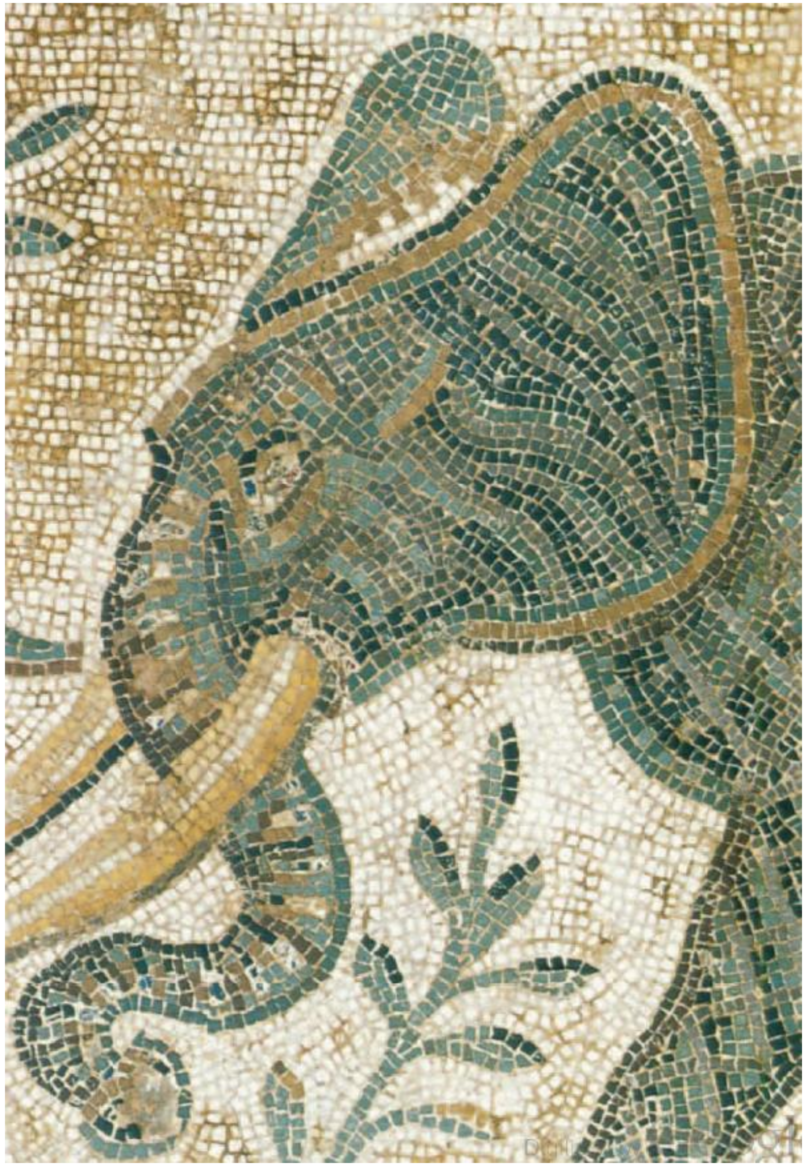
Piecing together the mosaic: Remarks on the mathematics of Zdeněk Strakoš

Jörg Liesen
Institute of Mathematics, TU Berlin
August 1, 2017

On mosaics

mosaic. A picture or pattern produced by arranging together small pieces of stone, tile, glass, etc.

(Oxford English Dictionary)



African elephant,
limestone, marble and glass,
Roman, 4th century,
Bardo Museum, Tunesia

From (A. B. Abed, ed., Stories in Stone, 2006)

June 1997 and June 2017

Put things into their proper context!

The Milovy Meeting in June 1997

9 skal

HOTEL DEVĚT SKAL MILOVY
☎ (0616) 66 42 82
66 42 84
592 03 Sněžné na Moravě

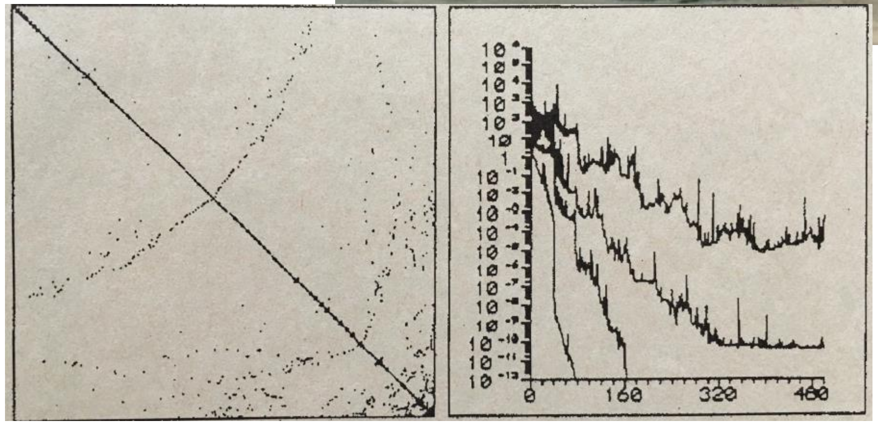
Jméno: Jörg Liesen

HOTELOVÝ PRŮKAZ

Pobyt od: do:
Číslo pokoje: A115

Číslo stolu:
Směna:

Přejeme příjemný pobyt na Vysočině
★★★



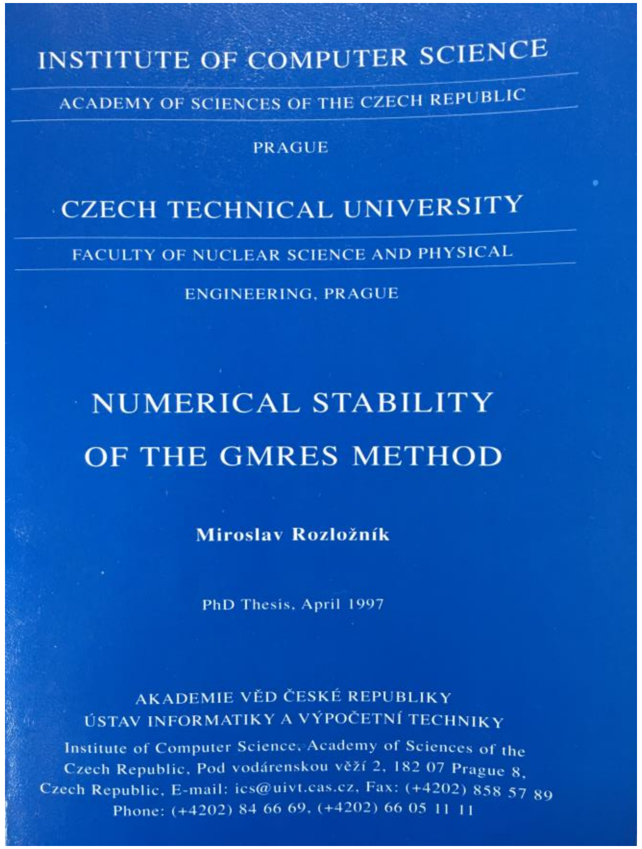
From: Daniel Szyld <impc97@uivt.cas.cz>
Date: Wed, 9 Jul 1997 16:20:04 -0400 (EDT)
Subject: Report on Czech-U.S. Workshop on Iterative Methods

Report on the
Czech-U.S. Workshop on Iterative Methods and Parallel Computing (IMPC'97),
June 16-21, 1997, Milovy, Czech Republic.

Submited by Daniel Szyld and Zdenek Strakos

Over 120 scientists from 20 countries met at the hotel Devet Skal in Milovy (literally ''Nine Rocks") near the Bohemian-Moravian Highlands (Central part of the Czech Republic). About half the participants were from the Czech Republic and the United States. The rest came from almost every (Eastern and Western) European country as well as Turkey, and Israel. All enjoyed the moderate weather, the wonderful atmosphere of camaraderie, and the lake view (and a few even dared a swim in it).

Miro's PhD thesis



- Miro's PhD thesis (completed in April 1997) served as a role model for my own thesis writing (completed in November 1998).
- Quoting Miro: "I also want to give special thanks to Zdeněk for setting high standards and then expecting them from me."

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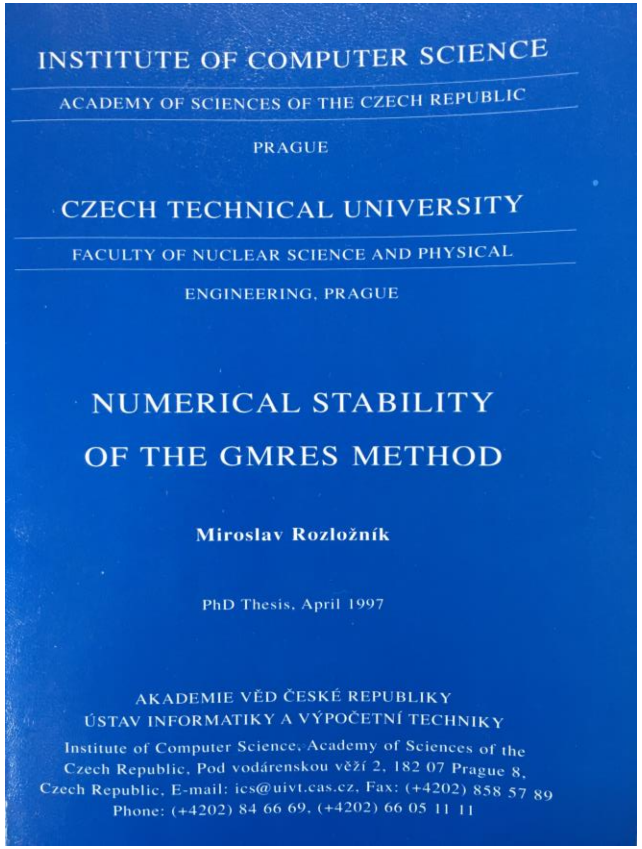
INSTITUTE OF COMPUTER SCIENCE
ACADEMY OF SCIENCES OF THE CZECH REPUBLIC
PRAGUE
CZECH TECHNICAL UNIVERSITY
FACULTY OF NUCLEAR SCIENCE AND PHYSICAL
ENGINEERING, PRAGUE

NUMERICAL STABILITY
OF THE GMRES METHOD

Miroslav Rozložník

PhD Thesis, April 1997

AKADEMIE VĚD ČESKÉ REPUBLIKY
ÚSTAV INFORMATIKY A VÝPOČETNÍ TECHNIKY
Institute of Computer Science, Academy of Sciences of the Czech Republic, Pod vodárenskou věží 2, 182 07 Prague 8, Czech Republic, E-mail: ics@uivt.cas.cz, Fax: (+4202) 858 57 89
Phone: (+4202) 84 66 69, (+4202) 66 05 11 11

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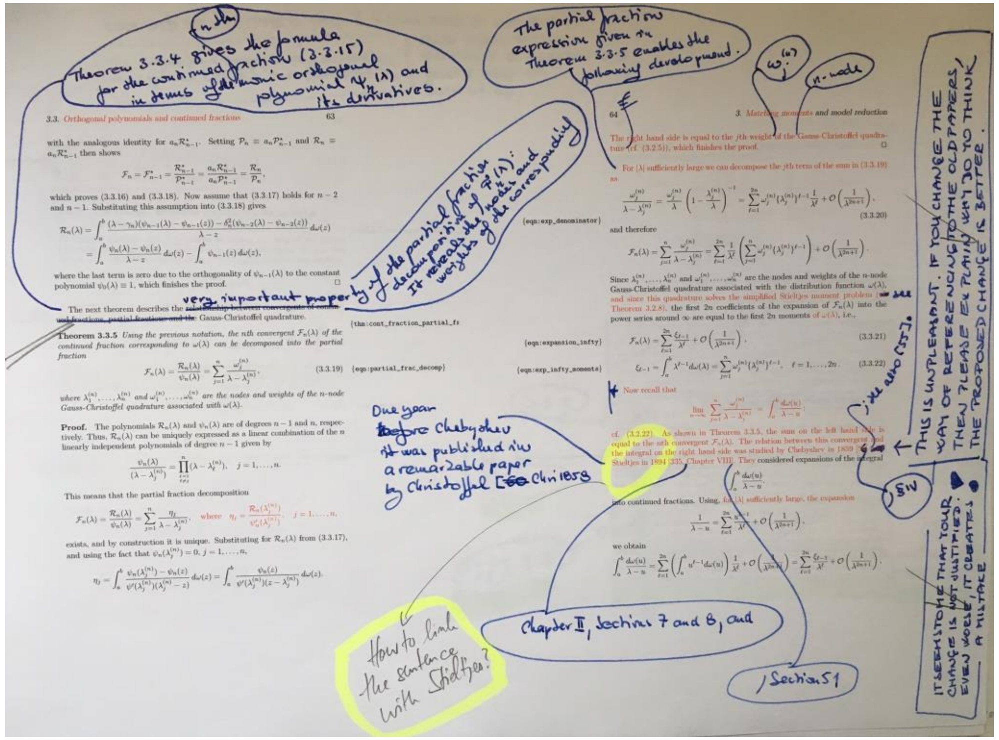
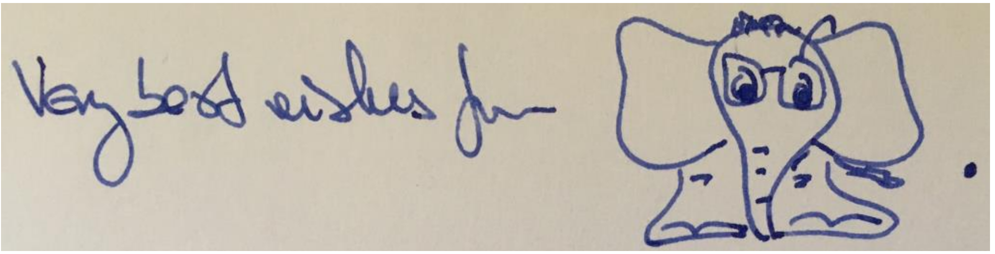
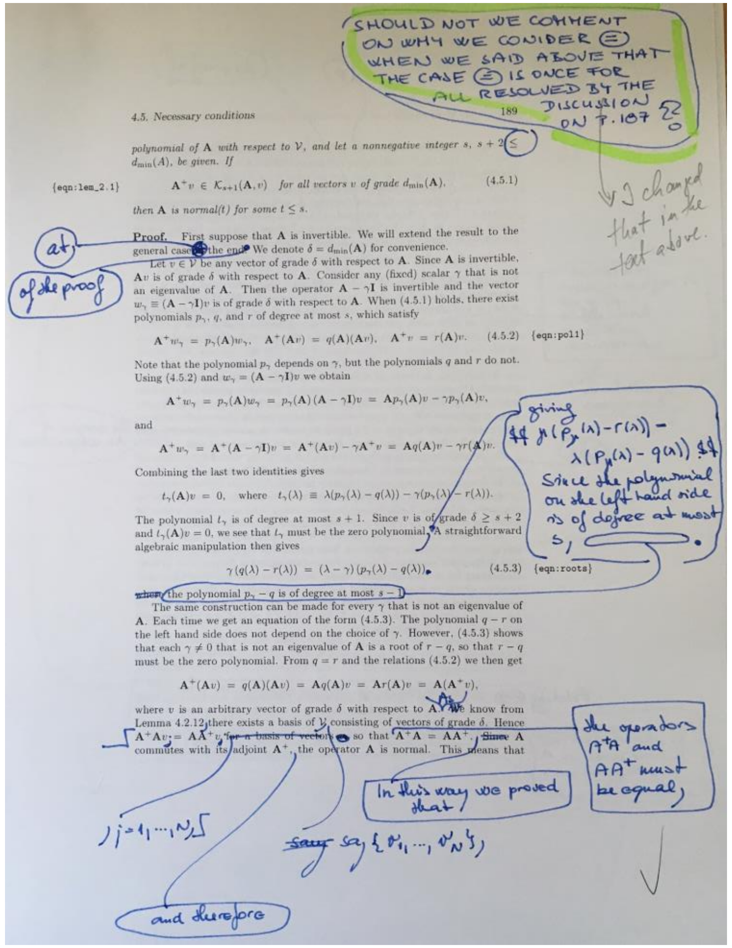
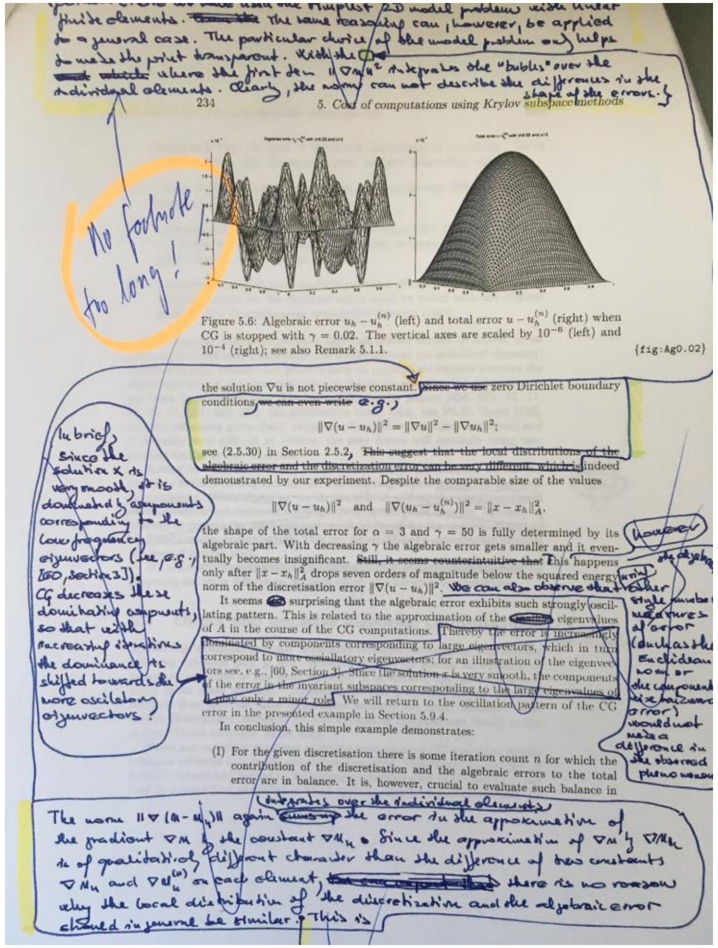
High standards and attention to detail

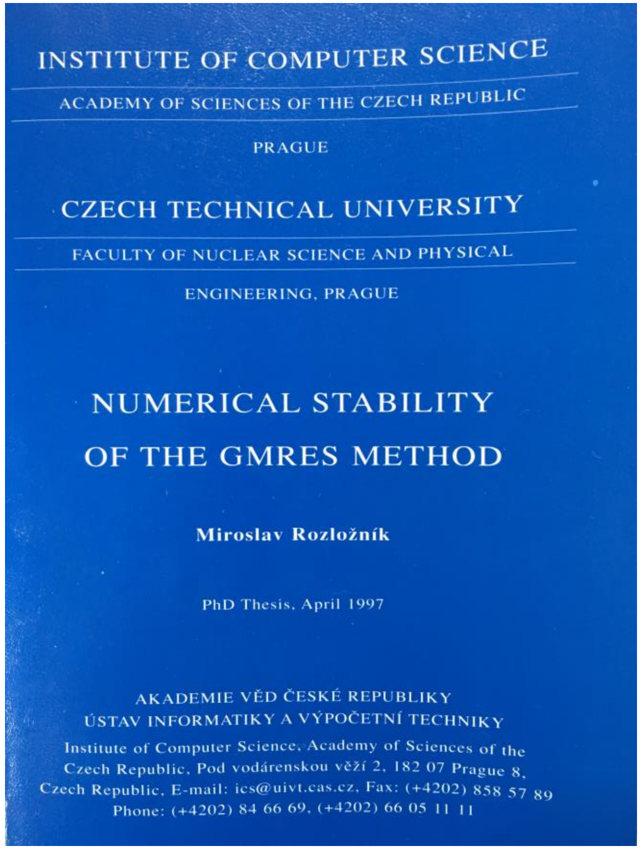






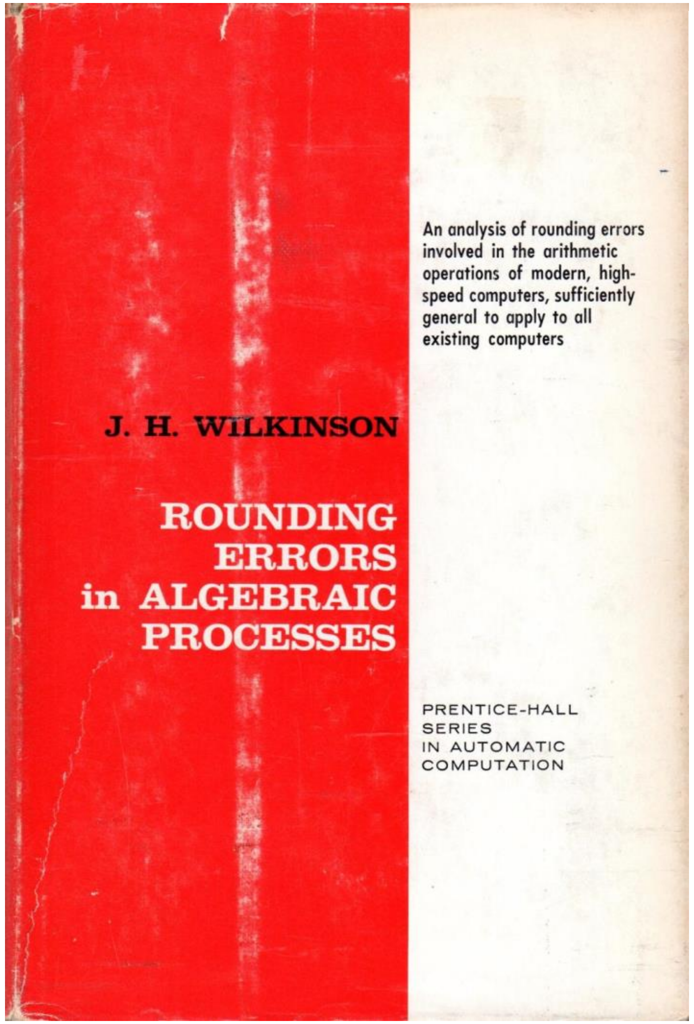


Miro’s PhD thesis



- One of the main results:
  Householder GMRES is backward stable.
- Important open question:
  Is modified Gram-Schmidt GMRES backward stable as well?

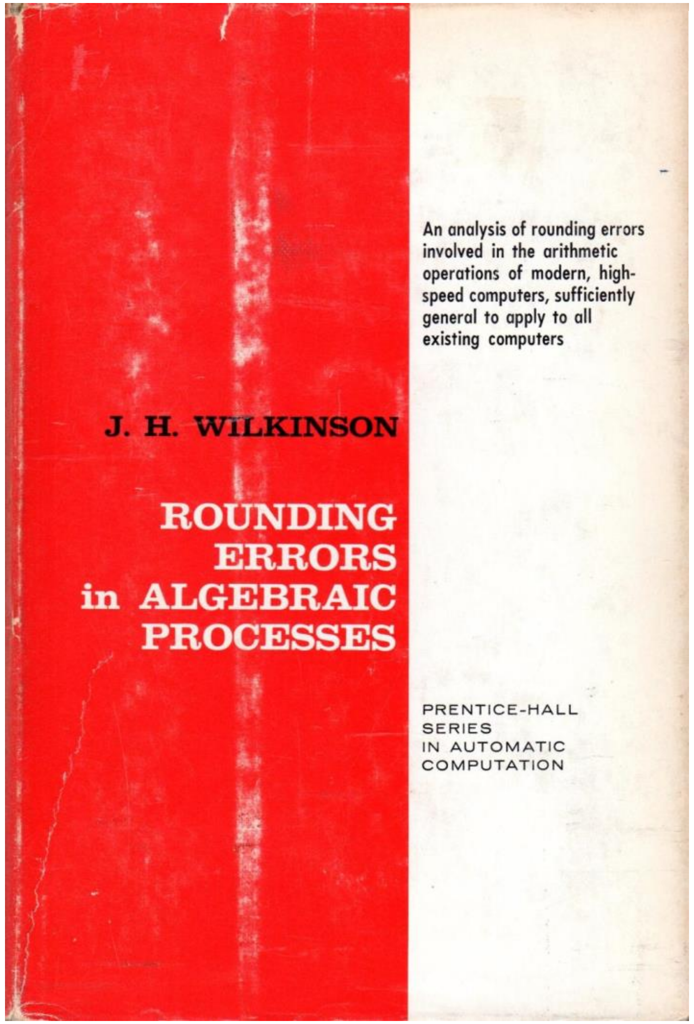
Origins of backward error analysis



(1963)

The idea of a *backward* error analysis, was to some extent implicit in the papers of von Neumann and Goldstine [18] and Turing [23]. It was described explicitly in Givens' paper [8] in the section on the calculation of the eigenvalues of a tri-diagonal matrix by the Sturm-sequence process. The error analysis in that paper seems not to have attracted as much attention as it deserved, possibly because it was not published in a readily available journal. Backward analysis has been used extensively by the author for the treatment of algebraic processes and has the advantage of suggesting automatically a convenient basis for comparison with the computed values.

Origins of backward error analysis



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(1964)

NUMERICAL COMPUTATION OF THE CHARACTERISTIC VALUES OF A REAL SYMMETRIC MATRIX

Wallace Givens

3.2 INVERSION OF THE ERROR PROBLEM

Considered in broad terms, a method of computation should be regarded as an operator which is applied to a selected one of a class of permissible M-component data vectors and which yields an ordered set of N numbers, that is, a solution vector.(1) This single-valued mapping of a region of the "data space" onto some region of the "solution space" is what we mean by a numerical method.

(1954)

A different section of Givens paper from 1954

TABLE 6

Order of the Given Matrix	Time to Reduce to Jacobi Form $\approx \frac{1}{3}(n-2)(n-1)(4n+83)10^{-3}$ seconds
10	3 seconds
20	19 seconds
30	55 seconds
40*	2 minutes
100*	26 minutes
1000*	15.7 days (of 24 hr)

*For matrices of order greater than about 36, the internal memory will not suffice, and the estimates will require sharp upward revision. The reliability of present machines would probably not permit a matrix of order 1000 to be reduced in any reasonable time.

- In 1954 it took 1.356.480 seconds to tridiagonalize a real symmetric matrix of order 1000.

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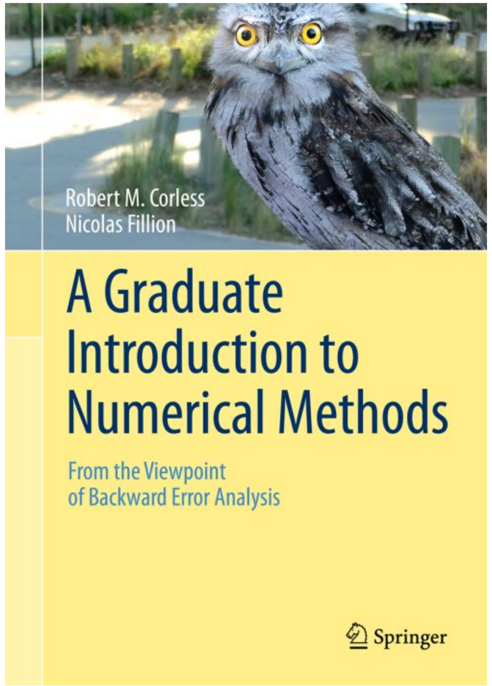
```
>> A=randn(1000);A=A+A';
>> tic, [Anew,Q]=tridiagonalize(A); toc,
Elapsed time is 28.552937 seconds.
>> norm(eye(1000)-Q*Q')
ans =
    1.2264e-14
>> norm(Anew-Q*A*Q')
ans =
    3.6638e-13
```

- In 1954 it took 1.356.480 seconds to tridiagonalize a real symmetric matrix of order 1000.
- Today it takes 28 seconds in MATLAB on my notebook with an ad hoc implementation based on Householder transformations.
- Computational devices (quickly) become outdated. Mathematics is timeless.

An analysis of rounding errors involved in the arithmetic operations of modern, high-speed computers, sufficiently general to apply to all existing computers

The ubiquitous nature of backward error analysis

- Backward error analysis was originally considered by Givens in the context of tridiagonalizing a real symmetric matrix.
- Through synthetization, abstraction and generalization it has become a mature theory which is applied throughout numerical mathematics.
- This is the strength of mathematics (cf. Mehrmann, Schilders & Strakoš, ICIAM Newsletter, 01/2017).



(2014)

"A good numerical method gives you
nearly the right solution
to nearly the right problem."

Backward error analysis for $Ax = b$

- Consider a linear algebraic system $Ax = b$ with $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$.
- If $\widehat{x}$ is an approximate solution, then $\|x - \widehat{x}\|$ is the (absolute) forward error.
- Backward error idea: Which linear algebraic system is solved exactly by the approximation $\widehat{x}$?
- If $(A + \Delta A)\widehat{x} = b + \Delta b$, then $\|\Delta A\|$ and $\|\Delta b\|$ are called the (absolute) backward errors in A and b.

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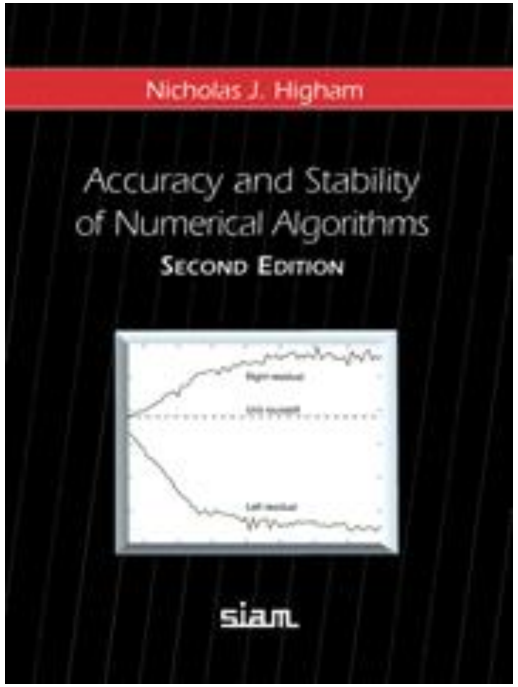

(2002)

"The data frequently contains uncertainties due to measurements, previous computations, or errors comitted in storing machine numbers on the computer.
If the backward error is no larger than these uncertainties, then the computed solution may hardly be criticised – it may be the solution we are seeking – for all we know."

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Theorem (Rigal & Gaches, 1967)
Let $r = b - A\widehat{x}$ be the residual, then the normwise relative backward error of $\widehat{x}$ is given by

$$\beta(\widehat{x}) = \min\{\varepsilon : (A + \Delta A)\widehat{x} = b + \Delta b \text{ with } \|\Delta A\|/\|A\| \leq \varepsilon \text{ and } \|\Delta b\|/\|b\| \leq \varepsilon\} = \frac{\|r\|}{\|b\| + \|A\|\|\widehat{x}\|},$$

where the second equality is attained by explicitly known the perturbations.

- A numerical method is normwise backward stable when its computed approximation $\widehat{x}$ satisfies

$$\beta(\widehat{x}) \approx \mathbf{u} \quad (= \text{machine precision}).$$

The GMRES method

- Consider $Ax = b$ with $A \in \mathbb{R}^{n \times n}$ nonsingular and $b \in \mathbb{R}^n$. Let $x_0 \in \mathbb{R}^n$ and $r_0 = b - Ax_0$.
- The GMRES method (Saad & Schultz, 1986) for $Ax = b$ generates a sequence x_k, $k = 1, 2, \ldots,$ with

$$x_k \in x_0 + \mathcal{K}_k(A, r_0) \quad \text{and} \quad \|r_k\| = \|b - Ax_k\| = \min_{z \in x_0 + \mathcal{K}_k(A, r_0)} \|b - Az\|,$$

  where $\mathcal{K}_k(A, r_0) = \text{span}\{r_0, Ar_0, \ldots A^{k-1} r_0\}$ is the kth Krylov subspace generated by A and r_0.
- GMRES is one of the most important iterative methods for general linear algebraic systems.

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- GMRES is one of the most important iterative methods for general linear algebraic systems.

- The paper of Saad & Schultz has 10.500 citations on Google Scholar (as of July 2017).
- Note: The 1992 paper of van der Vorst on Bi-CGStab, which in 2000 was named “the most-cited mathematics paper of the last decade” by the Institute of Scientific Information (ISI), currently has 5.300 citations on Google Scholar.

Backward stability of Householder GMRES

- GMRES uses Arnoldi's method for computing orthogonal bases of $\mathcal{K}_k(A, r_0)$.
- The Arnoldi decomposition in step k is given by $AV_k = V_{k+1}H_{k+1,k}$.
- Ideally, $V_k^T V_k = I_k$, but in finite precision computations we loose orthogonality.
- In the Householder implementation of Arnoldi we have $\|I_k - V_k^T V_k\| \approx \mathbf{u}$.
- One can show that the (final) computed x_n satisfies

$$\beta(x_n) = \frac{\|b - Ax_n\|}{\|b\| + \|A\|\|x_n\|} \approx \mathbf{u},$$

  i.e., Householder GMRES is normwise backward stable (see Miro's PhD thesis).
- In practice the cheaper modified Gram-Schmidt (MGS) implementation is used.

GMRES and linear least squares

- The GMRES minimization problem is a linear least squares problem:

$$\|r_k\| = \min_{z \in x_0 + \mathcal{K}_k(A, r_0)} \|b - Az\| = \min_{y \in \mathbb{R}^k} \|r_0 - By\| \quad \Longleftrightarrow \quad By \approx r_0,$$

where the columns of $B \in \mathbb{R}^{n \times k}$ form any basis of $A\mathcal{K}_k(A, r_0)$.

Theorem
If $\gamma > 0$ is a scaling parameter, then

SIAM J. Sci. Comput.
Vol. 23, No. 5, pp. 1503–1525

LEAST SQUARES RESIDUALS AND MINIMAL RESIDUAL METHODS*

J. LIESEN†, M. ROZLOŽNÍK‡, AND Z. STRAKOŠ§

$$\|r_k\| = \frac{\sigma_{\min}([r_0\gamma, B])}{\gamma} \prod_{j=1}^{k} \frac{\sigma_j([r_0\gamma, B])}{\sigma_j(B)}.$$

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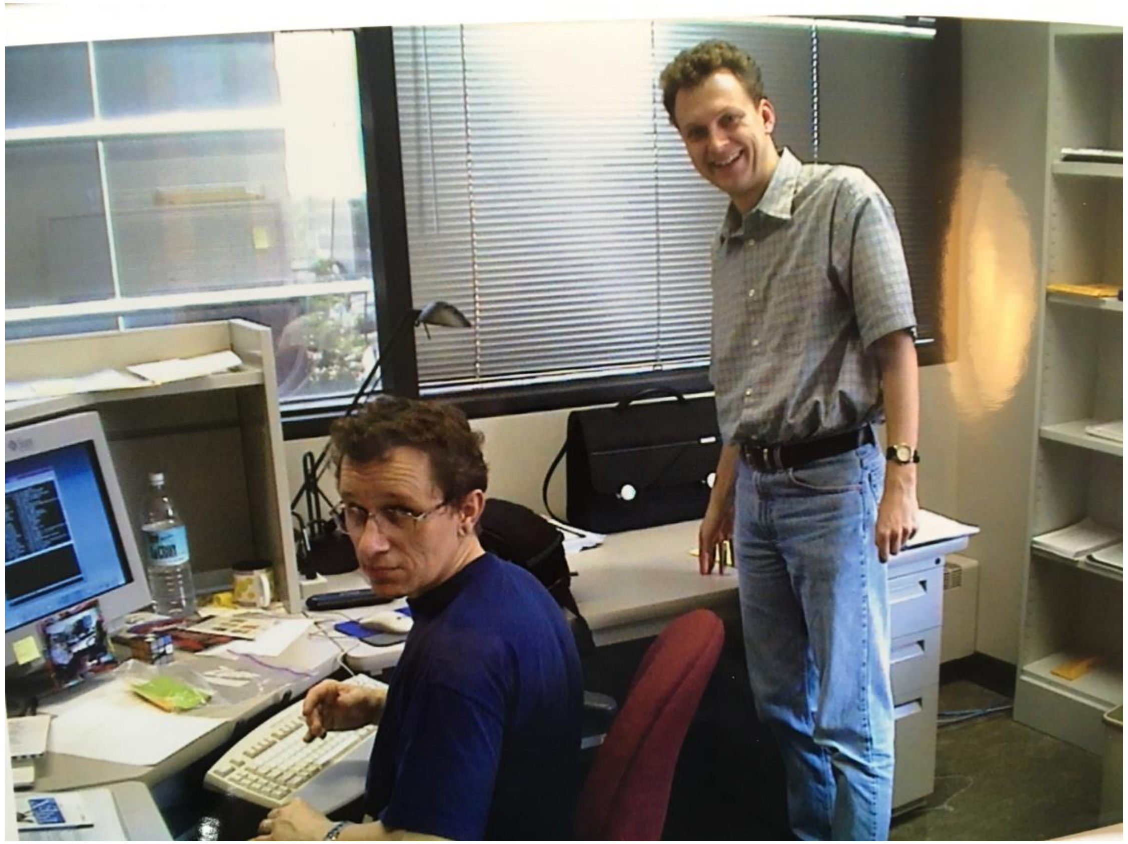
- With $\gamma = \|r_0\|^{-1}$ and $B = QR$ we obtain

$$\sigma_{\min}([r_0/\|r_0\|, Q]) \leq \frac{\|r_k\|}{\|r_0\|} \leq \sqrt{2}\sigma_{\min}([r_0/\|r_0\|, Q]),$$

which is useful for analyzing certain implementations, e.g., Simpler GMRES (Walker & Zhou, 1994).

Written in Bielefeld, Urbana, Atlanta, Zürich & Prague

(Emory University, 1999)



(University of Illinois, 2001)

Least squares and total least squares

- The least squares (LS) distance for the problem $By \approx r_0$ is given by

$$\min_{r,y} \|r\| \quad \text{subject to} \quad By = r_0 - r.$$

- This is a backward error like interpretation:
  The LS residual r is a minimal correction to the right hand side r_0 in order to make the corrected system compatible.

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- If we allow corrections to both B and r_0, we get the total least squares problem.
- For each parameter $\gamma > 0$, the scaled total least squares (STLS) distance is given by

$$\min_{s,E,z} \|[s, E]\|_F \quad \text{subject to} \quad (B + E)z\gamma = r_0\gamma - s.$$

- The STLS distance is equal to $\sigma_{\min}([r_0\gamma, B])$, which is an important quantity in the GMRES context.

Laying the foundations

- In order to understand this situation, Zdeněk and Chris Paige completely reworked the foundations of LS and STLS problems:

Theorem
For $\gamma \to 0$ the STLS solution becomes the LS solution, and

$$\lim_{\gamma \to 0} \frac{\sigma_{\min}([r_0\gamma, B])}{\gamma} = \|r\| \quad \text{(LS distance)}.$$


Numer. Math. (2002) 91: 117–146
Digital Object Identifier (DOI) 10.1007/s002110100314
Numerische Mathematik

Scaled total least squares fundamentals

Christopher C. Paige[1,*], Zdeněk Strakoš[2,**]

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Numerische Mathematik

Scaled total least squares fundamentals

Christopher C. Paige[1,*], Zdeněk Strakoš[2,**]

Theorem
With $\theta(\gamma) = \sigma_{\min}([r_0\gamma, B])/\sigma_{\max}(B)$ and $\delta(\gamma) = \sigma_{\min}([r_0\gamma, B])/\sigma_{\min}(B)$,

$$\left(1 + \frac{\gamma^2\|y\|^2}{1-\theta(\gamma)^2}\right)^{1/2} \le \frac{\|r\|\gamma}{\sigma_{\min}([r_0\gamma, B])} \le \left(1 + \frac{\gamma^2\|y\|^2}{1-\delta(\gamma)^2}\right)^{1/2}.$$

Numer. Math. (2002) 91: 93–115
Digital Object Identifier (DOI) 10.1007/s002110100317
Numerische Mathematik

Bounds for the least squares distance using scaled total least squares

Christopher C. Paige[1,*], Zdeněk Strakoš[2,**]

Backward error and loss of orthogonality in MGS-GMRES

- A classical result of Björk (1967) implies that in the finite precision MGS-Arnoldi algorithm we have

$$\|I - V_k^T V_k\|_F \approx \kappa([r_0\gamma, AV_{k-1}])\, \mathbf{u}.$$

Theorem
With certain (optimal) scalings $\gamma > 0$ and diagonal $D_{k-1} > 0$, and when $\delta_k \ll 1$,

$$\frac{1}{\sqrt{2}} \leq \kappa([r_0\gamma, AV_{k-1}D_{k-1}])\, \frac{\|b - Ax_k\|}{\|b\| + \|A\|\|x_k\|} \leq 2.$$

SIAM J. Sci. Comput.
Vol. 23, No. 6, pp. 1898–1923

RESIDUAL AND BACKWARD ERROR BOUNDS IN MINIMUM RESIDUAL KRYLOV SUBSPACE METHODS*

CHRISTOPHER C. PAIGE† AND ZDENĚK STRAKOŠ‡

- Thus, the product of the loss of orthogonality and the normwise relative backward error satisfies

$$\|I - V_k^T V_k\|_F\, \frac{\|b - Ax_k\|}{\|b\| + \|A\|\|x_k\|} \approx \mathcal{O}(1)\, \mathbf{u}.$$

All this holds in finite precision MGS-GMRES

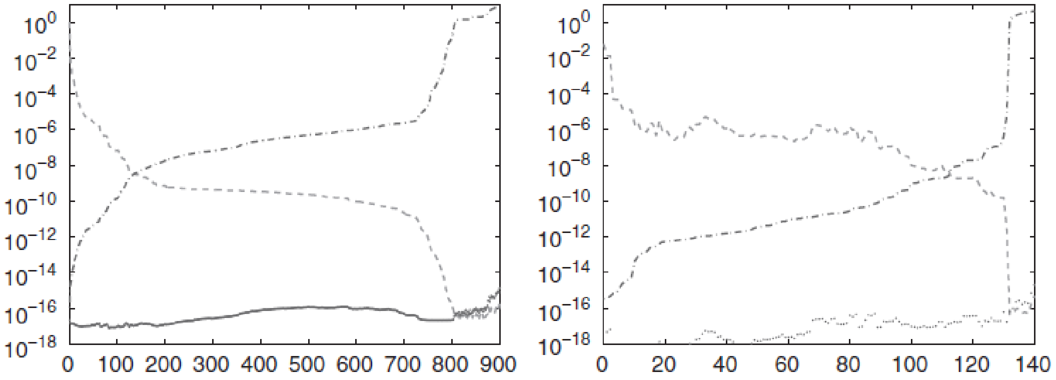


Figure 5.30 Results of MGS GMRES computations with the matrices Sherman2 (left) and West132 (right) from Matrix Market. Throughout the computation the product (dots) of the normwise relative backward error (dashed line) and the loss of orthogonality among the MGS Arnoldi vectors (dashed-dotted line) are close to (or below) machine precision.

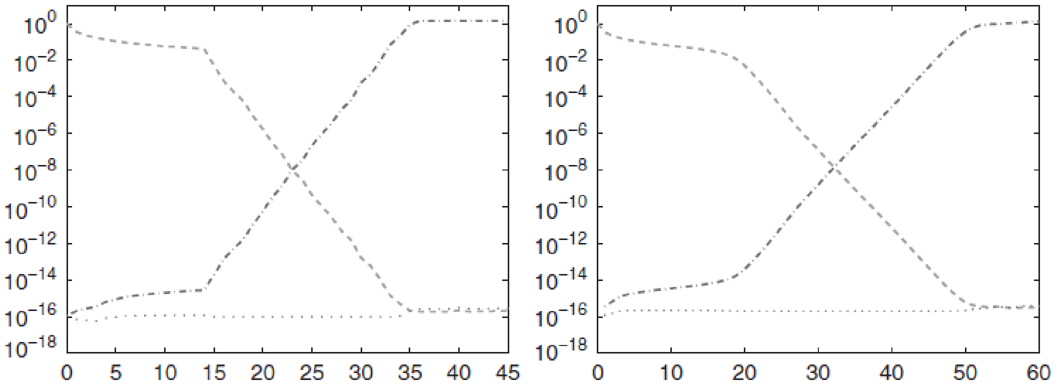
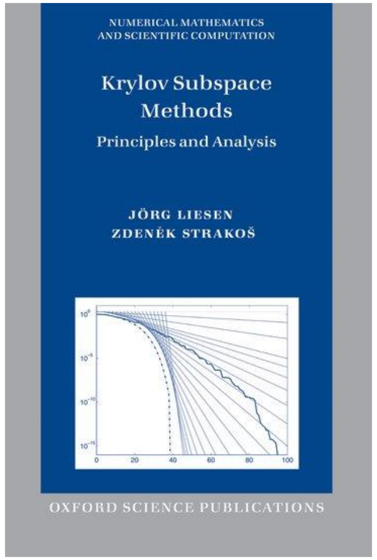


Figure 5.31 The same as in Figure 5.30 for matrices from the convection–diffusion model problem with dominating convection (see Section 5.7.5), the discontinuous inflow boundary conditions, the vertical wind (left) and the curl wind (right).

Pictures from (L. & Strakoš, Krylov Subspace Methods, Oxford University Press, 2013)



Completing the MGS-GMRES mosaic

SIAM J. MATRIX ANAL. APPL.
Vol. 28, No. 1, pp. 264–284

MODIFIED GRAM–SCHMIDT (MGS), LEAST SQUARES, AND BACKWARD STABILITY OF MGS-GMRES*

CHRISTOPHER C. PAIGE†, MIROSLAV ROZLOŽNÍK‡, AND ZDENĚK STRAKOŠ‡

8.2. Backward stability of MGS-GMRES for $Ax = b$ in (1.1). Even though MGS-GMRES always computes a backward stable solution $\bar{y}_k$ for the least squares problem (7.3), see section 8.1, we still have to prove that $\bar{V}_k\bar{y}_k$ will be a backward stable solution for the original system (1.1) for some k (we take this k to be $\hat{m}-1$ in (6.1)), and this is exceptionally difficult. Usually we want to show we have a backward stable solution when we *know* we have a small residual. The analysis here is different in that we will first prove that $\bar{B}_{\hat{m}}$ is numerically rank deficient, see (8.4), but to prove backward stability, we will then have to *prove* that our residual will be small, amongst other things, and this is far from obvious. Fortunately two little known researchers have studied this arcane area, and we will take ideas from [17]; see Theorem 2.4. To simplify the development and expressions we will absorb all small constants into the $\tilde{\gamma}_{kn}$ terms below.

Completing the MGS-GMRES mosaic

Using the usual approach of combining (8.15) with the definitions

$$\Delta b_k' \equiv -\frac{\|b\|_2}{\|b\|_2 + \|A\|_F \|\bar{x}_k\|_2} \tilde{r}_k(\bar{y}_k), \quad \Delta A_k' \equiv \frac{\|A\|_F \|\bar{x}_k\|_2}{\|b\|_2 + \|A\|_F \|\bar{x}_k\|_2} \frac{\tilde{r}_k(\bar{y}_k)\bar{x}_k^T}{\|\bar{x}_k\|_2^2},$$

shows $\quad (A + \Delta A_k + \Delta A_k')\bar{x}_k = b + \Delta b_k(\bar{y}_k) + \Delta b_k',$

$$\|\Delta A_k + \Delta A_k'\|_F \leq \tilde{\gamma}_{kn} \|A\|_F, \quad \|\Delta b_k(\bar{y}_k) + \Delta b_k'\|_2 \leq \tilde{\gamma}_{kn} \|b\|_2,$$

proving that the MGS-GMRES solution $\bar{x}_k$ is backward stable for (1.1).

Completing the MGS-GMRES mosaic

Using the usual approach of combining (8.15) with the definitions

$$\Delta b'_k \equiv -\frac{\|b\|_2}{\|b\|_2 + \|A\|_F\|\bar{x}_k\|_2}\tilde{r}_k(\bar{y}_k), \quad \Delta A'_k \equiv \frac{\|A\|_F\|\bar{x}_k\|_2}{\|b\|_2 + \|A\|_F\|\bar{x}_k\|_2}\frac{\tilde{r}_k(\bar{y}_k)\bar{x}_k^T}{\|\bar{x}_k\|_2^2},$$

shows $(A + \Delta A_k + \Delta A'_k)\bar{x}_k = b + \Delta b_k(\bar{y}_k) + \Delta b'_k,$

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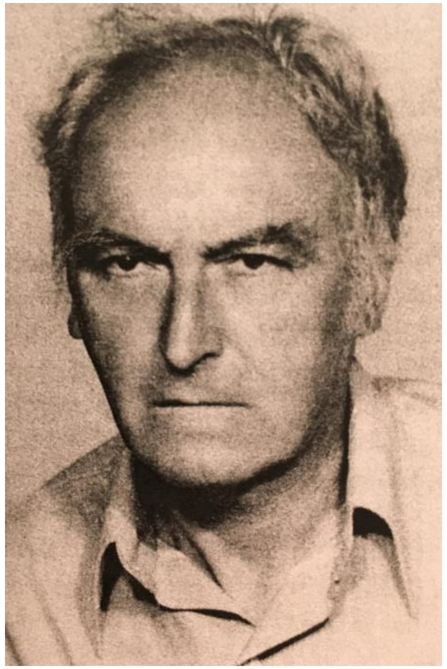
A higher level of understanding –
when Truth and Beauty become one.



Vlastimil Pták
(1925–1999)

Completing the MGS-GMRES mosaic

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A higher level of understanding –
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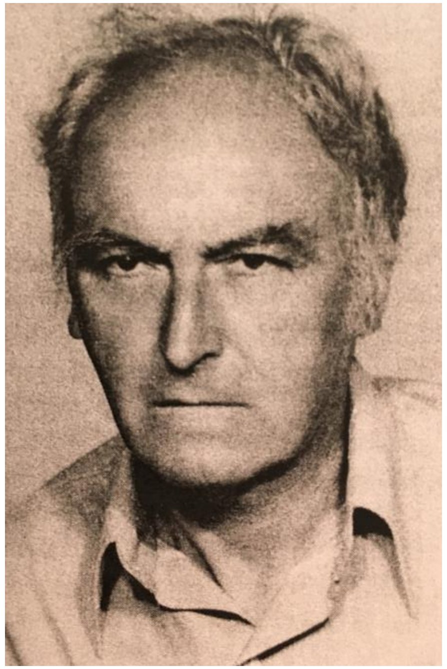
V. Pták, *Circumstances of the submission of my paper in 1956*, LAA 310 (2000):
"It is a comforting thought that the validity of mathematical theorems cannot be affected by ideological disputes ...
Even on the shelves of mathematical libraries the 1956 volume of Acta Szeged is conspicuous by the poorer quality of the paper. The presence of a foreign army on the territory of Hungary made it difficult to keep the usual standard ..."



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The convergence behavior of GMRES

BIT
1998, Vol. 38, No. 4, pp. 636–643

0006-3835/98/3804-0636 $12.00

KRYLOV SEQUENCES OF MAXIMAL LENGTH AND CONVERGENCE OF GMRES *

M. ARIOLI[1] , V. PTÁK[2] and Z. STRAKOŠ[2] †

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KRYLOV SEQUENCES OF MAXIMAL LENGTH AND CONVERGENCE OF GMRES *

M. ARIOLI[1], V. PTÁK[2] and Z. STRAKOŠ[2] †

- Complete characterization of all matrices A with prescribed eigenvalues and right hand sides b, such that GMRES attains a prescribed convergence curve.
- In particular, any nonincreasing convergence curve is possible for GMRES for a matrix having any eigenvalues.

Theorem 5.7.8

Consider N given positive numbers $f_0 \geq f_1 \geq \cdots \geq f_{N-1} > 0$ and N nonzero complex numbers $\lambda_1, \ldots, \lambda_N$, not necessarily distinct. Let $A \in \mathbb{C}^{N\times N}$ and $b \in \mathbb{C}^N$. Then the following three assertions are equivalent:

(1) *The eigenvalues of A are $\lambda_1, \ldots, \lambda_N$ and GMRES applied to $Ax = b$ with $x_0 = 0$ yields the residual norms $\|r_n\| = f_n$ for $n = 0, 1, \ldots, N-1$.*

(2) *$A = W_N R_N C R_N^{-1} W_N^*$ and $b = W_N h$, where*
- *W_N is a unitary matrix,*
- *C is the companion matrix of the polynomial*

$$q(\lambda) \equiv (\lambda - \lambda_1)\cdots(\lambda - \lambda_N) \equiv \lambda^N - \sum_{j=0}^{N-1} \alpha_j \lambda^j$$

(this matrix is stated in (5.7.24) above),
- *$h = [g_1, \ldots, g_N]^T$, where $g_n \equiv (f_{n-1}^2 - f_n^2)^{1/2}$, $n = 1, \ldots, N$, and we set $f_N \equiv 0$,*
- *R_N is nonsingular and upper triangular such that $R_N s = h$, where*

$$s = [\xi_1, \ldots, \xi_N]^T \quad \text{and} \quad p(\lambda) \equiv \left(1 - \frac{\lambda}{\lambda_1}\right)\cdots\left(1 - \frac{\lambda}{\lambda_N}\right) \equiv 1 - \sum_{j=1}^{N} \xi_j \lambda^j,$$

i.e. $\quad \xi_n = -\alpha_n/\alpha_0, \quad n = 1, \ldots, N-1, \quad$ *and* $\quad \xi_N = 1/\alpha_0.$

(3) *$A = W_N Y C^{-1} W_N^*$ and $b = W_N h$, where W_N, C and h are defined as in (2),*

$$Y \equiv R_N C^{-1} = \left[\begin{array}{c|c} g_1 & \\ \vdots & R \\ g_{N-1} & \\ \hline g_N & 0 \end{array}\right],$$

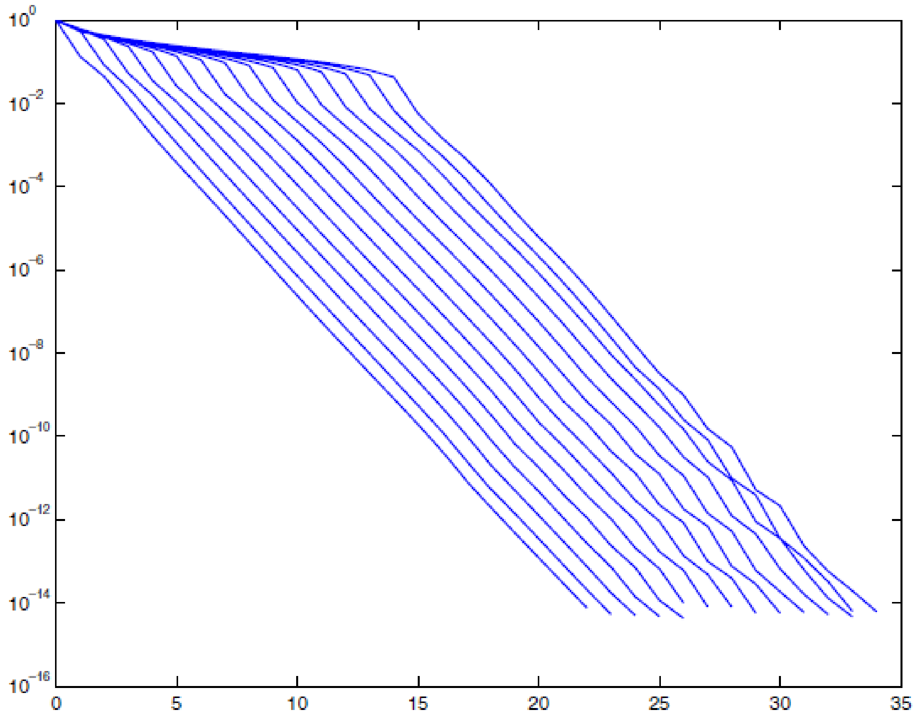
where $g_1, \ldots, g_N$ are defined as in (2), and R is an $(N-1)\times(N-1)$ nonsingular upper triangular matrix.

The convergence behavior of GMRES

SIAM J. Sci. Comput.
Vol. 26, No. 6, pp. 1989–2009

GMRES CONVERGENCE ANALYSIS FOR A CONVECTION-DIFFUSION MODEL PROBLEM*

J. LIESEN† AND Z. STRAKOŠ‡



- The length initial phase of stagnation depends on the boundary conditions in the model problem.
- The convergence of GMRES in particular for nonnormal matrices may depend strongly on the right hand side.

The convergence behavior of GMRES

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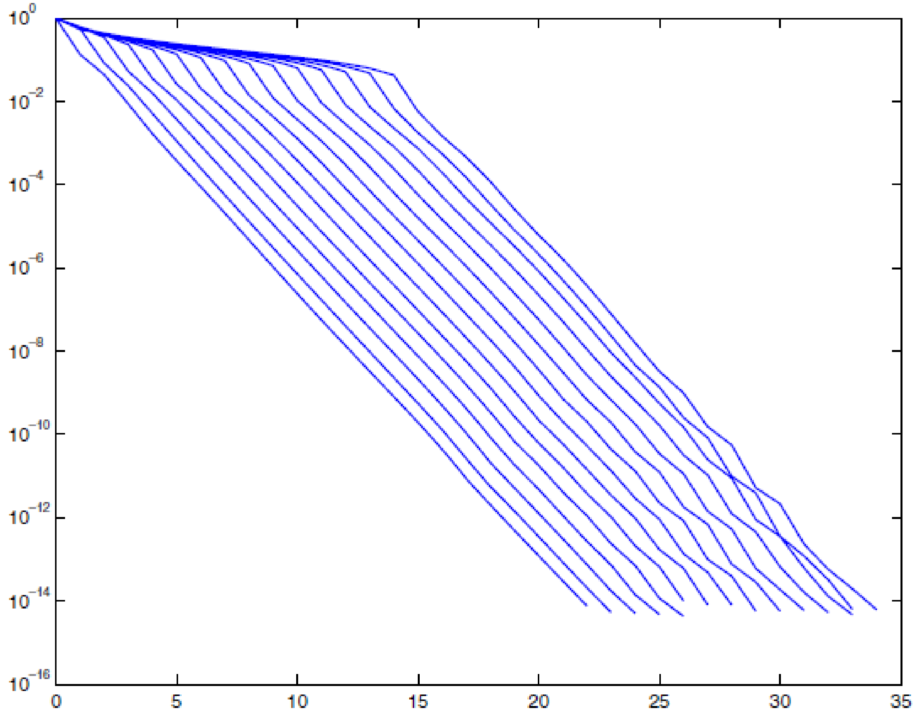

Some pieces have been placed in the GMRES convergence mosaic, but the overall picture is still unclear.

Piecing together the mosaic

"Goal: To get some *understanding* when and why things work, and when and why they do not."

GMRES convergence for
convection-diffusion problems

Backward stability
of GMRES

All nonincreasing convergence
curves are possible for GMRES

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The Strakoš matrix & finite precision CG

Error estimation in the CG method

Gauss quadrature & the CG method

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Acta Numerica (2006), pp. 471–542
doi: 10.1017/S096249290626001X

The Lanczos and conjugate gradient algorithms in finite precision arithmetic

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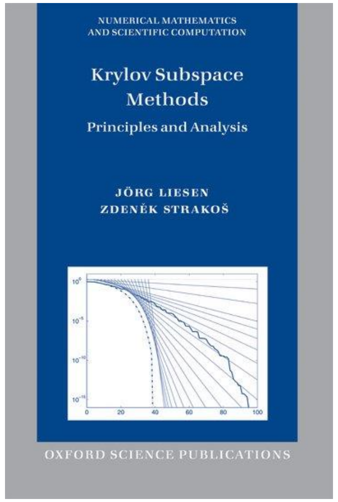
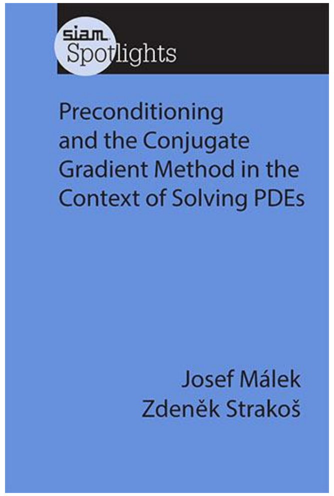
Dedicated to Chris Paige for his fundamental contributions to the rounding error analysis of the Lanczos algorithm

The Lanczos and conjugate gradient algorithms were introduced more than five decades ago as tools for numerical computation of dominant eigenvalues of symmetric matrices and for solving linear algebraic systems with symmetric positive definite matrices, respectively. Because of their fundamental relationship with the theory of orthogonal polynomials and Gauss quadrature of the Riemann–Stieltjes integral, the Lanczos and conjugate gradient algorithms represent very interesting general mathematical objects, with highly nonlinear properties which can be conveniently translated from algebraic language into the language of mathematical analysis, and *vice versa*. The algorithms are also very interesting numerically, since their numerical behaviour can be explained by an elegant mathematical theory, and the interplay between analysis and algebra is useful there too.

Motivated by this view, the present contribution wishes to pay a tribute to those who have made an understanding of the Lanczos and conjugate gradient algorithms possible through their pioneering work, and to review recent solutions of several open problems that have also contributed to knowledge of the subject.





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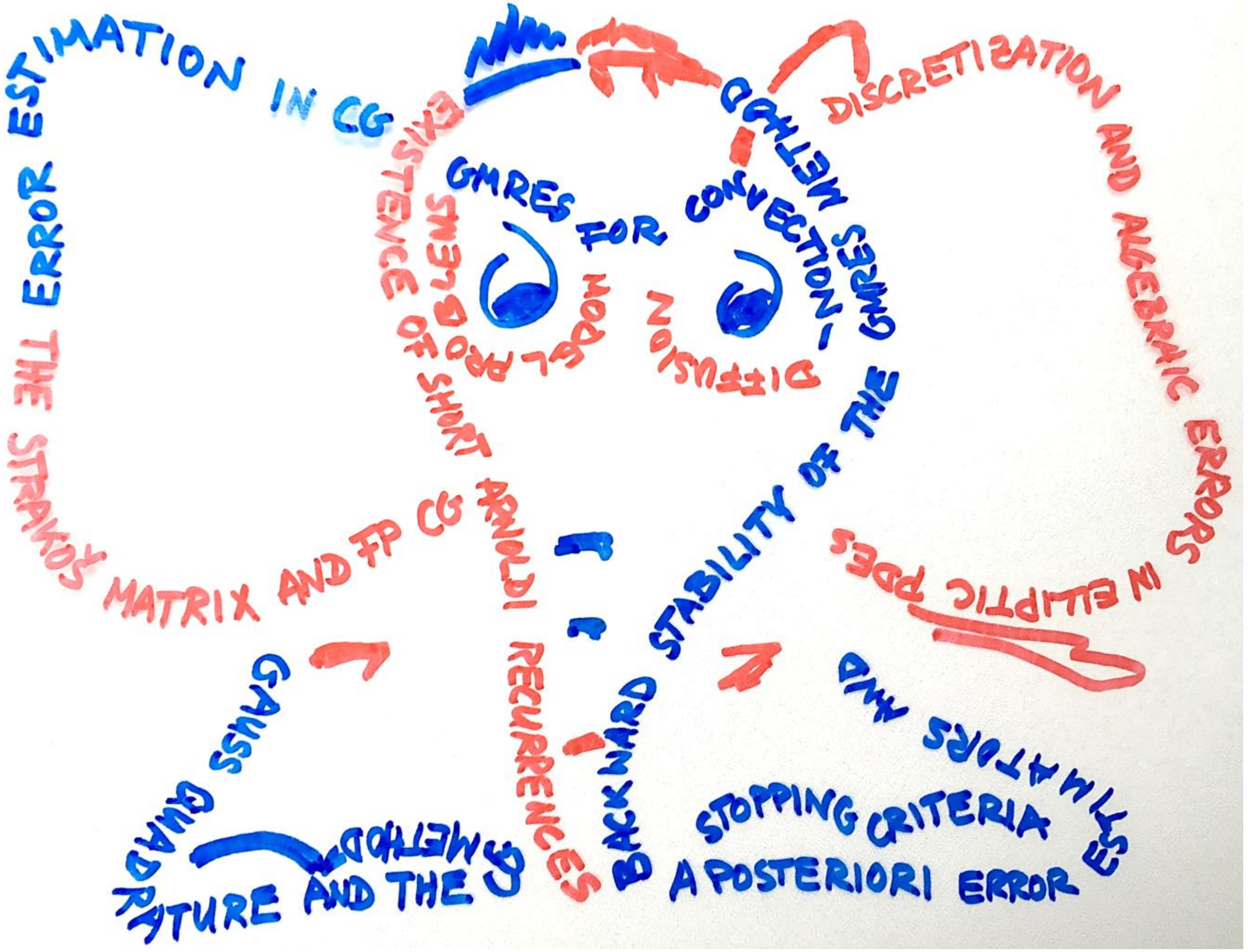

Srdečné blahoprání k narozeninám!