided they exist - unique in the class of weak solutions ?)
- Partial answer was provided independently by P. Germain and B. Desjardins : Strong solutions are unique in the class of weak solutions provided the weak solutions are slightly better (than the weak solution whose existence can be proved)

Relative energy inequality

$$\int_{\Omega}\left(\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2+E(\varrho\,|\,r)\right)\,\mathrm{d}x\Big|_0^\tau \tag{21}$$

$$+\int_0^\tau\int_\Omega \mathbb{S}\Big(\nabla_x(\mathbf{u}-\mathbf{U})\Big):\nabla_x\Big(\mathbf{u}-\mathbf{U}\Big)\,\mathrm{d}x\,\mathrm{d}t$$

$$\leq\int_0^\tau\int_\Omega\Big(\mathbb{S}(\nabla_x\mathbf{U}):\nabla_x\Big(\mathbf{U}-\mathbf{u}\Big)\,\mathrm{d}x\mathrm{d}t$$

$$+\int_0^\tau\int_\Omega\Big(\varrho\partial_t\mathbf{U}+\varrho\mathbf{u}\cdot\nabla_x\mathbf{U}\Big)\cdot(\mathbf{U}-\mathbf{u})\,\mathrm{d}x\mathrm{d}t$$

$$-\int_0^\tau\int_\Omega p(\varrho)\mathrm{div}_x\mathbf{U}\,\mathrm{d}x\,\mathrm{d}t$$

$$+\int_0^\tau\int_\Omega\left(\frac{r-\varrho}{r}\partial_t p(r)-\frac{\varrho}{r}u\cdot\nabla_x p(r)\right)\,\mathrm{d}x\,\mathrm{d}t$$

for all

$$r\in C^1_c([0,T]\times\overline{\Omega}),\ r>0,\ \mathbf{U}\in C^1_c([0,T]\times\Omega).$$

Dissipative solutions

Relative energy

$$\mathcal{E}(\varrho, \mathbf{u} \,\Big|\, r, \mathbf{U}) = \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + E(\varrho\,|\,r) \right) \,\mathrm{d}x$$

Functional spaces

$\varrho(t,x) \geq 0$ for a.a. $(t,x) \in (0,T)\times\Omega$, $\varrho \in L^\infty(0,T;L^1(\Omega))$,
$\varrho\mathbf{u} \in L^\infty(0,T;L^1(\Omega;R^3))$, $\varrho\mathbf{u}^2 \in L^\infty(0,T;L^1(\Omega))$,
$\mathbf{u} \in L^2(0,T;W_0^{1,2}(\Omega;R^3))$, $p(\varrho) \in L^\infty(0,T;L^1(\Omega))$.

Relative energy inequality

$$\mathcal{E}(\varrho, \mathbf{u} \,\Big|\, r, \mathbf{U})(\tau) + \int_0^\tau \int_\Omega \mathbb{S}\Big(\nabla_x(\mathbf{u}-\mathbf{U})\Big) : \nabla_x\Big(\mathbf{u}-\mathbf{U}\Big) \,\mathrm{d}x\,\mathrm{d}t$$

$$\leq \mathcal{E}(\varrho_0, \mathbf{u}_0 \,\Big|\, r(0), \mathbf{U}(0)) + \int_0^\tau \mathcal{R}\Big(\varrho, \mathbf{u} \,\Big|\, r, \mathbf{U}\Big)\mathrm{d}t$$

where the remainder $\mathcal{R}$ is given by the r.h.s. of formula (21) and the test functions are the same as in formula (21).

Existence of Weak and dissipative solutions

Existence of disspative solutions Feireisl, Sun, N., 2012

Under assumptions on initial data(11) and pressure $p \in C^1[0, \infty)$, (10), there is at least one dissiptive solution.

Weak solutions are dissipative : Feireisl, Jin, N., 2012

Under assumptions on initial data(11) and pressure $p \in C^1[0, \infty)$, any weak solution of the compressible Navier-Stokes system (1–5) is a dissipative one.

Thermodynamic stability conditions and relative energy function

Thermodynamic stability conditions

$$p'(\varrho) > 0, \quad \varrho > 0 \tag{22}$$

Relative energy function E under thermodynamic stability conditions

Under condition (22),

$$E(\varrho, r) = H(\varrho) - H'(r)(\varrho - r) - H(r)$$

$$E(\varrho, r) \geq 0, \quad E(\varrho, r) = 0 \Leftrightarrow \varrho = r$$

Relative entropy with (r, U) strong solution of CNSE

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + E(\varrho\,|\,r)\right)\,\mathrm{d}x\Big|_0^\tau \tag{23}$$

$$+\int_0^\tau\int_\Omega \mathbb{S}\Big(\nabla_x(\mathbf{u}-\mathbf{U})\Big) : \nabla_x\Big(\mathbf{u}-\mathbf{U}\Big)\,\mathrm{d}x\,\mathrm{d}t$$

$$\leq \int_0^\tau\int_\Omega (\varrho - r)\Big(\partial_t\mathbf{U} + \mathbf{U}\cdot\nabla_x\mathbf{U}\Big)\cdot(\mathbf{U}-\mathbf{u})\,\mathrm{d}x\mathrm{d}t$$

$$-\int_0^\tau\int_\Omega \varrho(\mathbf{u}-\mathbf{U})\cdot\nabla_x\mathbf{U}\Big)\cdot(\mathbf{U}-\mathbf{u})\,\mathrm{d}x\mathrm{d}t$$

$$-\int_0^\tau\int_\Omega \Big(p(\varrho) - p'(r)(\varrho - r) - p(r)\Big)\mathrm{div}_x\mathbf{U}\,\mathrm{d}x\,\mathrm{d}t$$

$$+\int_0^\tau\int_\Omega \frac{r-\varrho}{r}(\mathbf{u}-U)\cdot\nabla_x p(r)\,\mathrm{d}x\mathrm{d}t.$$

Relative energy and stability

Regularity of pressure

$$p \in C^1[0,\infty) \cap C^2(0,\infty), \quad p(0) = 0. \tag{24}$$

Weak strong uniqueness, stability I

Let $(\varrho, \mathbf{u})$ be a weak solution to the Navier-Stokes equations (1-6) with pressure obeying regularity (24) and thermodynamic stability conditions (22), emanating from finite energy initial data $(\varrho_0, \mathbf{u}_0)$ in the time interval $[0,T)$, $T > 0$ such that

$$0 < \underline{\varrho} < \varrho(t,x) < \overline{\varrho} < \infty. \tag{25}$$

Let $(r, \mathbf{U})$ be a strong solution of the same equations in the regularity class (27), with finite enrergy initial data $(r_0, \mathbf{U}_0)$. Then

$$\mathcal{E}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right) \leq c\mathcal{E}\left(\varrho_0, \mathbf{u}_0 \,\middle|\, r_0, \mathbf{U}_0\right).$$

No conditions on weak solution, minimum conditions on pressure

Weak strong uniqueness, stability II

Let the pressure verifies regularity (24), thermodynamic stability condition (22), and

$$p(\varrho) \leq c_1 + c_2\varrho + H(\varrho). \tag{26}$$

Let $(\varrho, \mathbf{u})$ be a weak solution to the compressible Navier-Stokes equations (1-5) emanating from the initial data $(\varrho_0, \mathbf{u}_0)$, and let $(r, \mathbf{U})$ be a strong solution of the same system emanating from the initial data $(0 < r_0, \mathbf{U}_0)$ in class (27). Then there exists $c = c(\Omega, T, \|r^{-1}\|_{0,\infty}, \|r\|_{1,\infty}, \|\mathbf{U}\|_{1,\infty})$ such that

$$\mathcal{E}\Big(\varrho, \mathbf{u} \,\Big|\, r, \mathbf{U}\Big) \leq c\mathcal{E}\Big(\varrho_0, \mathbf{u}_0 \,\Big|\, r_0, \mathbf{U}_0\Big).$$

Sufficient condition for (26)

$$0 < \frac{1}{p_\infty} \leq \liminf_{\varrho\to\infty} \frac{p(\varrho)}{\varrho^\gamma} \leq \limsup_{\varrho\to\infty} \frac{p(\varrho)}{\varrho^\gamma} \leq p_\infty < \infty, \text{ where } \gamma > 0,$$

Eistence of strong solutions corresponding to situations I,II

$$0 < \underline{r} \leq r \leq \bar{r} < \infty; \quad \mathbf{U} \in L^\infty((0,T) \times \Omega), \tag{27}$$

$$\partial_t r,\ \partial_t \mathbf{U},\ \nabla_x r,\ \nabla_x \mathbf{U} \in L^2(0,T; L^\infty(\Omega)),$$

Local in time (in large) or global in time (in small) existence of such solutions under additional compatibility conditions on initial data follows from the theory developed in 80'th by in works of Masumura, Nishida, Valli, Zajaczkowski

No conditions on weak solution, growth on pressure

Weak strong uniqueness, stability III

Let the pressure verifies regularity (24), thermodynamic stability condition (22), and (10) with $\gamma > 6/5$.
Let $(\varrho, \mathbf{u})$ be a weak solution in class (28) to the compressible Navier-Stokes equations (1-5) emanating from the finite energy initial data $(\varrho_0, \mathbf{u}_0)$, and let $(r, \mathbf{U})$ be a strong solution of the same system emanating from the initial data $(0 < r_0, \mathbf{U}_0)$. Then there exists $c > 0$ such that

$$\mathcal{E}\Big(\varrho, \mathbf{u} \Big| r, \mathbf{U}\Big) \leq c\mathcal{E}\Big(\varrho_0, \mathbf{u}_0 \Big| r_0, \mathbf{U}_0\Big).$$

Regularity of the strong solution II - corresponding to the theory of Cho, Choe, Kim, 90's

$$0 < \underline{r} \leq r \leq \overline{r} < \infty, \quad \mathbf{U} \in L^\infty((0,T) \times \Omega), \tag{28}$$

$$\nabla_x r \in L^2(0,T; L^q(\Omega; \mathbb{R}^3)), \ \nabla_x^2 \mathbf{U} \in L^2(0,T; L^q(\Omega)), \ q > \max\{(3, \frac{6\gamma}{5\gamma - 6})\}.$$

Regularity, Blow-up

Bow up criterion. Sun, Zhang, 2000

Let $(r, \mathbf{U})$ be a strong solution to CNSE in Cho,Choe, Kim's class on $[0, T_{\max})$. If $T_{\max} < \infty$ then

$$\lim_{T \to T_{\max}} \|\varrho\|_{L^\infty(Q_T)} \to \infty.$$

Regularity criterion

Let $(r, \mathbf{U})$ be a weak solution to CNSE (on $(0, T)$) emanating from the Cho,Choe,Kim's initial data. Suppose that r is bounded. Then $(\varrho, \mathbf{U})$ is a strong solution in the Cho, Choe, Kim's class.

Approximating system of PDEs for weak solutions

Approximating system

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) - \varepsilon \Delta \varrho = 0,$$

$$\partial_n \varrho|`_{\partial\Omega} = 0,$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x (p(\varrho) + \delta \varrho^4) + \varepsilon \nabla_x \varrho \cdot \nabla_x \mathbf{u}$$

$$= \mu \Delta \mathbf{u} + (\frac{\mu}{3} + \eta) \nabla_x \operatorname{div} \mathbf{u}$$

$$\mathbf{u}|_{\partial\Omega} = 0$$

Weak solutions are obtained letting first $\varepsilon \to 0$ and then $\delta \to 0$. This is not exploitable in the numerics !

Physical and numerical domain

The physical space is represented by a bounded domain $\Omega \subset R^3$. The numerical domains Ω_h are polyhedral domain, a union of tetrahedra K

$$\overline{\Omega}_h = \cup_{K \in \mathcal{T}_h} K.$$

If $K \cap L \neq \emptyset$, $K \neq L$, then $K \cap L$ is either a common face, or a common edge, or a common vortex. Either $\Omega = \Omega_h$ or

$$\sup_{x \in \partial\Omega} \mathrm{dist}(x, \partial\Omega_h) \leq ch, \tag{29}$$

or

$$\mathcal{V}_h \in \partial\Omega_h \text{ a vertex } \Rightarrow \mathcal{V}_h \in \partial\Omega. \tag{30}$$

Furthermore, we suppose that

$$\xi[K] \approx \mathrm{diam}[K] \approx h, \tag{31}$$

where $\xi[K]$ is the radius of the largest ball contained in K.

Discretisation

$K \in \mathcal{T}$ - regular partition of Ω_h into tetrahedrons of size h.

$\sigma = K|L \in \mathcal{E}_{\rm int}$ - set of internal faces, $\mathcal{E}$ - set of all faces.

$0 < t_1 < \ldots < t_n < \ldots < T$ - time discretisation of step Δt.

$\varrho(t_n, x) \approx \sum_{K \in \mathcal{T}} \varrho_K^n 1_K(x) \in Q_h(\Omega_h)$ - space of piecewise constants.

$\mathbf{u}(t_n, x) \approx \sum_{\sigma \in \mathcal{E}_{\rm int}} \mathbf{u}_\sigma^n \phi_\sigma(x) \in V_{h,0}(\Omega_h)$ - the CR space.

$$\int_\sigma \phi_\sigma \phi_{\sigma'} {\rm d}S = \delta_{\sigma,\sigma'}$$

Upwind :

$$\varrho_\sigma^{\rm up} = \left\{ \begin{array}{c} \varrho_K \text{ if } \mathbf{u}_\sigma \cdot \mathbf{n}_{\sigma,K} > 0 \\ \\ \varrho_L \text{ otherwise} \end{array} \right\}, \quad \text{where } \sigma = K|L.$$

Mean values :

$$V_K = \frac{1}{|K|} \int V {\rm d}x, \quad \hat{V} = \sum v_K 1_K(x), \quad V_\sigma = \frac{1}{|\sigma|} \int V {\rm d}S$$

Numerical scheme, Karper

$$\varrho^n \in Q_h(\Omega_h),\ \varrho^n \geq 0,\ \mathbf{u}^n \in V_{h,0}(\Omega_h; R^3), \quad n = 0, 1, \ldots, N, \tag{32}$$

$$\sum_{K \in \mathcal{T}} |K| \frac{\varrho_K^n - \varrho_K^{n-1}}{\Delta t} \phi_K + \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\mathrm{up}} (\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}) \phi_K = 0 \tag{33}$$

for any $\phi \in Q_h(\Omega_h)$ and $n = 1, \ldots, N$,

$$\sum_{K \in \mathcal{T}} \frac{|K|}{\Delta t} \left(\varrho_K^n \mathbf{u}_K^n - \varrho_K^{n-1} \mathbf{u}_K^{n-1} \right) \cdot \mathbf{v}_K + \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\mathrm{up}} \hat{\mathbf{u}}_\sigma^{n,\mathrm{up}} [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}] \cdot \mathbf{v}_K$$

$$- \sum_{K \in \mathcal{T}} p(\varrho_K^n) \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \mathbf{v}_\sigma \cdot \mathbf{n}_{\sigma,K} + \mu \sum_{K \in \mathcal{T}} \int_K \nabla \mathbf{u}^n : \nabla \mathbf{v} \, \mathrm{d}x \tag{34}$$

$+\frac{\mu}{3} \sum_{K \in \mathcal{T}} \int_K \mathrm{div} \mathbf{u}^n \mathrm{div} \mathbf{v} \, \mathrm{d}x = 0,$ for any $\mathbf{v} \in V_{h,0}(\Omega; R^3)$ and $n = 1, \ldots, N$.

$$\|(r,\mathbf{V})\|_{X_T(\mathbb{R}^3)} \equiv \|r\|_{C^1([0,T]\times R^3)} + \|\partial_t \nabla_x r\|_{C([0,T];L^6(R^3;R^3))} + \|\partial^2_{t,t} r\|_{C([0,T];L^6(R^3))}$$

$$\|\mathbf{V}\|_{C^1([0,T]\times R^3;R^3)} + \|\mathbf{V}\|_{C([0,T];C^2(R^3;R^3))} + \|\partial_t \nabla_x \mathbf{V}\|_{C([0,T];L^6(R^3;R^{3\times 3}))}$$

$$+\|\partial^2_{t,t}\mathbf{V}\|_{L^2(0,T;L^6(R^3))} \text{ and } r \geq \underline{r} > 0.$$

Case $\Omega = \Omega_h$, Gallouet, Herbin, Maltese, N., 2015

Let $(\varrho_h^n, \mathbf{u}_h^n) = (\varrho, \mathbf{u})$ be a family of numerical solutions of the numerical scheme (32–33) with $\gamma \geq 3/2$. Let $(r, \mathbf{V})$ be a classical solution of the compressible Navier-Stokes equations (1–6) in the class $X_T(\mathbb{R}^3)$. Then there exists $c > 0$ independent of h, Δt, ϱ, $\mathbf{u}$, such that

$$\mathcal{E}(\varrho^n, \mathbf{u}^n \Big| r, U) \leq c\Big(\mathcal{E}(\varrho_0, \mathbf{u}_0 \Big| r_0, \mathbf{U}_0) + h^\alpha + \sqrt{\Delta t}\Big),$$

where

$$\alpha = \frac{2\gamma - 3}{2} \text{ if } 3/2 \leq \gamma < 2, \;\; \alpha = \frac{1}{2} \text{ if } \gamma \geq 2,$$

$$\mathcal{E}(\varrho^n, \mathbf{u}^n \Big| r, \mathbf{V}) = \sum_{K\in\mathcal{T}} \int_K \Big(\frac{1}{2}\varrho_K^n (\mathbf{V}^n - \hat{\mathbf{u}}_K^n)^2 + E(\varrho_K^n | r^n)\Big)$$

FV/FD - MAC scheme Gallouet, Herbin, Maltese, N., 2016

The same result holds for the FV/FD scheme :

$$\mathcal{E}_\varepsilon\left(\varrho^n, \mathbf{u}^n \middle| r, U\right) \leq c\left(\mathcal{E}_\varepsilon\left(\varrho_0, \mathbf{u}_0 \middle| \overline{\varrho}, \mathbf{U}_0\right) + h^\alpha + \sqrt{\Delta t}\right).$$

Case $\Omega \neq \Omega_h$, Feireisl, Hosek, Maltese, N., 2016

Suppose that Ω is a bounded domain of class C^3. Let $(\varrho_h, \mathbf{u}_h) = (\varrho, \mathbf{u})$ be a family of numerical solutions of the numerical scheme (32–33) with $\gamma \geq 3/2$. Let $(r, \mathbf{V})$ be a weak solution solution of the compressible Navier-Stokes equations (1–6) with *bounded* density r emanating from initial data $(0 < r_0, \mathbf{V}_0) \in C^3(\overline{\Omega})$ that satisfy the compatibility condition

$$\mathbf{V}_0|_{\partial\Omega} = 0,\ \nabla p(r_0)|_{\partial\Omega} = [\mu\Delta\mathbf{V}_0 + \frac{\mu}{3}\nabla \mathrm{div}\mathbf{V}_0]|_{\partial\Omega}.$$

Then there exists $c > 0$ independent of h, Δt, ϱ, $\mathbf{u}$, such that

$$\mathcal{E}(\varrho^n, \mathbf{u}^n \Big| r, \mathbf{V}) \leq c\Big(\mathcal{E}(\varrho_0, \mathbf{u}_0 \Big| r_0, \mathbf{V}_0) + h^\alpha + \sqrt{\Delta t}\Big),$$

where

$$\alpha = \frac{2\gamma - 3}{2} \text{ if } 3/2 \leq \gamma < 2,\ \alpha = \frac{1}{2} \text{ if } \gamma \geq 2.$$

Convergence of the n. solutions to w. solutions

Case $\Omega = \Omega_h$, Karper, 2013, complemented for $\Omega \neq \Omega_h$ Michalek, Feireisl, Karper

Let $(\varrho_h, \mathbf{u}_h)$ be a family of numerical solutions of the numerical scheme (32–33) with $\Delta t = h$ and $\gamma > 3$. Then for a suitable subsequence

$$\varrho_h \rightharpoonup_* \varrho \text{ in } L^\infty(0,T; L^\gamma(\Omega)),$$

$$\mathbf{u}_h \rightharpoonup \mathbf{u} \text{ in } L^2(0,T; L^6(\Omega)),\ \nabla_h \mathbf{u}_h \rightharpoonup \nabla_x \mathbf{u} \text{ in } L^2(Q_T)$$

where $(\varrho, \mathbf{u})$ is a weak solution of problem (1–6).

Case $\Omega \neq \Omega_h$, Feireisl, Hosek, Maltese, N 2016

Let $\Omega \subset R^3$ be a bounded domain of class C^3. Let $\sup_{x\in\partial\Omega}(x, \partial\Omega_h) \leq ch$. Let the initial data $[\varrho_0, \mathbf{u}_0]$ belong to the regularity class

$$\varrho_0 \in C^3(\overline{\Omega}),\ \varrho_0 > 0 \text{ in } \overline{\Omega},\ \mathbf{u}_0 \in C^3(\overline{\Omega}; R^3),$$

and satisfy the compatibility conditions

$$\mathbf{u}_0|_{\partial\Omega} = 0,\ \nabla_x p(\varrho_0)|_{\partial\Omega} = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}_0)|_{\partial\Omega}.$$

Let $\{\varrho_h^n, \mathbf{u}_h^n\}_{h>0}$, $k = 0, 1, \ldots, [T/\Delta t]$, $h \approx \Delta t$, be a family of numerical solutions satisfying (32–34). Finally, suppose that

$$\varrho_h^n \leq \overline{\varrho} < \infty \text{ for all } h > 0,\ n = 0, 1, \ldots, [T/\Delta t]. \tag{35}$$

Then problem (1–6) admits a classical solution $[\varrho, \mathbf{u}]$ in $(0, T) \times \Omega$, and

$$\mathrm{ess}\sup_{t\in(0,T)} \int_{\Omega\cap\Omega_h} \left[\varrho_h |\widehat{\mathbf{u}}_h - \mathbf{u}|^2 + |\varrho_h - \varrho|^2\right](t, \cdot)\, \mathrm{d}x + \int_0^T \int_{\Omega\cap\Omega_h} |\nabla_h \mathbf{u}_h - \nabla_x \mathbf{u}|^2\, \mathrm{d}x\, \mathrm{d}t \tag{36}$$

$$\leq c\left(h^{1/2} + \int_\Omega \left[\varrho^0 |\widehat{\mathbf{u}}^0 - \mathbf{u}_0|^2 + |\varrho^0 - \varrho_0|^2\right]\, \mathrm{d}x\right).$$

Energy inequality - discrete case

$$\sum_K \frac{1}{2}\frac{|K|}{\Delta t}\Big(\varrho_K^n|\hat{\mathbf{u}}_K^n|^2-\varrho_K^{n-1}|\hat{\mathbf{u}}_K^{n-1}|^2\Big)+\sum_K\frac{|K|}{\Delta t}\Big(H(\varrho_K^n)-H(\varrho_K^{n-1})\Big)$$

$$+\sum_K\frac{|K|}{\Delta t}\varrho_K^{n-1}\frac{|\hat{\mathbf{u}}_K^n-\hat{\mathbf{u}}_K^{n-1}|^2}{2}+\sum_K\frac{|K|}{\Delta t}H''(\varrho_K^{n-1,n})\frac{|\varrho_K^n-\varrho_K^{n-1}|^2}{2}$$

$$+\sum_K\sum_{\sigma\in\mathcal{E}_K}\frac{1}{4}|\sigma|\varrho_\sigma^{n,\mathrm{up}}(\hat{\mathbf{u}}_K^n-\hat{\mathbf{u}}_L^n)^2\;|\mathbf{u}_\sigma\cdot\mathbf{n}_{\sigma,K}|$$

$$+\sum_K\sum_{\sigma\in\mathcal{E}_K}\frac{1}{4}|\sigma|H''(\varrho_{KL}^n)\Big(\varrho_K^n-\varrho_L^n\Big)^2\;|\mathbf{u}_\sigma\cdot\mathbf{n}_{\sigma,K}|$$

$$+\sum_K\Big(\mu\int_K|\nabla_x\mathbf{u}|^2\mathrm{d}x+(\frac{\mu}{3}+\eta)\int_K|\mathrm{div}\mathbf{u}|^2\mathrm{d}x\Big)\leq 0.$$

Discrete relative energy

$$\sum_K \frac{1}{2}\frac{|K|}{\Delta t}\Big(\varrho_K^n|\hat{\mathbf{u}}_K^n-\hat{\mathbf{U}}_{h,K}^n|^2-\varrho_K^{n-1}|\hat{\mathbf{u}}_K^{n-1}-\hat{\mathbf{U}}_{h,K}^{n-1}|^2\Big)$$

$$+\sum_K \frac{|K|}{\Delta t}\Big(E(\varrho_K^n|\hat{r}_K^n)-E(\varrho_K^{n-1}|\hat{r}_K^{n-1})\Big)$$

$$+\sum_K\Big(\mu\int_K|\nabla_x(\mathbf{u}^n-\mathbf{U}_h^n)|^2\mathrm{d}x+(\frac{\mu}{3}+\eta)\int_K|\mathrm{div}(\mathbf{u}^n-\mathbf{U}_h^n)|^2\mathrm{d}x\Big)$$

$$\preceq\sum_K\Big(\mu\int_K\nabla_x\mathbf{U}_h:\nabla_x(\mathbf{U}_h-\mathbf{u})\mathrm{d}x+(\frac{\mu}{3}+\eta)\int_K\mathrm{div}\mathbf{U}_h\mathrm{div}(\mathbf{U}_h-\mathbf{u})\mathrm{d}x\Big)$$

$$+\sum_K\frac{|K|}{\Delta t}\Big(\varrho_K^{n-1}(\hat{\mathbf{U}}_{h,K}^{n-1}-\hat{\mathbf{u}}_{h,K}^{n-1})\cdot(\hat{\mathbf{U}}_{h,K}^n-\hat{\mathbf{U}}_{h,K}^{n-1})+(\hat{r}_K^n-\varrho_K)(H'(\hat{r}_K^n)-H'(\hat{r}_K^{n-1})\Big)$$

$$+\sum_K\sum_{\sigma\in\mathcal{E}_K}|\sigma|\varrho_\sigma^{n,\mathrm{up}}\Big(\hat{\mathbf{U}}_{h,\sigma}^{n,\mathrm{up}}-\hat{\mathbf{u}}_\sigma^{n,\mathrm{up}}\Big)\cdot\hat{\mathbf{U}}_{h,K}(\mathbf{u}^n\cdot\mathbf{n}_{\sigma,K})$$

$$-\sum_K\sum_{\sigma\in\mathcal{E}_K}|\sigma|p(\varrho_K)(\hat{\mathbf{U}}_{h,\sigma}^n\cdot\mathbf{n}_{\sigma,K})-\sum_K\sum_{\sigma\in\mathcal{E}_K}|\sigma|\varrho_\sigma^{n,\mathrm{up}}H'(\hat{r}_K^n)(\mathbf{u}^n\cdot\mathbf{n}_{\sigma,K})$$

Sketch of the proof : Treatment of the Red term

$$\sum_K \sum_{\sigma\in\mathcal{E}_K} |\sigma| \varrho_\sigma^{n,\mathrm{up}} \left(\hat{\mathbf{U}}_{h,\sigma}^{n,\mathrm{up}} - \hat{\mathbf{u}}_\sigma^{n,\mathrm{up}} \right) \cdot \mathbf{U}_{h,K} (\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K})$$

$$\approx \sum_K \sum_{\sigma\in\mathcal{E}_K} |\sigma| \varrho_\sigma^{n,\mathrm{up}} \left(\hat{\mathbf{U}}_{h,\sigma}^{n,\mathrm{up}} - \hat{\mathbf{u}}_\sigma^{n,\mathrm{up}} \right) \cdot (\mathbf{U}_{h,K} - \mathbf{U}_\sigma)(\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K})$$

$$\approx \sum_K \sum_{\sigma\in\mathcal{E}_K} |\sigma| \varrho_\sigma^{n,\mathrm{up}} \left(\hat{\mathbf{U}}_{h,\sigma}^{n,\mathrm{up}} - \hat{\mathbf{u}}_\sigma^{n,\mathrm{up}} \right) \cdot (\mathbf{U}_{h,K} - \mathbf{U}_\sigma)(\hat{\mathbf{u}}_\sigma^{n,\mathrm{up}} \cdot \mathbf{n}_{\sigma,K})$$

$$\approx \sum_K \sum_{\sigma\in\mathcal{E}_K} |\sigma| \varrho_\sigma^{n,\mathrm{up}} \left(\hat{\mathbf{U}}_{h,\sigma}^{n,\mathrm{up}} - \hat{\mathbf{u}}_\sigma^{n,\mathrm{up}} \right) \cdot (\mathbf{U}_{h,K} - \mathbf{U}_\sigma)\hat{\mathbf{U}}_\sigma^{n,\mathrm{up}} \cdot \mathbf{n}_{\sigma,K}$$

$$+ \sum_K \sum_{\sigma\in\mathcal{E}_K} |\sigma| \varrho_\sigma^{n,\mathrm{up}} \left(\hat{\mathbf{U}}_{h,\sigma}^{n,\mathrm{up}} - \hat{\mathbf{u}}_\sigma^{n,\mathrm{up}} \right) \cdot (\mathbf{U}_{h,K} - \mathbf{U}_\sigma)(\hat{\mathbf{u}}_\sigma^{n,\mathrm{up}} - \hat{\mathbf{U}}_\sigma^{n,\mathrm{up}}) \cdot \mathbf{n}_{\sigma,K}$$

$$\approx \sum_{K} \sum_{\sigma\in\mathcal{E}_K} |\sigma| \varrho_\sigma^{n,\mathrm{up}} \left(\hat{\mathbf{U}}_{h,\sigma}^{n,\mathrm{up}} - \hat{\mathbf{u}}_\sigma^{n,\mathrm{up}} \right) \cdot (\mathbf{U}_{h,K} - \mathbf{U}_\sigma) \hat{\mathbf{U}}_\sigma^{n,\mathrm{up}} \cdot \mathbf{n}_{\sigma,K}$$

$$\sum_K \int_K r\partial_t \mathbf{U} \cdot (\mathbf{u} - \mathbf{U}) + \sum_K \int_K r\mathbf{U} \cdot \nabla \mathbf{U} \cdot (\mathbf{u} - \mathbf{U}) + \ldots = \ldots$$

$$\sum_K \int_K r\mathbf{U} \cdot \nabla \mathbf{U} \cdot (\mathbf{u} - \mathbf{U}_h) \approx \sum_K \int_K r_K \mathbf{U}_{h,K} \cdot \nabla \mathbf{U} \cdot (\mathbf{u}_K - \mathbf{U}_{h,K})$$

$$\approx \sum_K \sum_{\sigma\in\mathcal{E}_K} \int_\sigma r_K \mathbf{U}_{h,K} \cdot \mathbf{n}_{\sigma,K} (\mathbf{U} - \mathbf{U}_{h,K}) \cdot (\mathbf{u}_K - \mathbf{U}_{h,K})$$

$$\approx \sum_K \sum_{\sigma\in\mathcal{E}_K} |\sigma| \, r_K \mathbf{U}_{h,K} \cdot \mathbf{n}_{\sigma,K} (\mathbf{U}_\sigma - \mathbf{U}_{h,K}) \cdot (\mathbf{u}_K - \mathbf{U}_{h,K})$$

$$\approx \sum_K \sum_{\sigma\in\mathcal{E}_K} |\sigma| \, \hat{r}_\sigma^{\mathrm{up}} (\mathbf{U}_\sigma - \mathbf{U}_{h,K}) \cdot (\hat{\mathbf{u}}_\sigma^{\mathrm{up}} - \hat{\mathbf{U}}_{h,\sigma}^{\mathrm{up}}) \hat{\mathbf{U}}_\sigma^{\mathrm{up}} \cdot \mathbf{n}_{\sigma,K}$$

Projections

Projections

$$\Pi_h^V : W^{1,p}(\Omega) \to V_h, \quad \Pi_h^V(U) \equiv U_h \equiv \sum_{\sigma \in \mathcal{E}} \hat{U}_\sigma \phi_\sigma$$

$$\Pi_h^L : L^p(\Omega) \to L_h, \quad \Pi_h^L(r) \equiv r_h \equiv \sum_{K \in \mathcal{T}} \hat{r}_K 1_K$$

Estimates involving projections

Let $s = 1, 2$, $1 \leq p \leq \infty$. There exists $c > 0$ independent of h such that for all $K \in \mathcal{T}$:

$$\forall r \in W^{1,p}(K), \quad \|r_h - r\|_{L^p(K)} \leq ch\|\nabla_x r\|_{L^p(K)},$$

$$\forall U \in W^{s,p}(K), \quad \|U_h - U\|_{L^p(K)} \leq ch^s\|\nabla_x^s U\|_{L^p(K)}$$

$$\forall U \in W^{s,p}(K), \quad \|\nabla_x U_h - \nabla_x U\|_{L^p(K)} \leq ch^{s-1}\|\nabla_x^s U\|_{L^p(K)}.$$

Some auxiliary estimates

Poincaré type inequalities

Let $1 \le p \le \infty$. There exists $c > 0$ independent of h such that for all $K \in \mathcal{T}$:

$$\forall U \in W^{1,p}(K), \quad \|U_h - \hat{U}_{h,K}\|_{L^p(K)} \le ch\|\nabla_x U_h\|_{L^p(K)},$$

$$\forall U \in W^{1,p}(K), \quad \|U - \hat{U}_K\|_{L^p(K)} \le ch\|\nabla_x U\|_{L^p(K)}$$

$$\forall U \in W^{1,p}(K), \quad \|U_h - \hat{U}_\sigma\|_{L^p(K)} \le ch\|\nabla_x U\|_{L^p(K)}.$$

Sobolev type inequalities

Let $2 \le p \le 6$. There exists $c > 0$ independent of h such that for all $K \in \mathcal{T}$:

$$\forall U \in W^{1,p}(K), \quad \|U - \hat{U}_K\|_{L^p(K)} \le ch^{\frac{3}{p}-\frac{1}{2}}\|\nabla_x V\|_{L^2(K)},$$

$$\forall U \in W^{1,p}(K), \quad \|U - \hat{U}_\sigma\|_{L^p(K)} \le h^{\frac{3}{p}-\frac{1}{2}}\|\nabla_x U\|_{L^2(K)}$$

Compressible Navier-Stokes equations in low Mach number regime

We consider in $[0,T)\times\Omega$, $\Omega\subset R^3$ (a bounded Lipschitz domain or a periodic cell) the following system of equations

Continuity equation

$$\partial_t\varrho+\mathrm{div}_x(\varrho\mathbf{u})=0 \tag{37}$$

Momentum equation

$$\partial_t(\varrho\mathbf{u})+\mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u})+\frac{1}{\varepsilon^2}\nabla_x p(\varrho)=\mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{u}) \tag{38}$$

Boundary conditions

$$\mathbf{u}\Big|_{(0,T)\times\partial\Omega}=0 \quad \text{or periodic b.c.} \tag{39}$$

Initial conditions

$$\varrho(0,x)\equiv\varrho_{0,\varepsilon}(x)=\overline{\varrho}+\varepsilon\varrho^{(1)}_{0,\varepsilon}(x),\quad \mathbf{u}(0,x)=\mathbf{u}_{0,\varepsilon}(x). \tag{40}$$

Expected target system as $\varepsilon \to 0$

Incompressible Navier-Stokes equations

$$\overline{\varrho}\Big(\partial_t \mathbf{V} + \mathbf{V}\cdot\nabla_x \mathbf{V}\Big) + \nabla\Pi = 0, \quad \mathrm{div}\mathbf{V} = 0.$$

$\mathbf{V}(0) = \mathbf{V}_0,$ homogenous Dirichlet or periodic b.c.

Ill and Well prepared initial data

Ill Prepared :

$$\mathcal{E}_\varepsilon(\varrho_0, \mathbf{u}_0 | \overline{\varrho}, \mathbf{V}_0) = \int_\Omega \Big(\frac{1}{2}\varrho_0 |\mathbf{u}_0 - \mathbf{V}_0|^2 + \frac{1}{\varepsilon^2} E(\varrho_0|\overline{\varrho})\Big)\mathrm{d}x$$

is bounded as $\varepsilon \to 0$.

Well prepared :

$$\mathcal{E}_\varepsilon(\varrho_0, \mathbf{u}_0 | \overline{\varrho}, \mathbf{V}_0) = \int_\Omega \Big(\frac{1}{2}\varrho_0 |\mathbf{u}_0 - \mathbf{V}_0|^2 + \frac{1}{\varepsilon^2} E(\varrho_0|\overline{\varrho})\Big)\mathrm{d}x$$

tends to 0 as $\varepsilon \to 0$.

A result of Lions/Masmoudi

Consider the sequence of weak solutions $(\varrho_\varepsilon, \mathbf{u}_\varepsilon)$ of CNSE corresponding to ill prepared initial data $(\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon})$ such that $\mathbf{u}_{0,\varepsilon} \rightharpoonup \mathbf{u}_0$ in $W^{1,2}(\Omega)$. The there is a subsequence such that

$$\varrho_\varepsilon \to \overline{\varrho}, \quad \mathbf{u}_\varepsilon \rightharpoonup \mathbf{V}$$

and there is $\Pi \in \mathcal{D}'((0,T)\times\Omega)$ such that the couple $(\Pi, \mathbf{V})$ is a weak solution to the NSE with $\mathbf{V}_0 = \mathbf{H}(\mathbf{u}_0)$.

A result via the relative energy

Let $(\Pi, \mathbf{V})$ be a strong solution to the NSE with initial data $\mathbf{V}_0$ on interval $[0,T)$ and $(\varrho_\varepsilon, \mathbf{u}_\varepsilon)$ a weak solution of CNSE with ill prepared initial data. Then there is c independent of ε such that

$$\mathcal{E}_\varepsilon(\varrho_\varepsilon, \mathbf{u}_\varepsilon | \overline{\varrho}, \mathbf{V})(\tau) \le c\Big(\mathcal{E}_\varepsilon(\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon} | \overline{\varrho}, \mathbf{V}_0) + \varepsilon\Big)$$

$$\|(\Pi,\mathbf{V})\|_{X_T(\Omega)} \equiv \|\Pi\|_{C([0,T];C(\overline{\Omega}))} + \|\partial_t \Pi\|_{L^1(0,T;L^p(\Omega))} + \|\mathbf{V}\|_{C^1([0,T]\times\overline{\Omega};R^3)} + \|\mathbf{V}\|_{C([0,T];C^2(\overline{\Omega};R^3))} + \|\partial_t \nabla_x \mathbf{V}\|_{L^2(0,T;L^{6/5}(\Omega;R^{3\times 3}))} + \|\partial^2_{t,t}\mathbf{V}\|_{L^2(0,T;L^{6/5}(\Omega))} \text{ and } r \geq \underline{r} > 0.$$

FV/CR scheme Feireisl, Medvidova, Necasova, She, N., 2016

Let $(\varrho_h^n, \mathbf{u}_h^n) = (\varrho, \mathbf{u})$ be a family of numerical solutions of the FV/CR numerical scheme (32–33) with initial data $(\varrho^0, \mathbf{u}^0)$ obeying

$$\mathcal{E}_\varepsilon(\varrho^0, \mathbf{u}^0 | \overline{\varrho}, \mathbf{U}_0) \leq E_0 < \infty, \quad M/2 \leq \int_\Omega \varrho_0 \leq 2M, \quad M = \overline{\varrho}|\Omega|$$

and with $\gamma \geq 3/2$. Let $(\Pi, \mathbf{V})$ be a classical solution of the compressible Navier-Stokes equations (1–6) in the class $X_T(\Omega)$ corresponding to the initial velocity $\mathbf{U}_0$. Then there exists $c > 0$ independent of h, Δt, ε ϱ, $\mathbf{u}$, such that

$$\mathcal{E}_\varepsilon\left(\varrho^n, \mathbf{u}^n \middle| r, U\right) \leq c\left(\mathcal{E}_\varepsilon\left(\varrho_0, \mathbf{u}_0 \middle| \overline{\varrho}, \mathbf{U}_0\right) + h^\alpha + \sqrt{\Delta t} + \varepsilon\right),$$

where

$$\alpha = \min\left\{\frac{2\gamma - 3}{\gamma}, 1\right\}.$$

FV/FD - MAC scheme Maltese, N., 2016

The same result holds for the FV/FD scheme :

$$\mathcal{E}_\varepsilon(\varrho^n, \mathbf{u}^n \Big| r, U) \le c\Big(\mathcal{E}_\varepsilon(\varrho_0, \mathbf{u}_0 \Big| \overline{\varrho}, \mathbf{U}_0) + h^\alpha + \sqrt{\Delta t} + \varepsilon\Big).$$

Recall :

$$\mathcal{E}_\varepsilon(\varrho^n, \mathbf{u}^n \Big| r, \mathbf{V}) = \sum_{K\in\mathcal{T}} \int_K \Big(\frac{1}{2}\varrho_K^n(\mathbf{V}^n - \hat{\mathbf{u}}_K^n)^2 + \frac{1}{\varepsilon^2}E(\varrho_K^n|r^n)\Big)$$