

Non-standard FE analysis: local modifications

Barbara Wohlmuth and Uli Rüde^{*}

T. Horger, P. Pustejovska, P. Swierczynski

TU München, ^{*} FAU

wohlmuth@ma.tum.de

<http://www-m2.ma.tum.de>

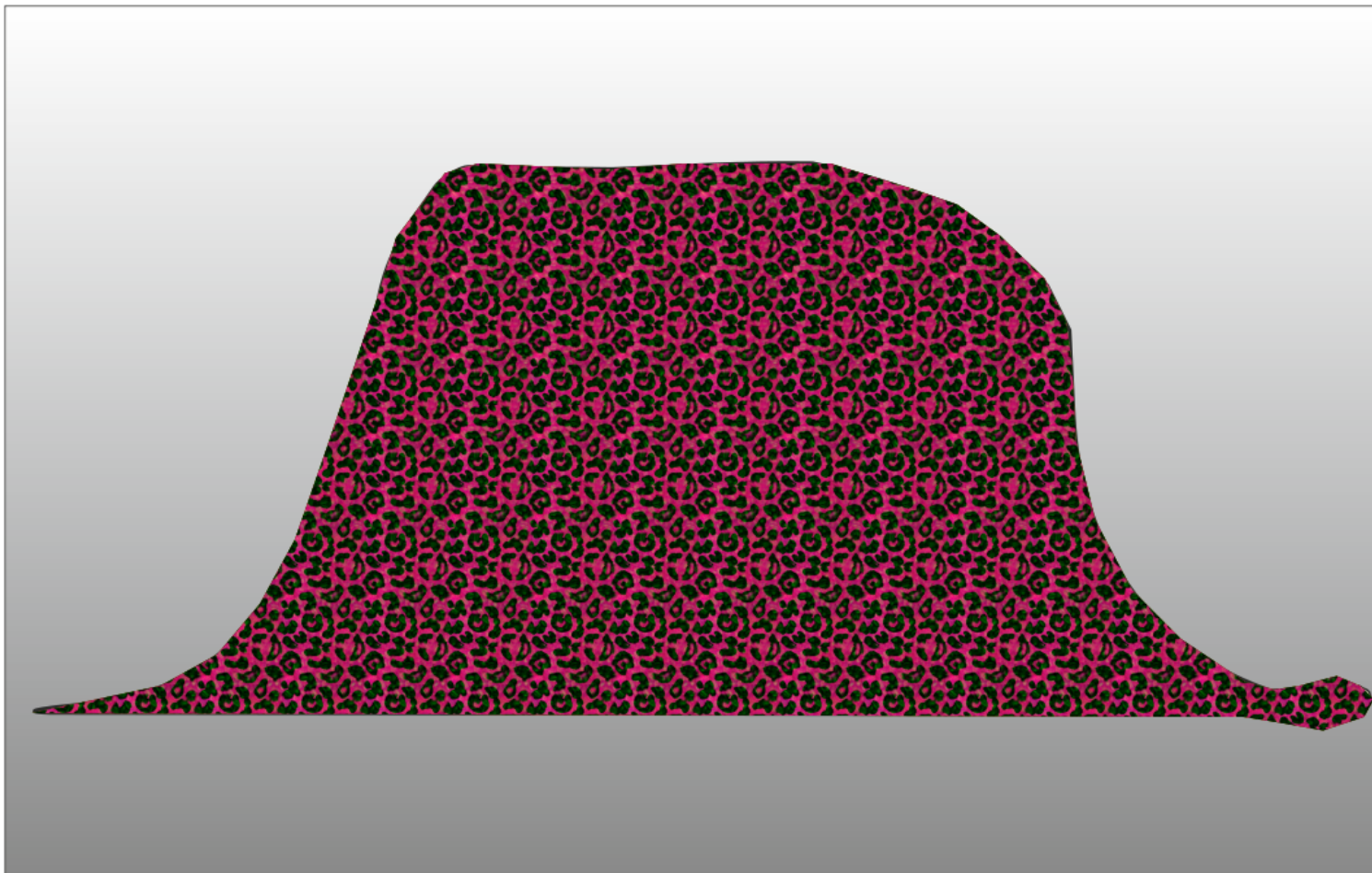
August 2, 2017

ERC-CZ MORE project: Implicitly constituted materials: Modeling, Analysis and Computing, Roztoky

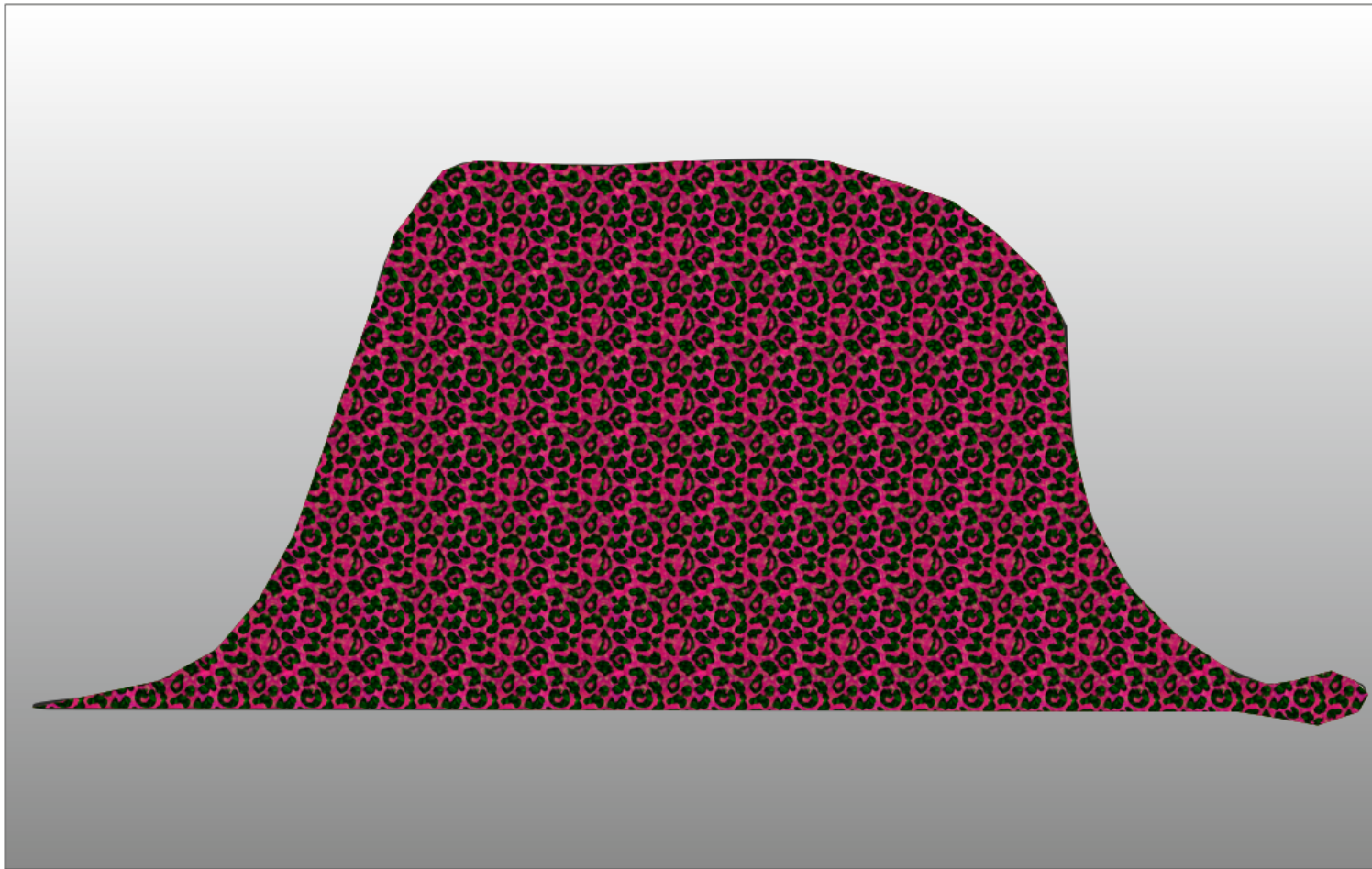
Overview

- Elephants in Math
- Pollution effect on domains with re-entrant corners
- **Energy corrected FE**
 - Construction
 - Theory
 - Algorithm
 - Higher order
 - Beyond the Laplacian

What do you see?

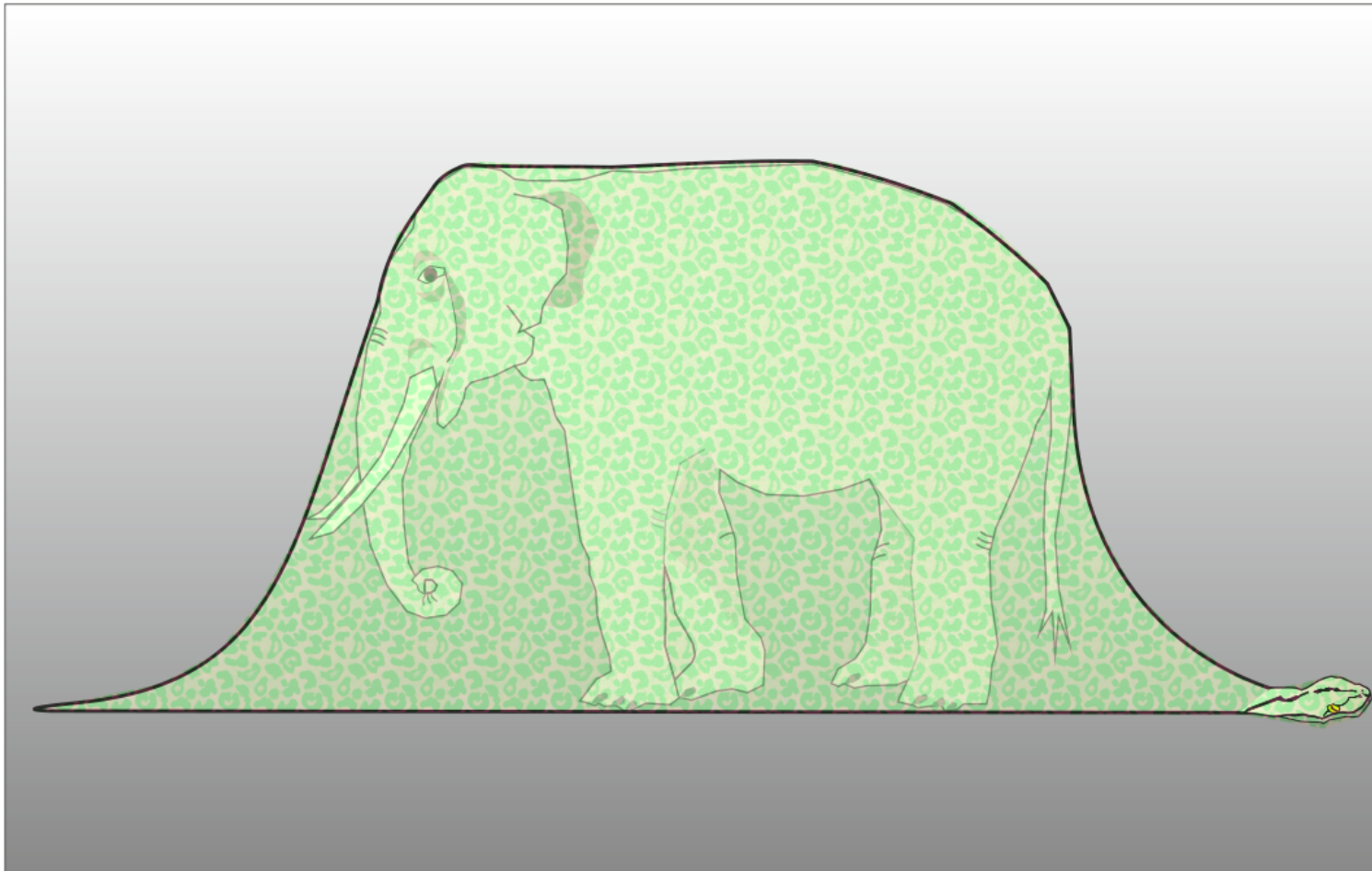


What do you see?



a hat?

What do you see?



... or a snake digesting an elephant?

Antoine de Saint-Exupéry: Le petit prince

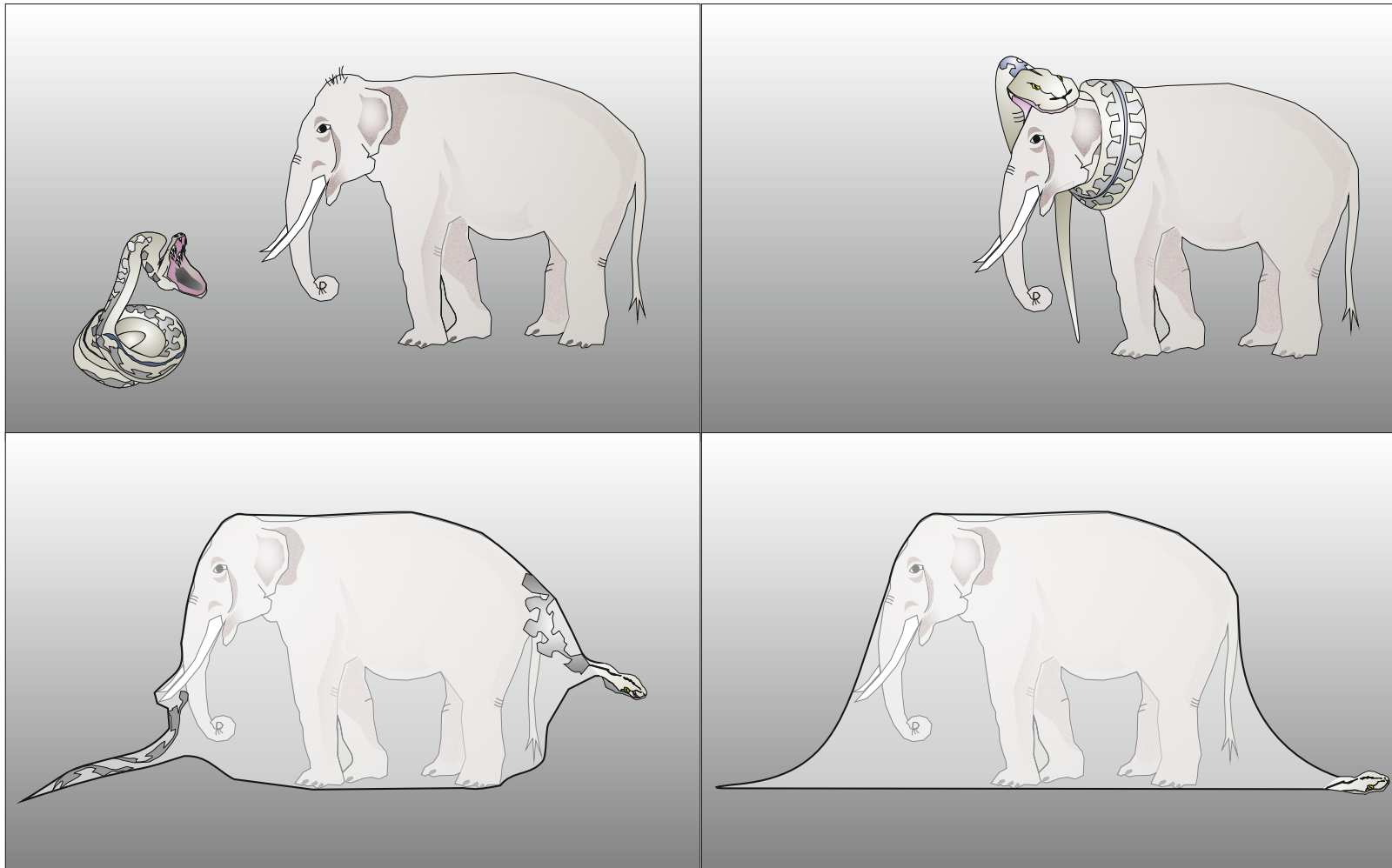
J'ai montré mon chef-d'oeuvre aux grandes personnes et je leur ai demandé si mon dessin leur faisait peur.

Elles m'ont répondu: "Pourquoi **un chapeau** ferait-il peur?"

Mon dessin ne représentait pas un chapeau. Il représentait **un serpent boa qui digérait un éléphant.**

Les grandes personnes m'ont conseillé de **laisser de côté les dessins de serpent boas . . .** et de **m'intéresser plutôt . . . au calcul . . .**

A snake devouring an elephant ...



... or a variational inequality?

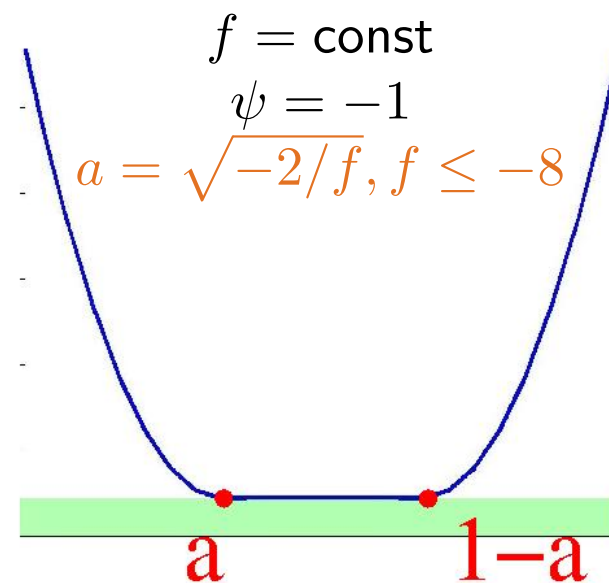
Obstacle model problem

- Constrained minimization problem:

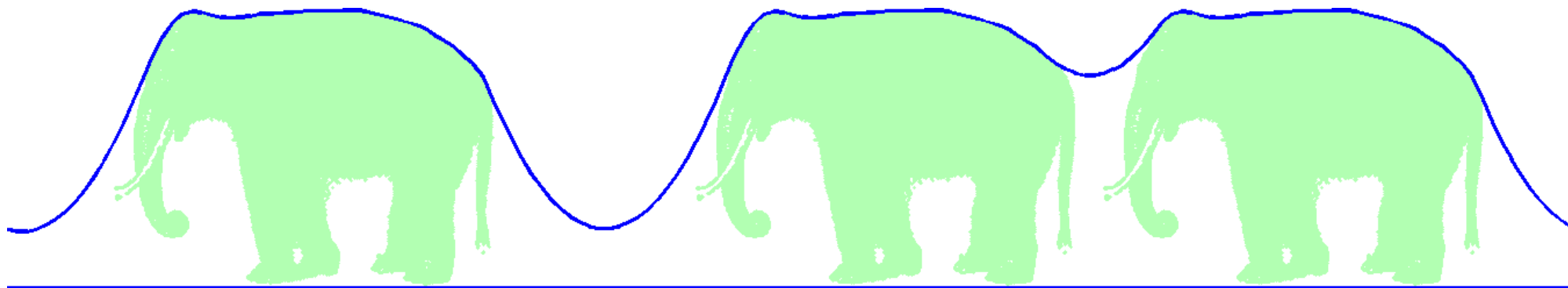
Find $u \in K := \{v \in H_0^1(0, 1); v \geq \psi\}$
such that

$$J(u) = \min_{v \in K} J(v)$$

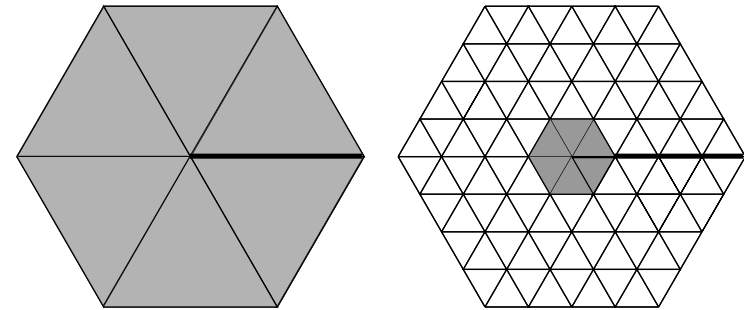
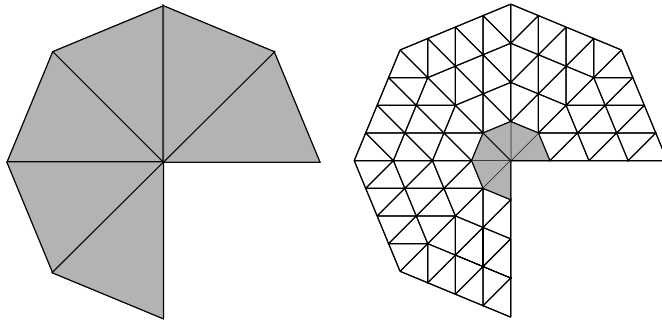
with $J(v) := \frac{1}{2} \int_0^1 (v')^2 ds - \int_0^1 f v ds$



- How to discretize and solve in the more general case?



Which domain pairs with which result?

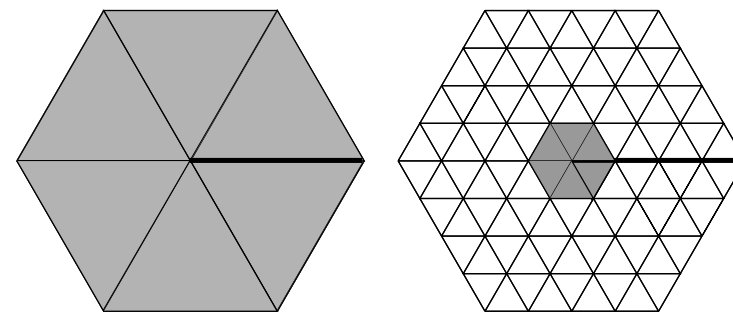
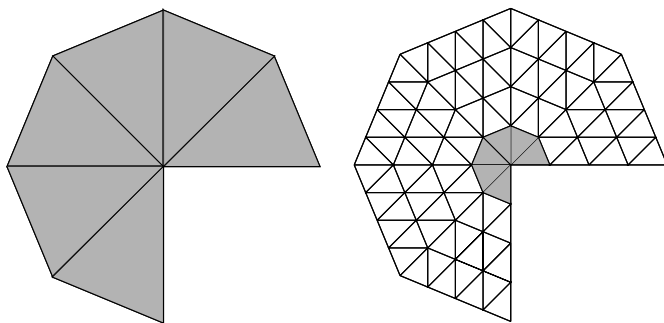


	linear FE	
l	error in weighted L^2 -norm	rate
0	9.734e-02	-
1	3.759e-02	1.37
2	1.678e-02	1.16
3	8.052e-03	1.06
4	3.962e-03	1.02
5	1.968e-03	1.01
6	9.811e-04	1.00
7	4.899e-04	1.00

	linear FE	
l	error in weighted L^2 -norm	rate
0	6.557e-02	-
1	2.200e-02	1.58
2	7.762e-03	1.50
3	2.852e-03	1.44
4	1.080e-03	1.40
5	4.167e-04	1.37
6	1.627e-04	1.36
7	6.394e-05	1.35

No **second order** convergence due to pollution effects

and now?



	linear FE	
l	error in weighted L^2 -norm	rate
0	6.557e-02	-
1	1.632e-02	2.01
2	4.079e-03	2.00
3	1.021e-03	2.00
4	2.556e-04	2.00
5	6.400e-05	2.00
6	1.603e-05	2.00
7	4.014e-06	2.00

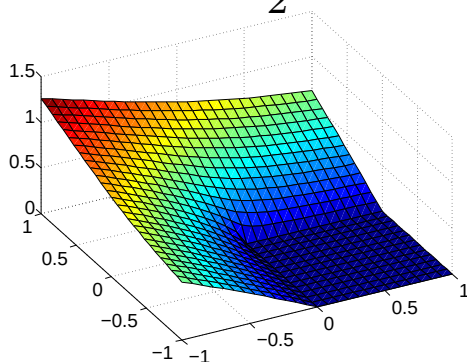
	linear FE	
l	error in weighted L^2 -norm	rate
0	9.734e-02	-
1	2.439e-02	2.00
2	6.071e-03	2.01
3	1.511e-03	2.01
4	3.761e-04	2.01
5	9.363e-05	2.01
6	2.331e-05	2.01
7	5.807e-06	2.01

Second order convergence and no pollution effects

Pollution effect for corner singularities

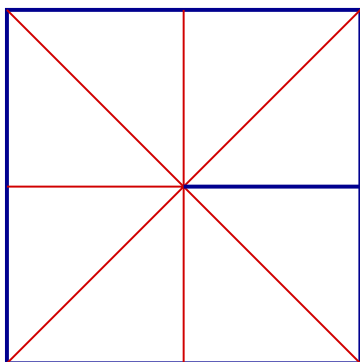
Corner singularities: $u \notin H^2(\Omega)$ but in $H_\alpha^2(\Omega)$, $\alpha > 1 - \frac{\pi}{\theta}$

- L-shape domain: $\theta = \frac{3}{2}\pi$



l	weighted L^2 -norm	rate
2	1.328	1.45
3	0.4909	1.44
4	0.1849	1.41
5	0.0709	1.38
6	0.0276	1.36

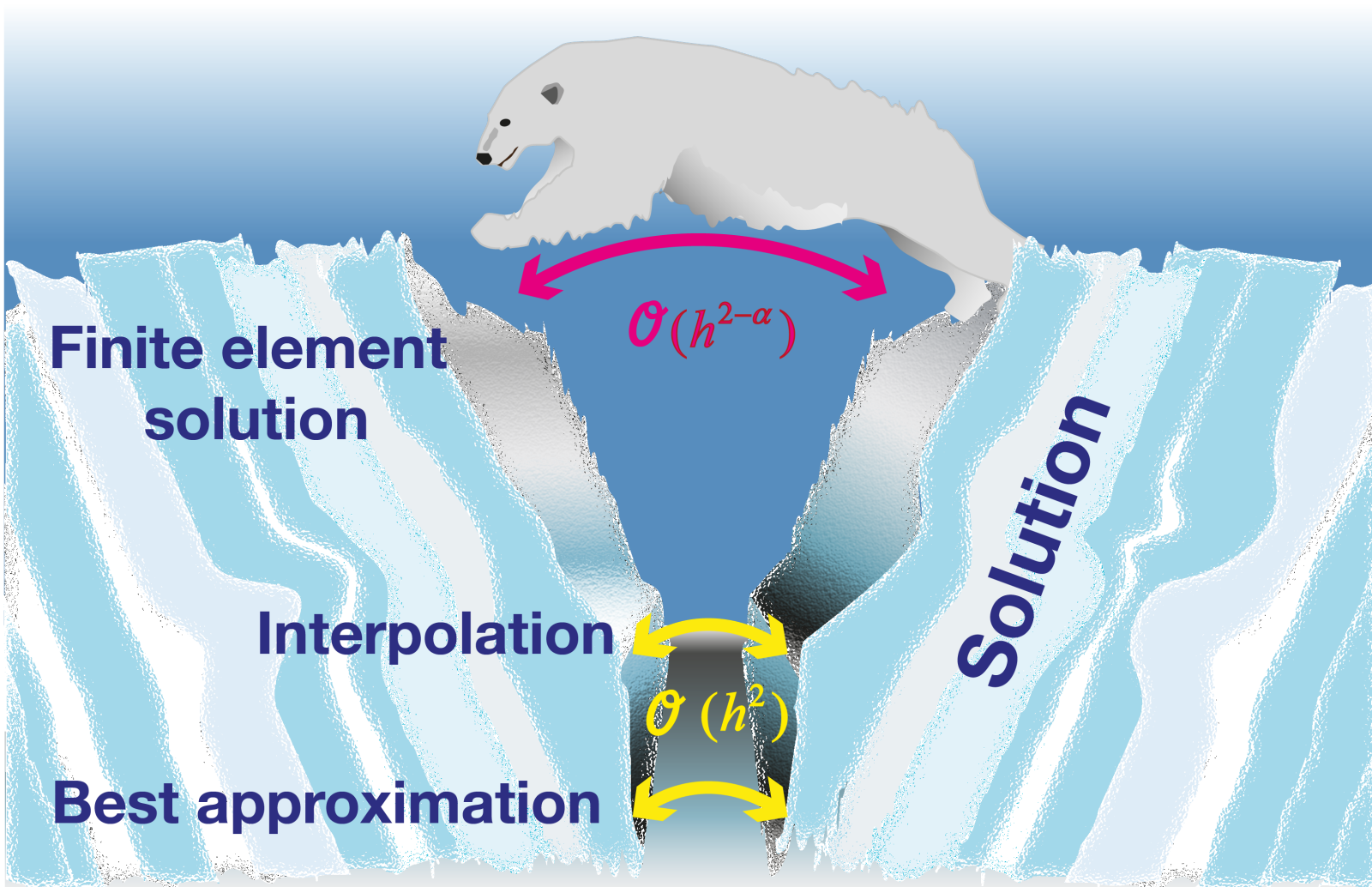
- Slit domain: $\theta = 2\pi$



l	weighted L^2 -norm	rate
2	3.0992	1.07
3	1.5628	0.99
4	0.7897	0.98
5	0.3974	0.99
6	0.1994	0.99

Standard FE solutions do not recover best approximation order of $\mathcal{O}(h^2)$

Can the pollution gap be bridged?



Local singularities in the solution reduce globally the accuracy

Pollution gap

Finite element approximation

Best term approximation

L^2 -norm

$$\|u - u_h\|_{L^2} = \mathcal{O}(h^{\frac{2\pi}{\theta}})$$

$$\inf_{v_h \in V_h^k} \|u - v_h\|_{L^2} = \mathcal{O}(h^{1+\frac{\pi}{\theta}})$$

H^1 -norm

$$\|u - u_h\|_{H^1} = \mathcal{O}(h^{\frac{\pi}{\theta}})$$

$$\inf_{v_h \in V_h^k} \|u - v_h\|_{H^1} = \mathcal{O}(h^{\frac{\pi}{\theta}})$$

weighted L^2 -norm

$$\|u - u_h\|_{L_\alpha^2} = \mathcal{O}(h^{\frac{2\pi}{\theta}})$$

$$\inf_{v_h \in V_h^k} \|u - v_h\|_{L_\alpha^2} = \mathcal{O}(h^{k+1})$$

weighted H^1 -norm

$$\|u - u_h\|_{H_\alpha^1} = \mathcal{O}(h^{\min(k, \frac{2\pi}{\theta})})$$

$$\inf_{v_h \in V_h^k} \|u - v_h\|_{H_\alpha^1} = \mathcal{O}(h^k)$$

How does the weight α depend on k and θ ?

Expansion of the solution

Let Ω be a polygonal with a single re-entrant corner of an angle $\theta \in (\pi, 2\pi)$, $m \in \mathbb{N}_0$, and $f \in H_{-\gamma}^m(\Omega)$ for $\gamma > 1 + m - \lambda_1$. Also, let $N \in \mathbb{N}$ be such that

$$\forall i = 1, \dots, N : \quad \lambda_i < 1 + m + \gamma, \quad \text{and} \quad \lambda_{N+1} > 1 + m + \gamma.$$

Then the unique solution $u \in H_0^1(\Omega)$ of $-\Delta u = f$ in Ω and $u = 0$ on $\partial\Omega$ belongs also to $H_{\gamma}^{m+2}(\Omega)$ and it can be decomposed into

$$u = \sum_{i=1}^N \mu_i^u s_i + U,$$

where $U \in H_{-\gamma}^{m+2}(\Omega) \cap H_0^1(\Omega)$ and $\mu_i^u \in \mathbb{R}$. Moreover, it satisfies:

$$\|u\|_{m+2,\gamma} \lesssim \|f\|_{m,-\gamma}, \quad \sum_{i=1}^N |\mu_i^u| + \|U\|_{m+2,-\gamma} \lesssim \|f\|_{m,-\gamma}.$$

U is called the smooth remainder, and μ_i^u the stress intensity factors. [Kondratjev 67]

How to eliminate the pollution effect?

- Standard remedies:**
- Graded meshes and adaptivity
 - Enrichment by singular functions
 - Weighted least squares formulation

Alternative approach: Energy correction [Egger+Rüde+Wohlmuth]

Find $u_h^m(\gamma) \in V_h$ such that $a_\gamma(u_h^m(\gamma), v_h) = \langle f, v_h \rangle_\Omega$, $v_h \in V_h$ with

$$a_\gamma(u_h^m, v_h) := a(u_h^m, v_h) - \gamma c_{\omega_h}(u_h^m, v_h)$$

V_h is lowest order conforming finite element space associated with the mesh $\mathcal{T}_h$

Question: How to define ω_h and γ ?

ω_h local patch with **fixed** number of elements

$$\bar{\omega}_h := \bigcup \{T \in \mathcal{T}_h; x_c \in \partial T\}$$

x_c is the **re-entrant** corner

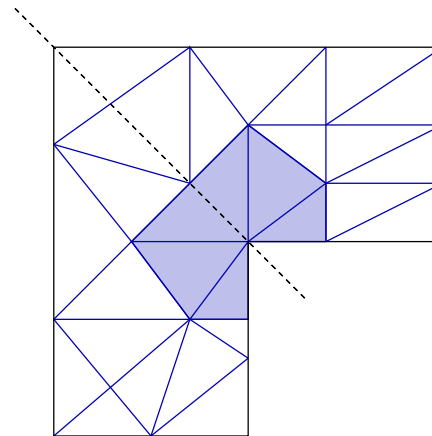
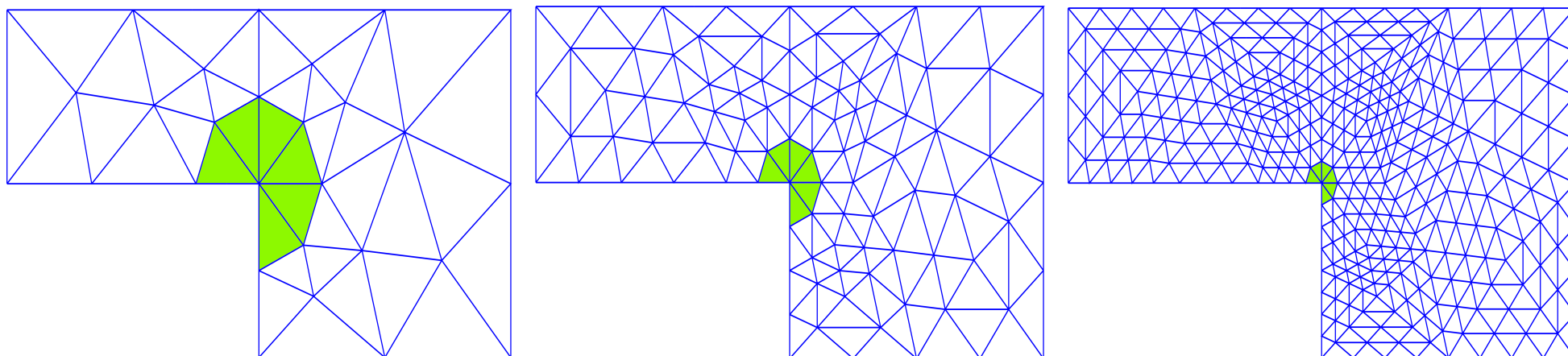


Illustration of mesh-size influence on ω_l



Characteristic properties:

- area of ω_l is $\mathcal{O}(h_l^2)$
- only **fixed number** of entries in the global stiffness matrix are modified (here 13)
- multiplication of the local stiffness matrix by a factor of $(1 - \gamma)$

Trivial to implement provided we know how to select the value for γ

Sufficient and necessary condition

Energy approximation plays an important role:

$$|a(u, u) - a_\gamma(u_l(\gamma), u_l(\gamma))| = |f(u) - f(u_l(\gamma))| \leq \|f\|_{0;-\alpha} \|u - u_l(\gamma)\|_{0;\alpha}$$

Observation: Necessary condition for $\mathcal{O}(h_l^2)$ convergence in the weighted L^2 -norm

$$(*) \quad a(s_i, s_j) - a_\gamma((s_i)_l(\gamma), (s_j)_l(\gamma)) = \mathcal{O}(h_l^2), \quad i, j = 1, 2$$

with the singular functions $s_i = \eta(r)r^{i\pi/\theta} \sin(\frac{i\pi}{\theta}\phi)$, $\eta(r)$ smooth cut-off function

Interpretation of (*): The energy has to be approximated accurately.

Theorem: The condition (*) is also **sufficient** for optimal order convergence.

Moreover if it holds for $i = j = 1$ and θ satisfies

- $\pi < \theta < \frac{3}{2}\pi$ or
- the coarse mesh is locally symmetric

then it also holds for **all** $i, j = 1, 2$

Main ingredients of the proof

- Introduce the weighted dual problem: $-\Delta v = r^{2\alpha}(u - u_l(\gamma))$
- Use the expansions $u = \sum_{i=1}^2 \lambda_i s_i + U$ and $v = \sum_{i=1}^2 \mu_i s_i + V$
- Introducing the modified Galerkin operator denoted by R^m , i.e., $u_l(\gamma) = R^m u$ and setting $c(\cdot, \cdot) := \gamma a_{\omega_l}(\cdot, \cdot)$, we get

$$\begin{aligned}
 \|u - R^m u\|_{0;\alpha}^2 &= \sum_{i,j} \lambda_i \mu_j (a(s_i - R^m s_i, s_j - R^m s_j) - c(R^m s_i, R^m s_j)) \\
 &\quad + \sum_i \lambda_i (a(s_i - R^m s_i, V - R^m V) - c(R^m s_i, R^m V)) \\
 &\quad + \sum_j \mu_j (a(U - R^m U, s_j - R^m s_j) - c(U, R^m s_j)) \\
 &\quad + a(U - R^m U, V - R^m V) - c(R^m U, R^m V)
 \end{aligned}$$

Each of the four terms on the right can be bounded by $h_l^2 \|f\|_{0;\alpha} \|u - R^m u\|_{0;\alpha}$ due to

- **Term I:** Condition $(*)$ and $\|r^{2\alpha}(u - R^m u)\|_{0;-\alpha} = \|u - R^m u\|_{0;\alpha}$
- **Term II+III:** A modified Galerkin orthogonality, Green's formula and optimal order a priori estimates in the smooth reminder U for a negatively weighed L^2 -norm.
- **Term IV:** Regularity estimates on the smooth reminders.

Why is the condition $(*)$ reduced to $j = i = 1$ in some cases?

- The case $i = 2 = j$ does automatically hold due to the regularity of s_2 .
- Observing that on a symmetric domain we have orthogonality between s_1 and s_2 , we can exploit symmetry arguments for $\{i, j\} = \{1, 2\}$.
 - Use a **smooth cut-off function** supported in the symmetry region of the mesh.
 - Define for $R^m s_1$ a mirrored finite element $R_*^m s_1$ such that the sum is **symmetric** on the symmetry part of the mesh and the difference is **discrete harmonic**.
 - Use a **discrete trace theorem** and **Wahlbin-type** localization techniques.

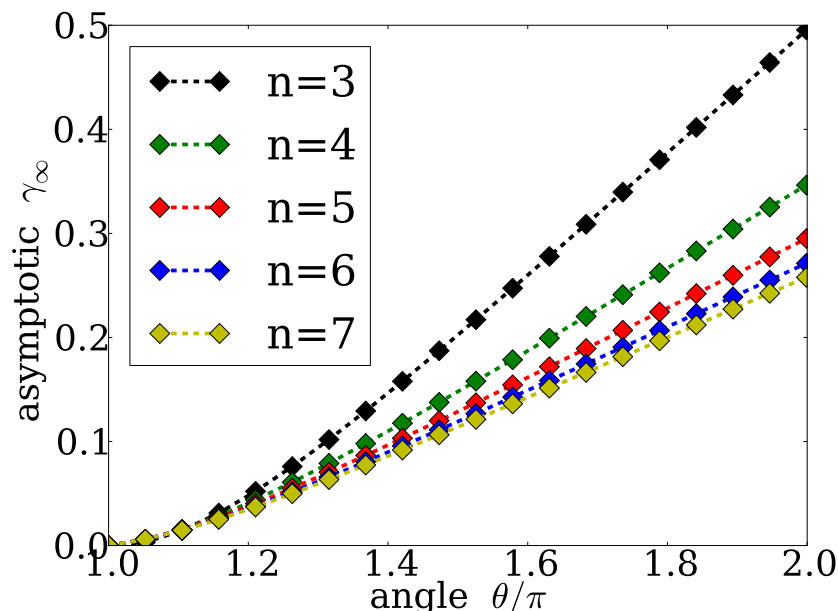
Existence and uniqueness result

Interpretation of the modification for $\gamma < 1$: Local softening of the stiffness

Theorem: There exists a unique $\gamma_\infty(\theta)$ depending on the local coarse mesh such that

$$\|u - u_l(\gamma_\infty(\theta))\|_{0;\alpha} + h_l \|\nabla(u - u_l(\gamma_\infty(\theta)))\|_{0;\alpha} \leq h_l^2 \|f\|_{0;-\alpha}, \quad 1 - \frac{\pi}{\theta} < \alpha < 1$$

if θ at x_c satisfies $\pi < \theta \leq \frac{3}{2}\pi$ or the coarse mesh is locally symmetric



Convergence rates



	standard FE		energy corrected FE	
l	$\gamma = 0$	rate	$\gamma = \gamma_\infty$	rate
0	6.557e-02	-	6.557e-02	-
1	2.200e-02	1.58	1.632e-02	2.01
2	7.762e-03	1.50	4.079e-03	2.00
3	2.852e-03	1.44	1.021e-03	2.00
4	1.080e-03	1.40	2.556e-04	2.00
5	4.167e-04	1.37	6.400e-05	2.00
6	1.627e-04	1.36	1.603e-05	2.00
7	6.394e-05	1.35	4.014e-06	2.00

	standard FE		energy corrected FE	
l	$\gamma = 0$	rate	$\gamma = \gamma_\infty$	rate
0	9.734e-02	-	9.734e-02	-
1	3.759e-02	1.37	2.439e-02	2.00
2	1.678e-02	1.16	6.071e-03	2.01
3	8.052e-03	1.06	1.511e-03	2.01
4	3.962e-03	1.02	3.761e-04	2.01
5	1.968e-03	1.01	9.363e-05	2.01
6	9.811e-04	1.00	2.331e-05	2.01
7	4.899e-04	1.00	5.807e-06	2.01

Energy correction yields **optimal** convergence order in weighted L^2 -norm

How to get γ_∞ theoretically?

Define the **energy defect** function on each level by:

$$g_l(\gamma) := a(s_1, s_1) - a((s_1)_l(\gamma), (s_1)_l(\gamma)) + \gamma a_{\omega_l}((s_1)_l(\gamma), (s_1)_l(\gamma))$$

Theorem: On each level there exists a unique $\gamma_l^* < 1$ such that $g_l(\gamma_l^*) = 0$.

Remark: The energy defect function is strictly **monotone**.

Theorem: The root γ_l^* of $g_l(\cdot)$ defines a Cauchy sequence. Moreover

$$|\gamma_l^* - \gamma_\infty| \leq C h_l^{2(1-\pi/\theta)}, \quad g_l(\gamma_\infty) = \mathcal{O}(h_l^2)$$

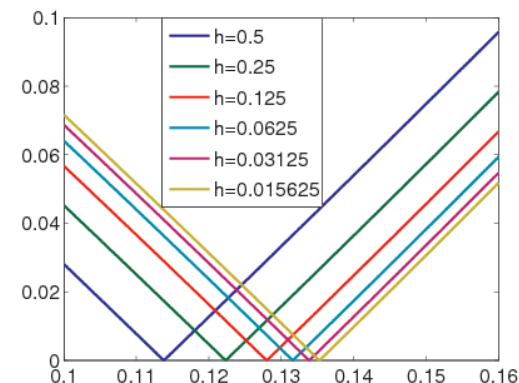
and **optimal convergence** rates are obtained

$$\|u - u_l(\gamma_\infty)\|_{0;\alpha} = \mathcal{O}(h_l^2)$$

Remark: On each level, we get the relation $g_{l-1}(\gamma) = 2^{2\pi/\theta} g_l(\gamma) + \mathcal{O}(h_l^2)$

Main ingredients of the proofs

- Absolute value of a re-scaled **energy defect function** on different levels.
- It can be observed numerically that the unique level dependent roots **converge**.



- Let a ball of radius κh_l be a subset of ω_l then for κ large enough, we have

$$a(s_1 - I_l s_1, s_1 - I_l s_1) \leq \gamma_0 \int_{\omega_l} \nabla I_l s_1 \nabla I_l s_1 \, dx, \quad 0 < \gamma_0 < 1 \text{ fixed}$$

- The energy defect function is C^1 with $g_l(0) \geq 0$ and $g_l(\gamma_0) \leq 0$ satisfying

$$c_1 h_l^{\frac{2\pi}{\theta}} \leq -g'_l(\gamma) \leq c_2 h_l^{\frac{2\pi}{\theta}}$$

- Exploit the scaling property of FE approximations on ω_0 and ω_1

$$s_{l+1}^1(\gamma)(x) = 2^{-\pi/\theta} s_l^0(\gamma)(2x), \quad \text{on } \omega_1$$

How to get γ_∞ algorithmically?

Observation: Newton algorithm applied on the energy defect function reads

$$\gamma_l^{k+1} = -\frac{a(u, u) - a(u_l(\gamma_l^k), u_l(\gamma_l^k))}{a_{\omega_l}(u_l(\gamma_l^k), u_l(\gamma_l^k))}$$

Algorithm: On each level $l \geq 1$ set $\tilde{\gamma}_l$ as

$$\tilde{\gamma}_l = -\frac{a(u, u) - a(u_{l-1}(\tilde{\gamma}_{l-1}), u_{l-1}(\tilde{\gamma}_{l-1}))}{a_{\omega_{l-1}}(u_{l-1}(\tilde{\gamma}_{l-1}), u_{l-1}(\tilde{\gamma}_{l-1}))}, \quad \tilde{\gamma}_0 := 0$$

Remark: On each level only **one** Newton step and **one** global solve is carried out.

Theorem: The inexact multi-level Newton converges. We find

$$|\tilde{\gamma}_l - \gamma_\infty| \leq C_1 |\tilde{\gamma}_{l-1} - \gamma_\infty|^2 + C_2 h_l^{2(1-\pi/\theta)}.$$

Moreover, we have optimal convergence rates, i.e., $\|u - u_l(\tilde{\gamma}_l)\|_{0;B} = \mathcal{O}(h_l^2)$

NO extra cost within multigrid framework **provided** that $a(u, u)$ is known

What to do if $a(u, u)$ is unknown?

Observation: $g_l(\gamma) - g_{l-1}(\gamma)$ does not depend on $a(u, u)$

Idea: Approximate the root of $g_l(\gamma) - g_{l-1}(\gamma)$ by one nested Newton step

Algorithm: On each level $l \geq 1$ set $\bar{\gamma}_l$ as

$$\bar{\gamma}_l = \frac{a(u_l(\bar{\gamma}_{l-1}), u_l(\bar{\gamma}_{l-1})) - a(u_{l-1}(\bar{\gamma}_{l-1}), u_{l-1}(\bar{\gamma}_{l-1}))}{a_{\omega_l}(u_l(\bar{\gamma}_{l-1}), u_l(\bar{\gamma}_{l-1})) - a_{\omega_{l-1}}(u_{l-1}(\bar{\gamma}_{l-1}), u_{l-1}(\bar{\gamma}_{l-1}))}, \quad \bar{\gamma}_0 = 0$$

Remark: On each level only **one** Newton step but **two** global solves are carried out.

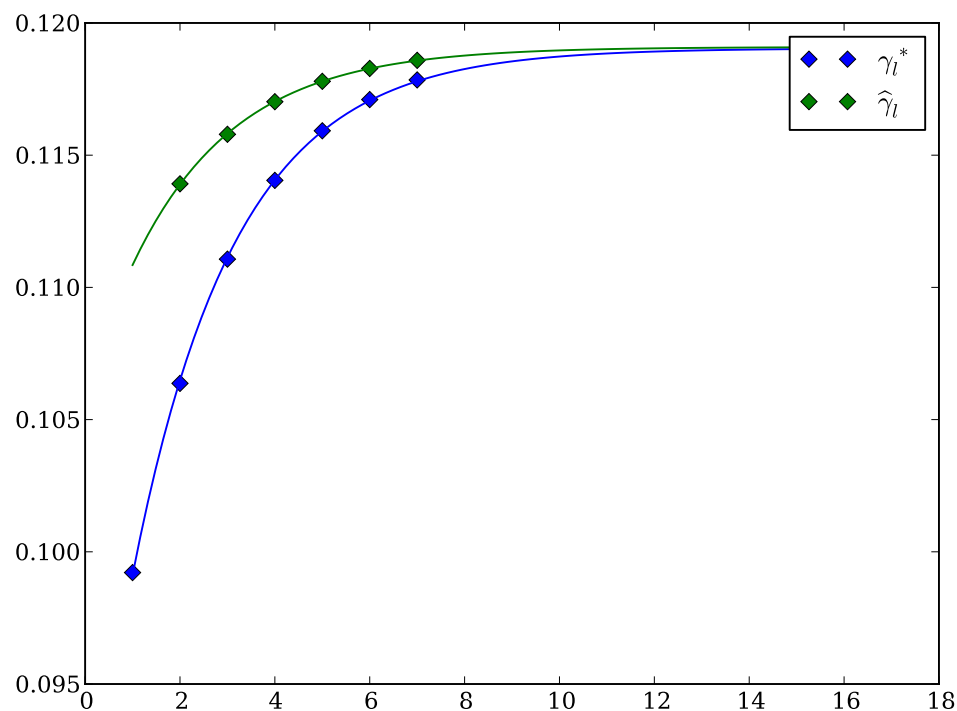
Theorem: The inexact multi-level Newton converges. We find

$$|\bar{\gamma}_l - \gamma_\infty| \leq C_1 |\bar{\gamma}_{l-1} - \gamma_\infty|^2 + C_3 h_l^{2(1-\pi/\theta)}.$$

Moreover, we have optimal convergence rates, i.e., $\|u - u_l(\bar{\gamma}_l)\|_{0;B} = \mathcal{O}(h_l^2)$

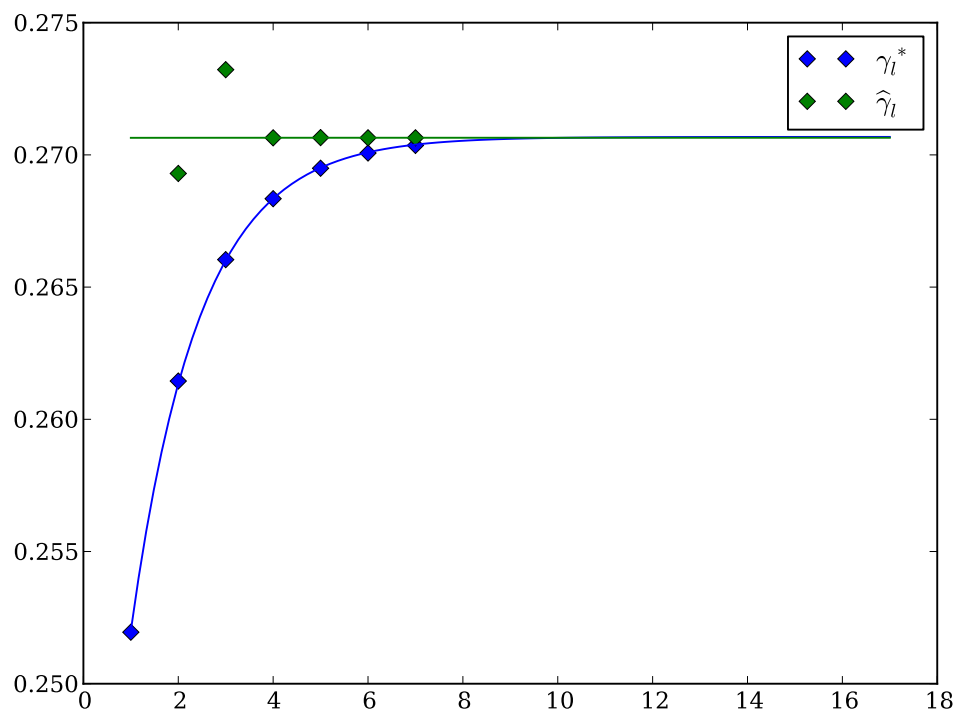
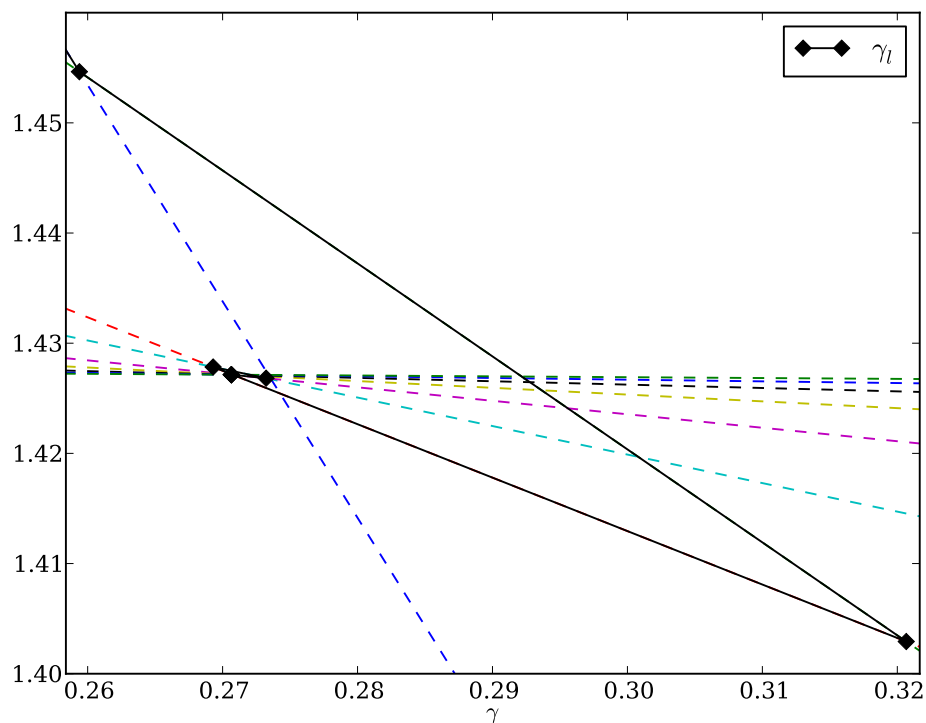
Question: Can we avoid the **two** global solves on each level?

Idea: Replace $u_{l-1}(\bar{\gamma}_{l-1})$ by $u_{l-1}(\bar{\gamma}_{l-2})$

$$\hat{\gamma}_l = \frac{a(u_l(\hat{\gamma}_{l-1}), u_l(\hat{\gamma}_{l-1})) - a(u_{l-1}(\hat{\gamma}_{l-2}), u_{l-1}(\hat{\gamma}_{l-2}))}{a_{\omega_l}(u_l(\hat{\gamma}_{l-1}), u_l(\hat{\gamma}_{l-1})) - a_{\omega_{l-1}}(u_{l-1}(\hat{\gamma}_{l-2}), u_{l-1}(\hat{\gamma}_{l-2}))}, \quad \hat{\gamma}_0 = \hat{\gamma}_{-1} = 0$$


Two-level algorithm with one solve

$$\hat{\gamma}_l = \frac{a(u_l(\hat{\gamma}_{l-1}), u_l(\hat{\gamma}_{l-1})) - a(u_{l-1}(\hat{\gamma}_{l-2}), u_{l-1}(\hat{\gamma}_{l-2}))}{a_{\omega_l}(u_l(\hat{\gamma}_{l-1}), u_l(\hat{\gamma}_{l-1})) - a_{\omega_{l-1}}(u_{l-1}(\hat{\gamma}_{l-2}), u_{l-1}(\hat{\gamma}_{l-2}))}, \quad \hat{\gamma}_0 = \hat{\gamma}_{-1} = 0$$

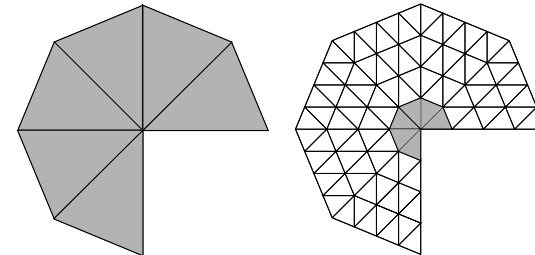


Domain with slit: $\hat{\gamma}_l$ and γ_l^* converge to γ_∞ with $\mathcal{O}(h_l)$

Convergence study for energy correction methods

L-shaped domain: weighted error norm

$$\|r^{1/3}(s - R_h s)\|_{L^2(\Omega)}$$

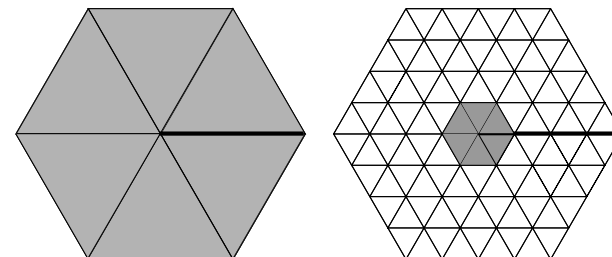


	no correction		asymptotic		root		iterative method	
level	$\gamma = 0$	rate	$\gamma = \gamma_\infty$	rate	$\gamma = \gamma_l^*$	rate	$\gamma = \hat{\gamma}_l$	rate
0	6.5567e-02	-	6.5567e-02	-	6.5567e-02	-	6.5567e-02	-
1	2.2000e-02	1.58	1.6319e-02	2.01	1.7034e-02	1.94	1.6377e-02	2.00
2	7.7619e-03	1.50	4.0794e-03	2.00	4.3064e-03	1.98	4.1623e-03	1.98
3	2.8522e-03	1.44	1.0211e-03	2.00	1.0791e-03	2.00	1.0418e-03	2.00
4	1.0797e-03	1.40	2.5563e-04	2.00	2.6983e-04	2.00	2.6058e-04	2.00
5	4.1670e-04	1.37	6.4003e-05	2.00	6.7462e-05	2.00	6.5188e-05	2.00
6	1.6268e-04	1.36	1.6027e-05	2.00	1.6870e-05	2.00	1.6309e-05	2.00
7	6.3940e-05	1.35	4.0141e-06	2.00	4.2196e-06	2.00	4.0806e-06	2.00

Convergence study for energy correction methods

Slit domain: weighted error norm

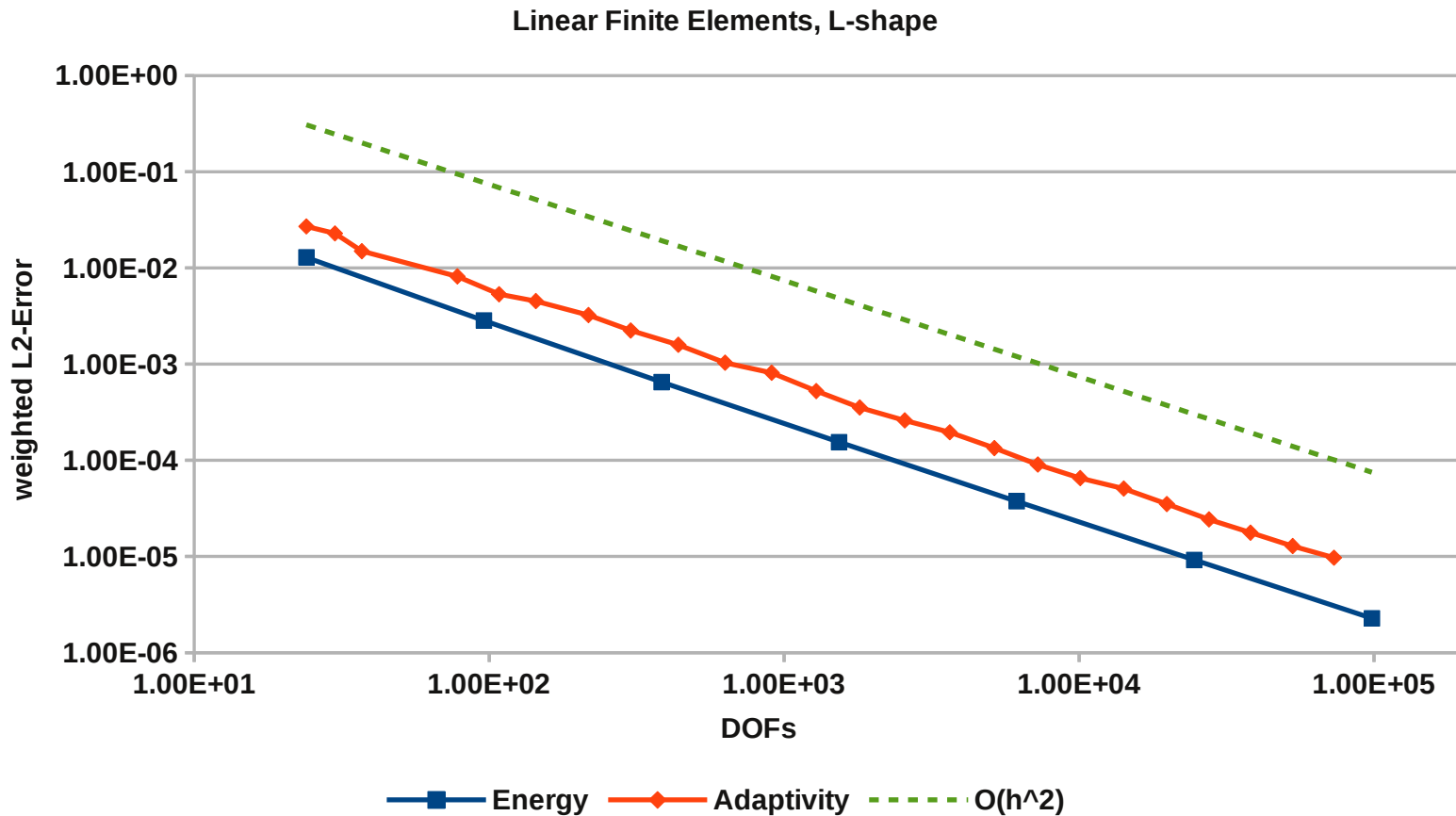
$$\|r^{1/2}(s - R_h s)\|_{L^2(\Omega)}$$



	no correction		asymptotic		root		iterative method	
level	$\gamma = 0$	rate	$\gamma = \gamma_\infty$	rate	$\gamma = \gamma_l^*$	rate	$\gamma = \hat{\gamma}_l$	rate
0	9.7336e-02	-	9.7336e-02	-	9.7336e-02	-	9.7336e-02	-
1	3.7585e-02	1.37	2.4393e-02	2.00	2.4912e-02	1.97	2.3531e-02	2.05
2	1.6784e-02	1.16	6.0714e-03	2.01	6.1803e-03	2.01	6.0846e-03	1.95
3	8.0519e-03	1.06	1.5111e-03	2.01	1.5303e-03	2.01	1.5070e-03	2.01
4	3.9621e-03	1.02	3.7610e-04	2.01	3.7956e-04	2.01	3.7612e-04	2.00
5	1.9680e-03	1.01	9.3629e-05	2.01	9.4313e-05	2.01	9.3632e-05	2.01
6	9.8111e-04	1.00	2.3314e-05	2.01	2.3463e-05	2.01	2.3315e-05	2.01
7	4.8989e-04	1.00	5.8072e-06	2.01	5.8413e-06	2.01	5.8071e-06	2.01

Uniform versus adaptively refined meshes

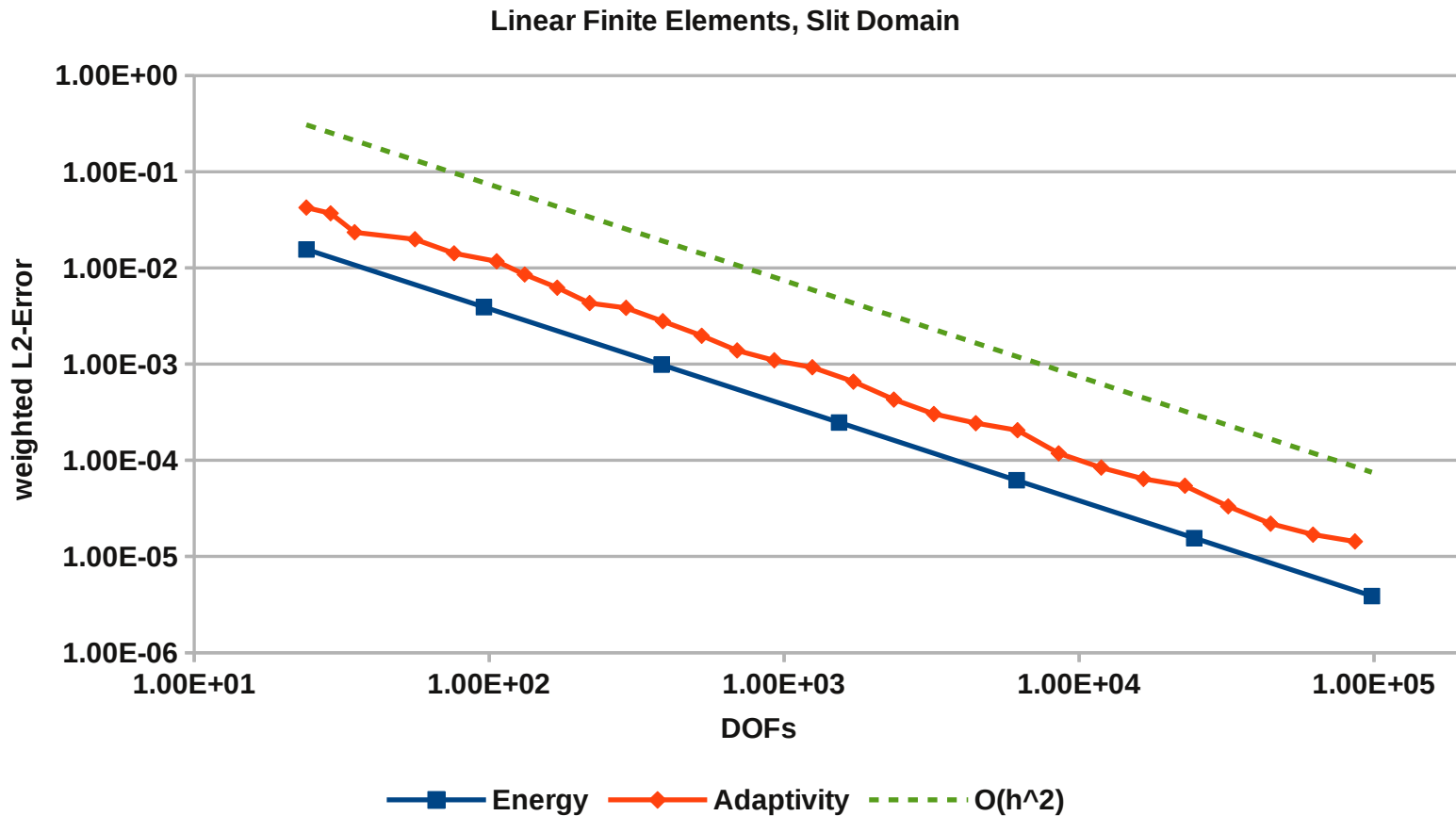
Example: L-shape domain



Adaptive refinement results in graded meshes

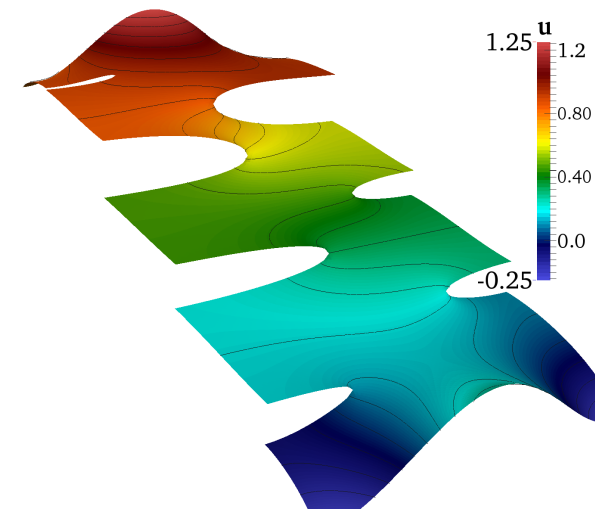
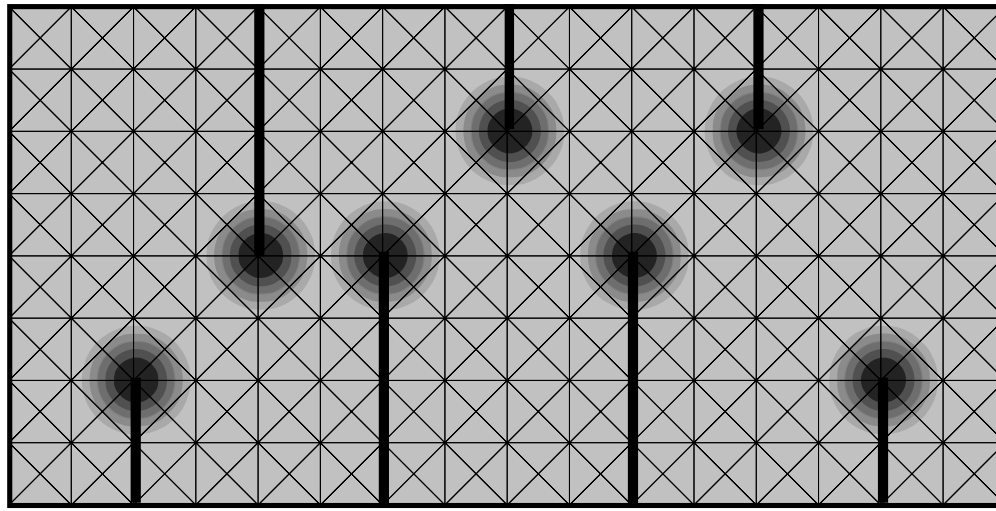
Uniform versus adaptively refined meshes

Example: Slit domain



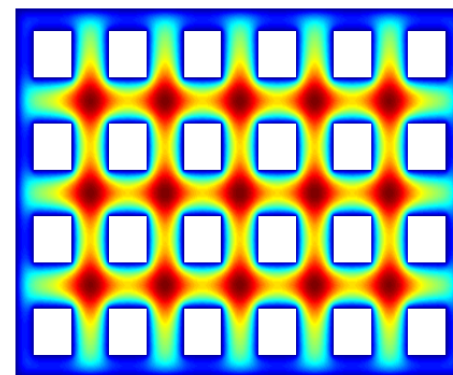
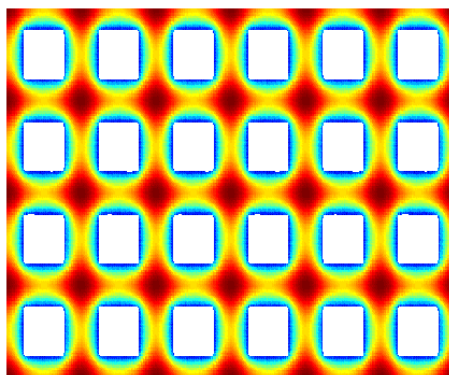
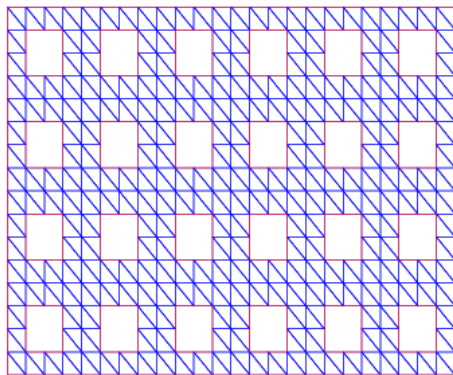
Adaptive refinement results in graded meshes

Multi re-entrant corners



	standard finite elements				energy-corrected finite elements			
l	weighted	rate	flux	rate	weighted	rate	flux	rate
0	1.700e-2	-	1.463e-2	-	7.233e-3	-	5.794e-3	-
1	8.663e-3	0.97	4.385e-3	1.74	1.810e-3	2.00	1.256e-3	2.20
2	4.324e-3	1.00	1.414e-3	1.63	4.541e-4	1.99	2.477e-4	2.34
3	2.154e-3	1.00	4.786e-4	1.56	1.136e-4	2.00	4.583e-5	2.43
4	1.075e-3	1.00	1.658e-4	1.53	2.822e-5	2.01	7.515e-6	2.61
5	5.373e-4	1.00	5.7876e-5	1.52	6.474e-6	2.12	6.325e-7	3.57

Neumann and Dirichlet boundary data



Neumann boundary data:

level	$\gamma = 0$	rate	$\gamma = \gamma_\infty$	rate
1	2.75702	—	2.21389	—
2	1.00287	1.35	0.49796	2.15
3	0.36959	1.44	0.11487	2.11
4	0.13919	1.40	0.02746	2.06
5	0.05336	1.38	0.00672	2.03

Dirichlet boundary data:

level	$\gamma = 0$	rate	$\gamma = \gamma_\infty$	rate
1	2.41536	—	2.00729	—
2	0.84611	1.51	0.46510	2.11
3	0.30408	1.48	0.10903	2.09
4	0.11269	1.43	0.02624	2.06
5	0.04278	1.40	0.00642	2.03

The rates do not but the value of γ_∞ does depend on the type of the boundary condition

Where do we stand?

- The **accuracy** of many quantities of interest, e.g.,
 - weighted L^2 -norm errors
 - eigenvalues, stress intensity factors**cannot** be qualitatively better than the **accuracy in the energy**.
- **Energy-corrected FEMs**
 - modify the PDE operator on the discrete level
 - require only a **local re-scaling** of the stiffness matrix.and recover **optimal order** convergence on **uniformly** refined meshes
- The idea can be generalized to
 - Higher order elements
 - Optimal control problems
 - Different PDEs (including parabolic problems)and the **number** of required parameter **only** depends on the PDE and order.

Second order convergence and sketch of a proof

Theorem: Let $1 - \frac{\pi}{\omega} < \alpha < 1$. If the above mentioned modification is used with γ_∞ , the following upper and lower bound for $\lambda_{h,m}$ hold,

$$|\lambda_{h,m} - \lambda_m| \leq Ch^2 \lambda_m^{\alpha+1}.$$

Idea: Follow the lines of conforming finite elements in the lowest order case [Raviart, Thomas 83], [Babuska, Osborn 89,91]

$$\begin{aligned} \lambda_{h,m} &\leq \max_{v \in E_{m,h}} \frac{a_{ec}(v, v)}{(v, v)} = \max_{v \in V_m} \frac{a_{ec}(R_h v, R_h v)}{(R_h v, R_h v)} \\ &= \max_{v \in V_m} \frac{a(v, v) + a_{ec}(R_h v, R_h v) - a(v, v)}{(R_h v, R_h v)} \\ &= \max_{v \in V_m} \frac{a(v, v)}{(v, v)} \max_{v \in V_m} \frac{(v, v)}{(R_h v, R_h v)} + \max_{v \in V_m} \frac{a_{ec}(R_h v, R_h v) - a(v, v)}{(R_h v, R_h v)} \\ &\leq \lambda_m \frac{1 + Ch^2 \lambda_m^{2+\alpha}}{1 - Ch^2 \lambda_m^{\alpha+1}} \lesssim \lambda_m (1 + Ch^2 \lambda_m^{\alpha+1}) + Ch^2 \lambda_m^{2+\alpha} (1 + Ch^2 \lambda_m^{\alpha+1}). \end{aligned}$$

Eigenvalue convergence on a slit domain

- no correction, i.e., $\gamma = 0$:

	Exact	Mesh 1	Mesh 2	Mesh 3	Mesh 4
1.Eigenvalue Error Rate:	1.034074	1.141259 0.107185	1.080043 0.045969 1.221355	1.055071 0.020997 1.130428	1.044072 0.009997 1.070539
7.Eigenvalue Error Rate:	12.264895	13.976904 1.712008	12.757176 0.492280 1.798137	12.439548 0.174652 1.494993	12.326897 0.062001 1.494107

- with correction: $\gamma = 0.269920207767$:

	Exact	Mesh 1	Mesh 2	Mesh 3	Mesh 4
1.Eigenvalue Error Rate:	1.034074	1.063604 0.029530	1.041806 0.007732 1.933264	1.036132 0.002058 1.909102	1.034648 0.000574 1.842170
7.Eigenvalue Error Rate:	12.264895	13.895886 1.630990	12.671946 0.407050 2.002469	12.367674 0.102778 1.985664	12.290884 0.025988 1.983591

Optimal convergence for the stress intensity factor

Kondratiev expansion: $u = \sum_{0 < n < \frac{\theta}{\pi}} k_n s_n + U, k_n \in \mathbb{R}, U \in H^2(\Omega)$

Stress-intensity factors k_n are given by

$$k_n = \frac{1}{n\pi} \left(\int_{\Omega} f s_{-n} + u \Delta s_{-n} dx \right).$$

Theorem: stress-intensity factors converge with $|k_n - k_n^h| = \mathcal{O}(h^2)$

L-shape domain: analytical $k_1 = 1$, error given in percent

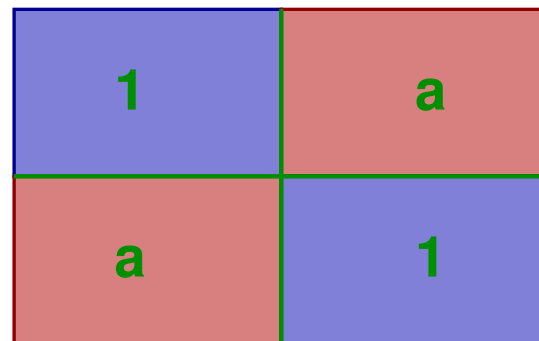
$l \setminus \gamma$	0.00000	rate	0.10000	rate	0.13802	rate	0.15000	rate	0.20000	rate
1	2.08912	—	0.75388	—	0.21929	—	0.04750	—	0.68716	—
2	1.30421	0.68	0.42965	0.81	0.07749	1.50	0.03592	0.40	0.52226	0.40
3	0.50188	1.38	0.15953	1.43	0.02070	1.90	0.02413	0.57	0.21703	1.27
4	0.19475	1.37	0.05972	1.42	0.00481	2.11	0.01294	0.90	0.08944	1.28
5	0.07636	1.35	0.02288	1.38	0.00111	2.12	0.00593	1.13	0.03629	1.30
6	0.03009	1.34	0.00888	1.37	0.00024	2.18	0.00255	1.22	0.01460	1.31

Heterogeneous coefficients

$$-\nabla \cdot (\kappa(x) \nabla u) = f \text{ in } \Omega$$

Heterogeneous coefficients: low regularity

$$u \in H^{1+s(a)}(\Omega), s(a) \rightarrow 0, \text{ for } a \rightarrow \infty$$



$a = 10$

standard FE		energy corrected FE	
$\gamma = 0$	Rate	$\gamma = \gamma_l$	Rate
3.061e-02	-	2.729e-02	-
1.389e-02	1.14	7.195e-03	1.92
6.977e-03	0.99	1.836e-03	1.97
3.792e-03	0.88	4.636e-04	1.99
2.142e-03	0.82	1.170e-04	1.99
1.230e-03	0.80		
7.110e-04	0.79		
4.125e-04	0.79		
2.397e-04	0.78		

$a = 100$

standard FE		energy corrected FE	
$\gamma = 0$	Rate	$\gamma = \gamma_l$	Rate
1.4784e-02	-	1.2247e-02	-
8.5842e-03	0.78	3.2825e-03	1.90
5.8509e-03	0.55	8.4232e-04	1.96
4.3313e-03	0.43		
3.3299e-03	0.38		
2.6147e-03	0.35		
2.0824e-03	0.33		
2.0824e-03	0.33		
1.6758e-03	0.31		

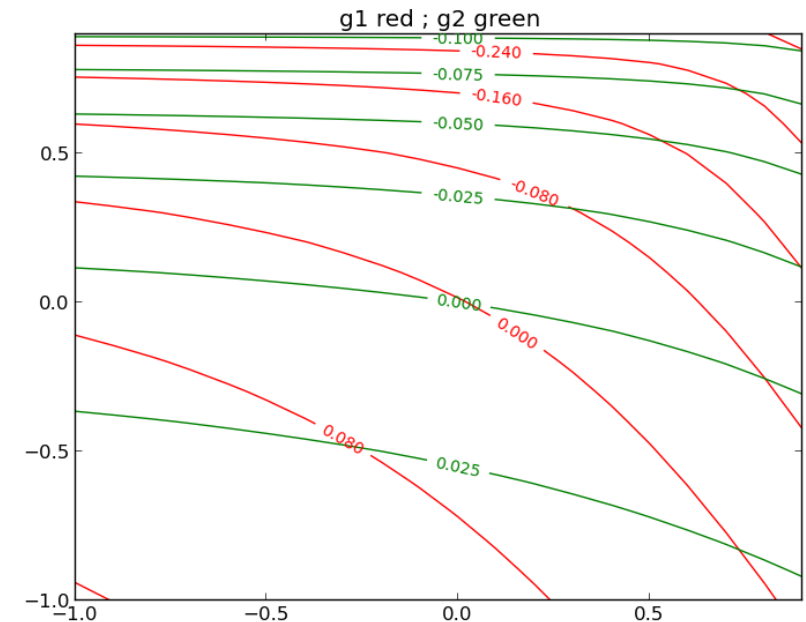
Comparison for quadratic FEM in a slit domain

Idea: Use a **two** parameter modification
Vector valued energy correction function

$$g(\gamma_1, \gamma_2) := \begin{pmatrix} g_1(\gamma_1, \gamma_2) \\ g_2(\gamma_1, \gamma_2) \end{pmatrix}$$

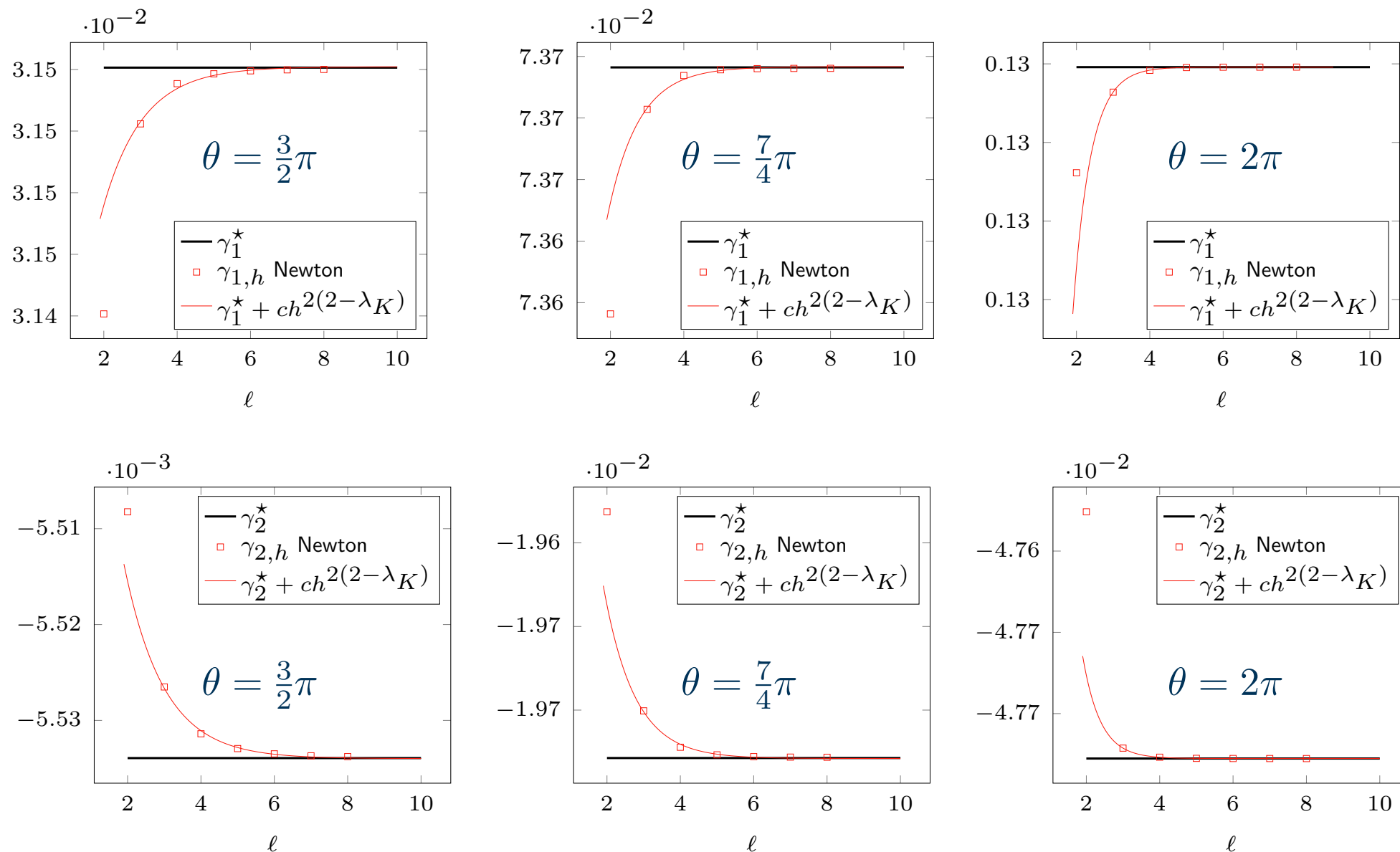
g_i takes the energy defect of the i th singular fct

Observation: **Third** order convergence order



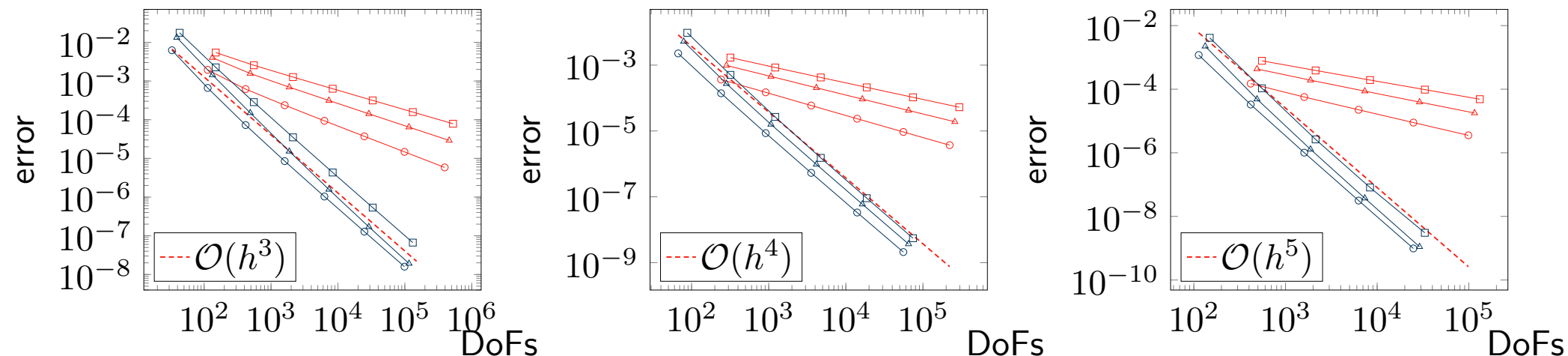
h	standard	rate	one param	rate	γ	two params	rate
0.5773	5.1185e-03	-	4.4776e-03	-	0.0951855	8.5753e-03	-
0.2886	2.4148e-03	1.08	1.1746e-03	1.93	0.0950813	7.6883e-04	3.48
0.1443	1.1968e-03	1.01	2.9408e-04	2.00	0.0950433	7.7440e-05	3.31
0.0721	5.9654e-04	1.00	7.3418e-05	2.00	0.0950374	8.3179e-06	3.22
0.0360	2.9784e-04	1.00	1.8344e-05	2.00	0.0950367	9.4368e-07	3.14
0.0180	1.4881e-04	1.00	4.5852e-06	2.00	0.0950366	1.1778e-07	3.00

Convergence of the level dependent correction factors



Higher order FEM in domain with a re-entrant corner

Error decay in weighted L^2 -norm: $k = 2$ (left), $k = 3$ (middle), $k = 4$ (right)



Different re-entrant corner angles: $\theta = \frac{3\pi}{2}$ (square), $\theta = \frac{7\pi}{4}$ (triangle), $\theta = 2\pi$ (circle)

Standard FE: Convergence rate is k independent and non-optimal

Energy corrected FE: Convergence rate is of optimal order $k + 1$

Optimal order convergence in L^2 -norm can be recovered by a post-process:

$$u_h^{\text{post}} := u_h + \sum_{i=1}^K \mu_i^h (s_i - (s_i)_h), \quad \mu_i^h := \frac{1}{i\pi} \int_{\Omega} f s_{-i} + u_h \Delta s_{-i} \, dx$$

Energy corrected FEM for optimal control

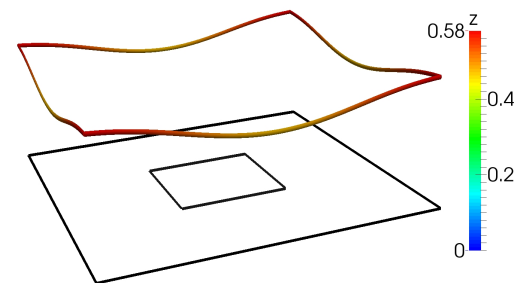
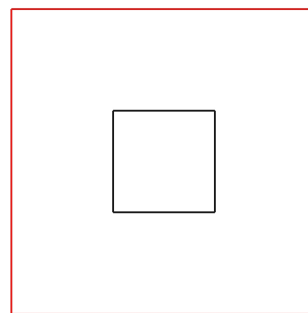
Variational formulation Find $(u_h, p_h) \in V_h \times Q_h \subset H^1(\Omega) \times H_0^1(\Omega)$:

$$\begin{aligned} a_h(u_h, v_h) + b_h(v_h, p_h) &= \langle \bar{u}, v_h \rangle_\Omega & \forall v_h \in V_h, \\ b_h(u_h, q_h) &= 0 & \forall q_h \in Q_h, \end{aligned}$$

with the bilinear forms

$$\begin{aligned} a_h(u_h, v_h) &:= \langle u_h, v_h \rangle_\Omega + \varrho \langle \nabla u_h, \nabla v_h \rangle_\Omega - \gamma \varrho \langle \nabla u_h, \nabla v_h \rangle_{\omega_h}, \\ b_h(v_h, q_h) &:= -\langle \nabla v_h, \nabla q_h \rangle_\Omega + \gamma \langle \nabla v_h, \nabla q_h \rangle_{\omega_h}. \end{aligned}$$

Observation: Correction parameter $\gamma := \gamma_\infty$ from the Laplace problem with Dirichlet bc's does work.



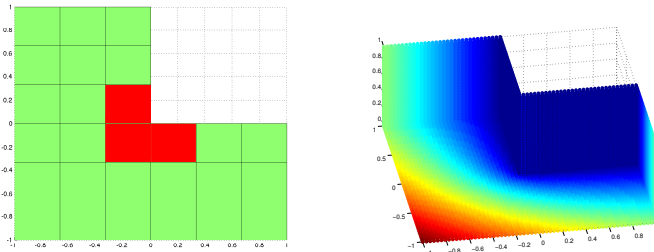
Energy corrected FEM for optimal control

Application – optimal heating

$\gamma = 0$				$\gamma = 0.13826$			
$\ u - u_h\ _0$	eoc	$\ p - p_h\ _0$	eoc	$\ u - u_h\ _0$	eoc	$\ p - p_h\ _0$	eoc
1.95772e-01	–	1.39284e-01	–	1.27634e-01	–	1.36700e-01	–
7.93318e-02	1.30	4.11260e-02	1.76	3.33816e-02	1.93	3.86096e-02	1.82
3.08926e-02	1.36	1.23075e-02	1.74	9.73307e-03	1.78	1.01883e-02	1.92
1.19190e-02	1.37	3.86805e-03	1.67	2.93579e-03	1.73	2.66291e-03	1.94
4.57471e-03	1.38	1.28335e-03	1.59	9.03594e-04	1.70	7.10945e-04	1.91
1.72261e-03	1.41	4.38927e-04	1.55	2.79789e-04	1.69	1.94948e-04	1.87
6.09002e-04	1.50	1.46131e-04	1.59	8.50900e-05	1.72	5.38358e-05	1.86
$\ u - u_h\ _{0,1/3}$	eoc	$\ p - p_h\ _{0,1/3}$	eoc	$\ u - u_h\ _{0,1/3}$	eoc	$\ p - p_h\ _{0,1/3}$	eoc
3.04166e-01	–	1.62870e-01	–	1.84747e-01	–	1.59856e-01	–
1.15021e-01	1.40	4.41601e-02	1.88	3.81633e-02	2.28	4.20132e-02	1.93
4.33599e-02	1.41	1.20065e-02	1.88	8.83978e-03	2.11	1.04577e-02	2.01
1.64886e-02	1.39	3.38209e-03	1.83	2.10365e-03	2.07	2.57662e-03	2.02
6.28839e-03	1.39	1.00468e-03	1.75	5.08950e-04	2.05	6.35398e-04	2.02
2.36179e-03	1.41	3.13132e-04	1.68	1.23488e-04	2.04	1.56045e-04	2.03
8.34196e-04	1.50	9.76391e-05	1.68	2.93878e-05	2.07	3.72298e-05	2.07

What else is possible?

- 3D results for the Laplacian (same correction parameter as in 2D)



level	DOF	no correction		energy correction	
		$\gamma = 0$	rate	$\gamma = 0.0662$	rate
0	160	0.001733	-	0.001255	-
1	1408	0.000636	1.38	0.000320	1.89
2	11776	0.000246	1.34	8.22e-05	1.92
3	96256	9.64e-05	1.34	2.12e-05	1.93
4	778240	3.80e-05	1.34	5.52e-06	1.94

- Linear elasticity (more than one correction parameter)

l	standard FE		energy corrected FE	
	$\gamma = 0$	rate	$\gamma = \gamma_\infty$	rate
2	2.4098e-02	1.28	7.5363e-03	2.03
3	1.0185e-02	1.24	1.8842e-03	2.00
4	4.3673e-03	1.22	4.7072e-04	2.00
5	1.8874e-03	1.21	1.1746e-04	2.00
6	8.1907e-04	1.20	2.9262e-05	2.01

Energy corrected FEM for Stokes equations

Let $\Omega \subset \mathbb{R}^2$ be bounded, polygonal, $\mathbf{f} \in L^2_{-\alpha}(\Omega)^2$

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega. \end{aligned}$$

Characterization of solutions: let $\omega \in (\pi, 2\pi)$ and $1 - \beta < \operatorname{Re}(\lambda) < 1 + \alpha$, then

$$\mathbf{u} = \sum_{i=1}^N \sum_{j=1}^{I_i} \sum_{k=0}^{\kappa_{i,j}-1} c_{i,j,k} \mathbf{s}_{i,j,k}^u + \mathbf{U}, \quad p = \sum_{i=1}^N \sum_{j=1}^{I_i} \sum_{k=0}^{\kappa_{i,j}-1} c_{i,j,k} s_{i,j,k}^p + P$$

with constants $c_{i,j,k}$ and

$$\mathbf{s}_{i,j,k}^u = \eta(r) r^{\lambda_i} \sum_{l=0}^k \frac{1}{l!} (\ln r)^l \begin{pmatrix} \varphi_{j,k-l}^{(i)}(\theta) \\ \psi_{j,k-l}^{(i)}(\theta) \end{pmatrix}, \quad s_{i,j,k}^p = \eta(r) r^{\lambda_i-1} \sum_{l=0}^k \frac{1}{l!} (\ln r)^l \xi_{j,k-l}^{(i)}(\theta)$$

with $(\mathbf{U}, P) \in H^2_{-\alpha}(\Omega)^2 \times H^1_{-\alpha}(\Omega)$

[Dauge 1989], [Ortl, Sändig 1995]

Energy corrected FEM for Stokes equations

Eigenvalues λ_i have to satisfy the condition

$$\lambda_i^2 \sin^2(\omega) - \sin^2(\lambda_i \omega) = 0, \quad \lambda_i \neq 0, \quad \lambda_i \in \mathbb{C}.$$

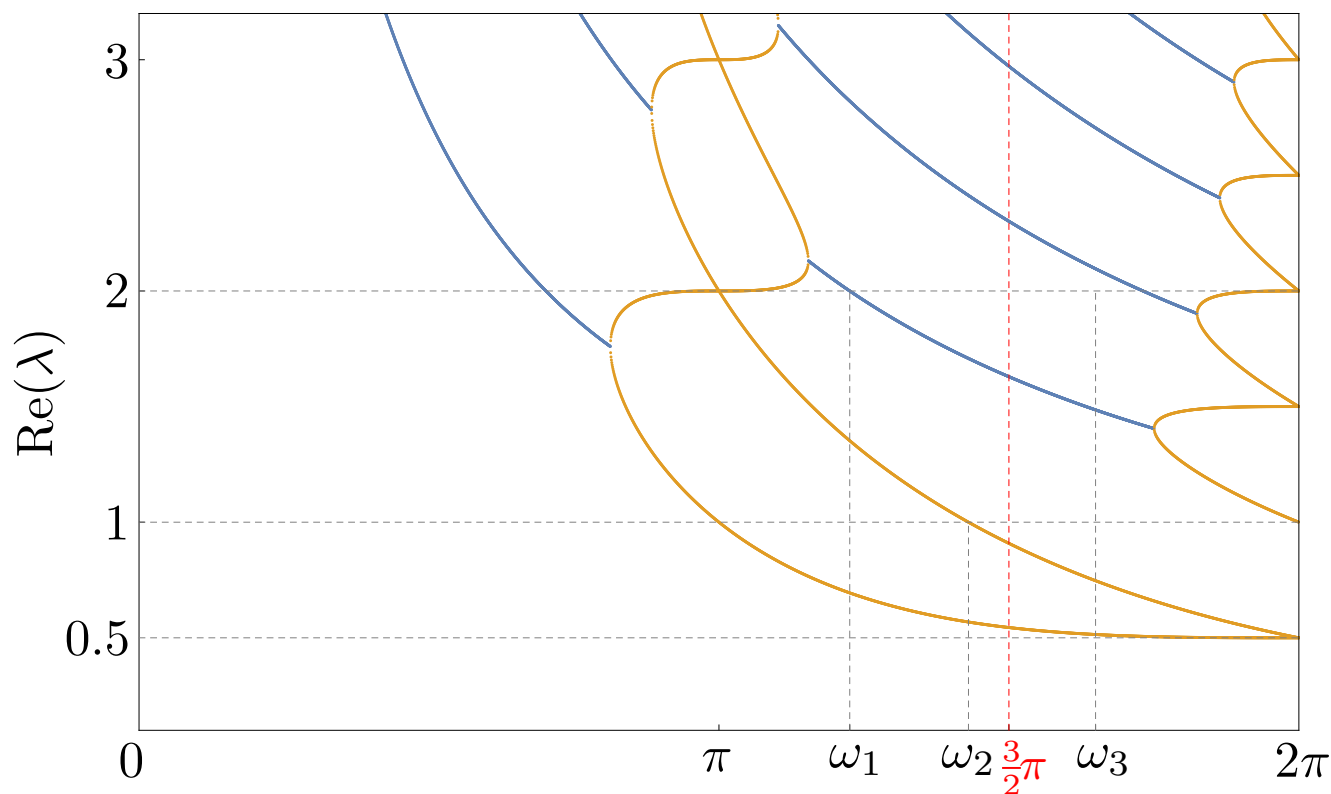


Figure: $\text{Re}(\lambda)$ as a function of ω

Energy corrected FEM for Stokes equations

Singular functions for the L-shaped domain $\omega = 3/2\pi$

$$u = \sum_{i=1}^2 \eta(r) r^{\lambda_i} (\varphi^{(i)}(\theta), \psi^{(i)}(\theta))^{\top} + U, \quad p = \sum_{i=1}^2 \eta(r) r^{\lambda_i-1} \xi^{(i)}(\theta) + P$$

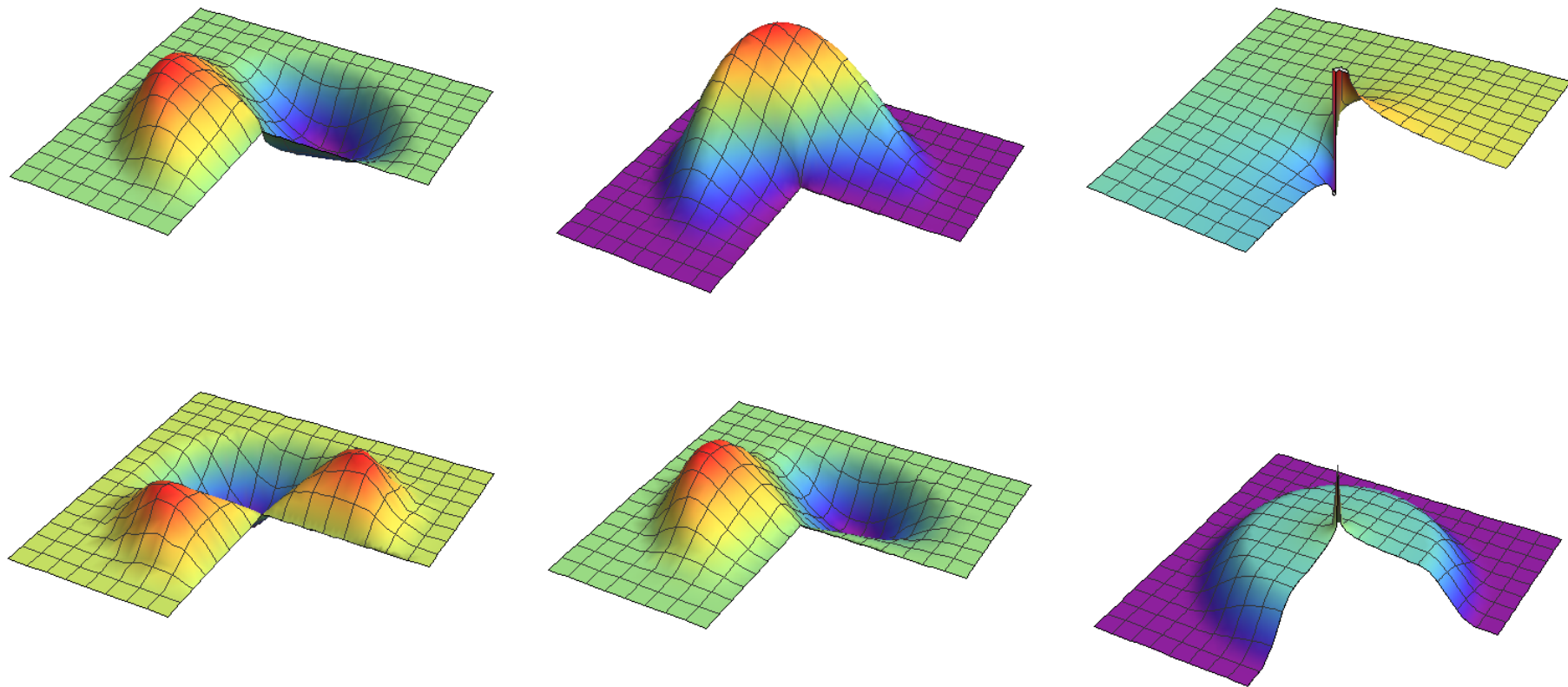


Figure: Sing. functions $(s_{i,r}^u, s_{i,\theta}^u, s_i^p)$ for $i = 1, 2$

Energy corrected FEM for Stokes equations

Variational formulation for **stable FE pair** (e.g. Mini, P2–P1, ...)

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$:

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) &= \langle \mathbf{f}, \mathbf{v}_h \rangle_\Omega & \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b(\mathbf{u}_h, q_h) &= 0 & \forall q_h \in Q_h, \end{aligned}$$

with $a_h(\mathbf{u}_h, \mathbf{v}_h) := a(\mathbf{u}_h, \mathbf{v}_h) - \gamma_1 a_{\omega_1}(\mathbf{u}_h, \mathbf{v}_h) - \gamma_2 a_{\omega_2}(\mathbf{u}_h, \mathbf{v}_h)$.

Variational formulation for **stabilized FE** (e.g. P1–P1 with PSPG stab.)

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$:

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) &= \langle \mathbf{f}, \mathbf{v}_h \rangle_\Omega & \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b(\mathbf{u}_h, q_h) - c_h(p_h, q_h) &= 0 & \forall q_h \in Q_h, \end{aligned}$$

with $c_h(p_h, q_h) := c(p_h, q_h) - (1 - \gamma_1)^{-1} c_{\omega_1}(p_h, q_h) - (1 - \gamma_2)^{-1} c_{\omega_2}(p_h, q_h)$,

→ motivated from the equivalence of Mini element and PSPG.

Energy corrected FEM for Stokes equations

Necessary condition for optimal convergence in weighted $L^2(\Omega)$ norm

$$g_{i,h}(\gamma_1, \gamma_2) := a(\mathbf{s}_i^u, \mathbf{s}_i^u) - \left[a_h(\mathbf{s}_{i,h}^u, \mathbf{s}_{i,h}^u) + c_h(s_{i,h}^p, s_{i,h}^p) \right] = \mathcal{O}(h^2)$$

for $i = 1$ if $\omega \in (\pi, 1.43\pi]$ and for $i = 1, 2$ if $\omega \in (1.43\pi, 1.65\pi)$.

Construction: Define

- $\gamma^* = (\gamma_1^*, 0)^\top$ with $\gamma_1^* \in (0, 1)$ as the limit of the root of $g_{1,h}(\cdot)$ if $\omega \in (\pi, 1.43\pi]$
- $\gamma^* \in (-1, 1)^2$ as the limit of the root of $\mathbf{g}_h(\cdot) := (g_{1,h}(\cdot), g_{2,h}(\cdot))^\top$ if $\omega \in (1.43\pi, 1.65\pi)$

Theorem: Let $\mathbf{V}_h \times Q_h$ be a stable pair of low order. For the energy corrected finite element solution $(\mathbf{u}_h, p_h)$ with γ^* holds

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,\alpha} + \|p - p_h\|_{0,\alpha} \lesssim h \|\mathbf{f}\|_{0,-\alpha} \quad \text{and} \quad \|\mathbf{u} - \mathbf{u}_h\|_{0,\alpha} \lesssim h^2 \|\mathbf{f}\|_{0,-\alpha}.$$

Numerical results L-shaped domain

P2–P1 element $(\mathbf{u}, p) = (s_1^u, s_1^p)$, weights $\alpha = 1 - \lambda_1$, $\beta = 3/2 - \lambda_1$

$\gamma = 0$				$\gamma = (-0.18050, 0.06703)^\top$			
$\ \mathbf{u} - \mathbf{u}_h\ _0$	eoc	$\ p - p_h\ _0$	eoc	$\ \mathbf{u} - \mathbf{u}_h\ _0$	eoc	$\ p - p_h\ _0$	eoc
6.24297e-02	–	3.76799e-01	–	6.77854e-02	–	3.68949e-01	–
3.03536e-02	1.04	2.00709e-01	0.91	3.02091e-02	1.17	1.81270e-01	1.03
1.20753e-02	1.33	1.27266e-01	0.66	1.06982e-02	1.50	1.09426e-01	0.73
4.90503e-03	1.30	8.06119e-02	0.66	3.71669e-03	1.53	7.34415e-02	0.58
2.06841e-03	1.25	5.23079e-02	0.62	1.27895e-03	1.54	5.01889e-02	0.55
9.04139e-04	1.19	3.47209e-02	0.59	4.38827e-04	1.54	3.43913e-02	0.55
4.06348e-04	1.15	2.33942e-02	0.57	1.50458e-04	1.54	2.35772e-02	0.54
$\ \mathbf{u} - \mathbf{u}_h\ _{0,\alpha}$	eoc	$\ p - p_h\ _{0,\beta}$	eoc	$\ \mathbf{u} - \mathbf{u}_h\ _{0,\alpha}$	eoc	$\ p - p_h\ _{0,\beta}$	eoc
2.94081e-02	–	4.73148e-01	–	5.04587e-02	–	3.42732e-01	–
9.38624e-03	1.65	1.72502e-01	1.46	1.71967e-02	1.55	1.26726e-01	1.44
3.21263e-03	1.55	7.58580e-02	1.19	4.44481e-03	1.95	4.19851e-02	1.59
1.36660e-03	1.23	3.50865e-02	1.11	1.12266e-03	1.99	1.43575e-02	1.55
6.20441e-04	1.14	1.65028e-02	1.09	2.78808e-04	2.01	5.00219e-03	1.52
2.88024e-04	1.11	7.78711e-03	1.08	6.85885e-05	2.02	1.75797e-03	1.51
1.34837e-04	1.09	3.67251e-03	1.08	1.67725e-05	2.03	6.20125e-04	1.50

Theoretical versus numerical convergence rates

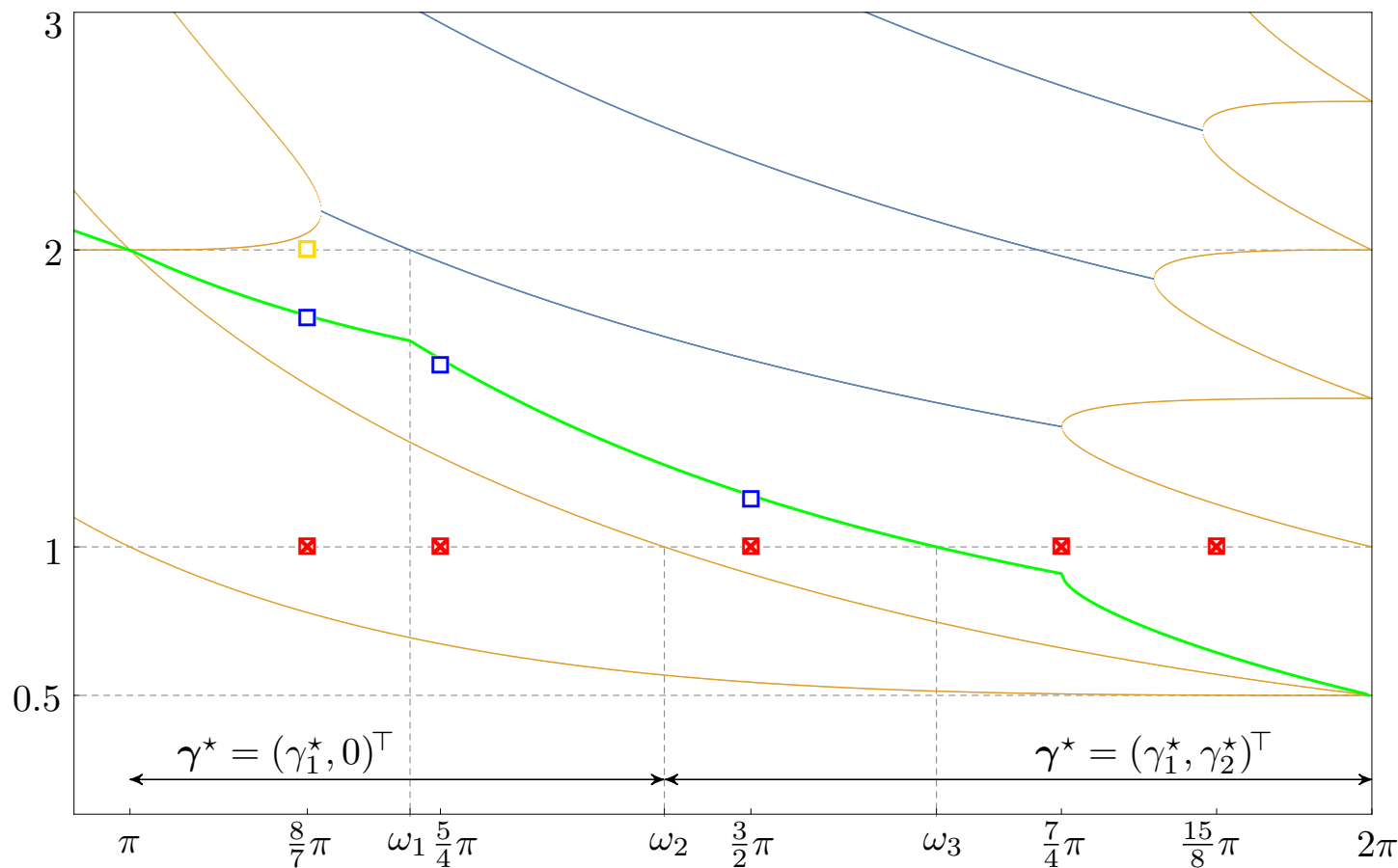
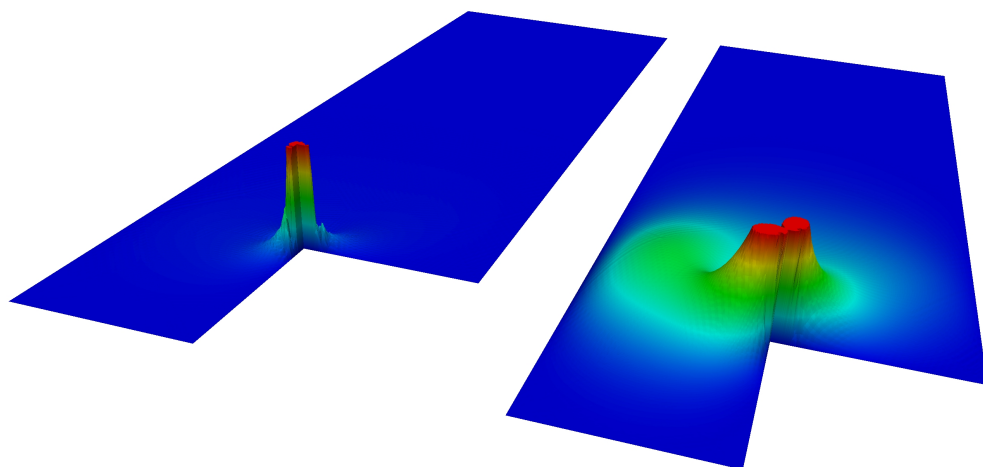


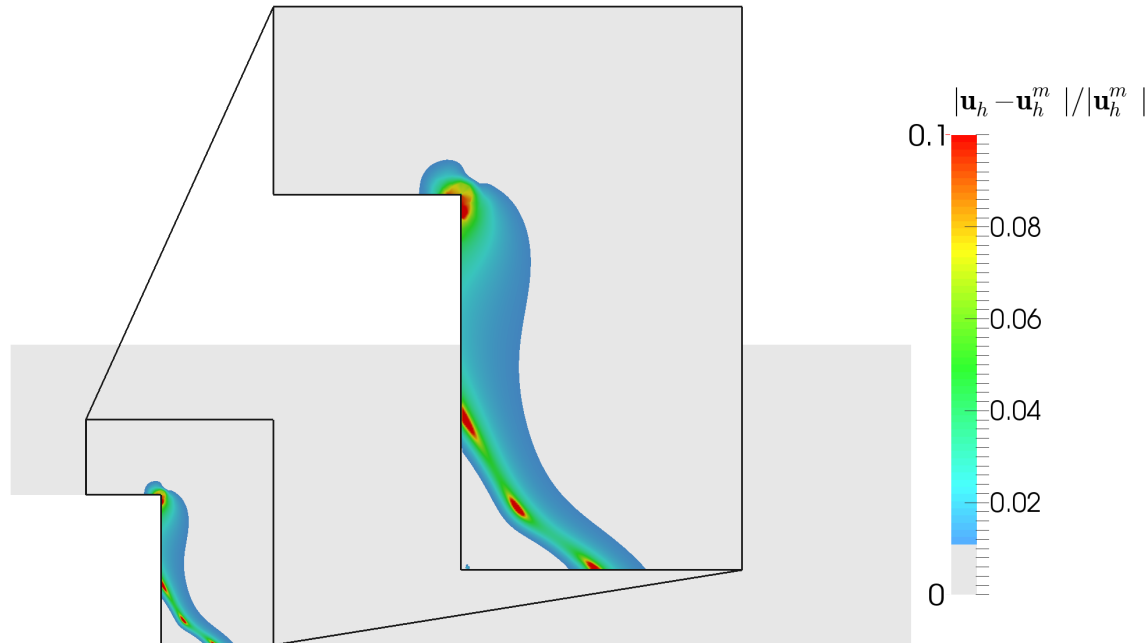
Fig.: Convergence rates of $\|\mathbf{u} - \mathbf{u}_h^m\|_{L_\alpha^2(\Omega)}$ for the Mini element ($\times$) and the Taylor-Hood element ($\square$), shifted by -1 . Red: $\alpha = 1 - \lambda_1$ and Blue: $\alpha = \lambda_3 - 1$. The green line represent $\lambda_1 + \lambda_3 - 1$.

Local modification – global effect



error in the
energy corrected (left) and in the
standard (right) FE solution

difference between energy
corrected and standard FE
solution



Parabolic problem on non-convex domain

Consider the heat-equation: $u_t - \Delta u = f$ in $\Omega \times (0, T]$ plus BC + IC

Regularity result: by [Banasiak, Roach 1991]: Let $u(t)$ be the weak solution of the heat equation on a polygonal domain $\Omega \subset \mathbb{R}^2$ with a re-entrant corner with θ as angle. Let also $f \in H^1((0, T); L_{-\alpha}^2(\Omega))$, $f(0) + \Delta u_0 \in H_0^1(\Omega)$ and $\alpha \geq 1 - \frac{\pi}{\theta}$. Then there exists $U \in H^1(0, T; H_{-\alpha}^2(\Omega))$ and $k_1(t) \in C([0, T]) \cap C^1((0, T))$ such that

$$u(t, \mathbf{x}) = U(t, \mathbf{x}) + k_1(t)s_1(\mathbf{x}),$$

where s_1 is the first singular function of the Laplacian.

Semi-discrete finite element discretisation:

Find $u_h^m \in \mathcal{V}_h^k$ for a.e. $t \in (0, T]$ s.t.

$$\langle u_{h,t}^m, v_h \rangle_{\Omega} + a_{\gamma}(u_h^m, v_h) = \langle f, v_h \rangle_{\Omega}, \quad \text{for all } v_h \in \mathcal{V}_h^k,$$

where the same modification as in the Laplacian is applied.

A priori result for the lowest order case

Optimal accuracy in the weighted norms for the semi-discretized solution:

$$\|u(t) - u_h^m(t)\|_{L_{\alpha}^2(\Omega)} = \mathcal{O}(h^2)$$

Explicit Euler on uniform meshes: $dt \leq ch^2$

$$\|u(T) - u_h^m\|_{L_{\alpha}^2(\Omega)} = \mathcal{O}(h^2 + dt)$$

l	$\ u - u_h^m\ _0$	rate	$\ u - u_h^m\ _{0,\alpha}$	rate	$\ u - \tilde{u}_h^m\ _0$	rate
1	4.4255e-01		3.2725e-01		9.0874e-02	
2	1.3907e-01	1.67	8.4712e-02	1.95	2.1572e-02	2.07
3	4.3758e-02	1.67	2.1778e-02	1.96	5.5544e-03	1.96
4	1.3809e-02	1.66	5.6008e-03	1.96	1.3906e-03	2.00
5	4.3534e-03	1.67	1.4372e-03	1.96	3.4440e-04	2.01
6	1.3658e-03	1.67	3.6688e-04	1.97	8.4871e-05	2.02

Post-processing allows to recover second order in h convergence in the L^2 -norm

What about explicit higher order time integration

Observation: Graded meshes require a too severe stepsize restriction in time

Explicit Euler scheme time stepping $\rightarrow \mathcal{O}(\Delta t)$

$$\mathbf{u}_{h,n+1}^m = \mathbf{u}_{h,n}^m + \Delta t \mathbf{M}^{-1} [\mathbf{F}^n - \mathbf{S}^m \mathbf{u}_{h,n}^m]$$

Heun's method (2nd order explicit Runge-Kutta scheme) $\rightarrow \mathcal{O}(\Delta t^2)$

$$\tilde{\mathbf{u}}_{h,n+1}^m = \mathbf{u}_{h,n}^m + \Delta t \mathbf{M}^{-1} [\mathbf{F}^n - \mathbf{S}^m \mathbf{u}_{h,n}^m]$$

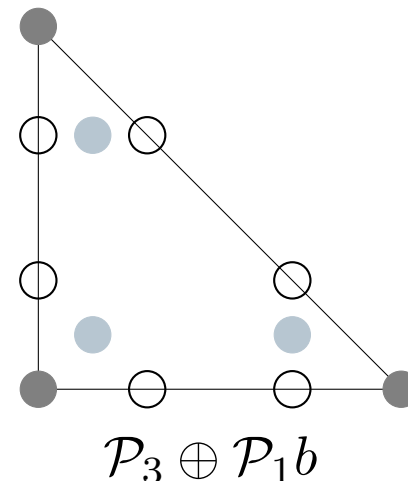
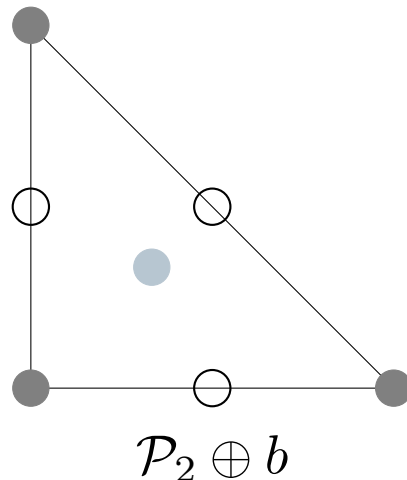
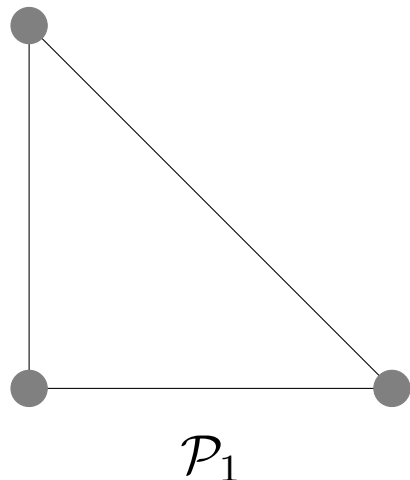
$$\mathbf{u}_{h,n+1}^m = \mathbf{u}_{h,n}^m + \frac{1}{2} \Delta t \mathbf{M}^{-1} [(\mathbf{F}^n - \mathbf{S}^m \mathbf{u}_{h,n}^m) + (\mathbf{F}^{n+1} - \mathbf{S}^m \tilde{\mathbf{u}}_{h,n+1}^m)],$$

$\mathbf{M}$ - mass matrix, $\mathbf{S}^m$ - **energy-corrected** stiffness matrix

Can we use mass lumping, i.e., a diagonal M also for higher order?

Mass-lumping for higher order FEM

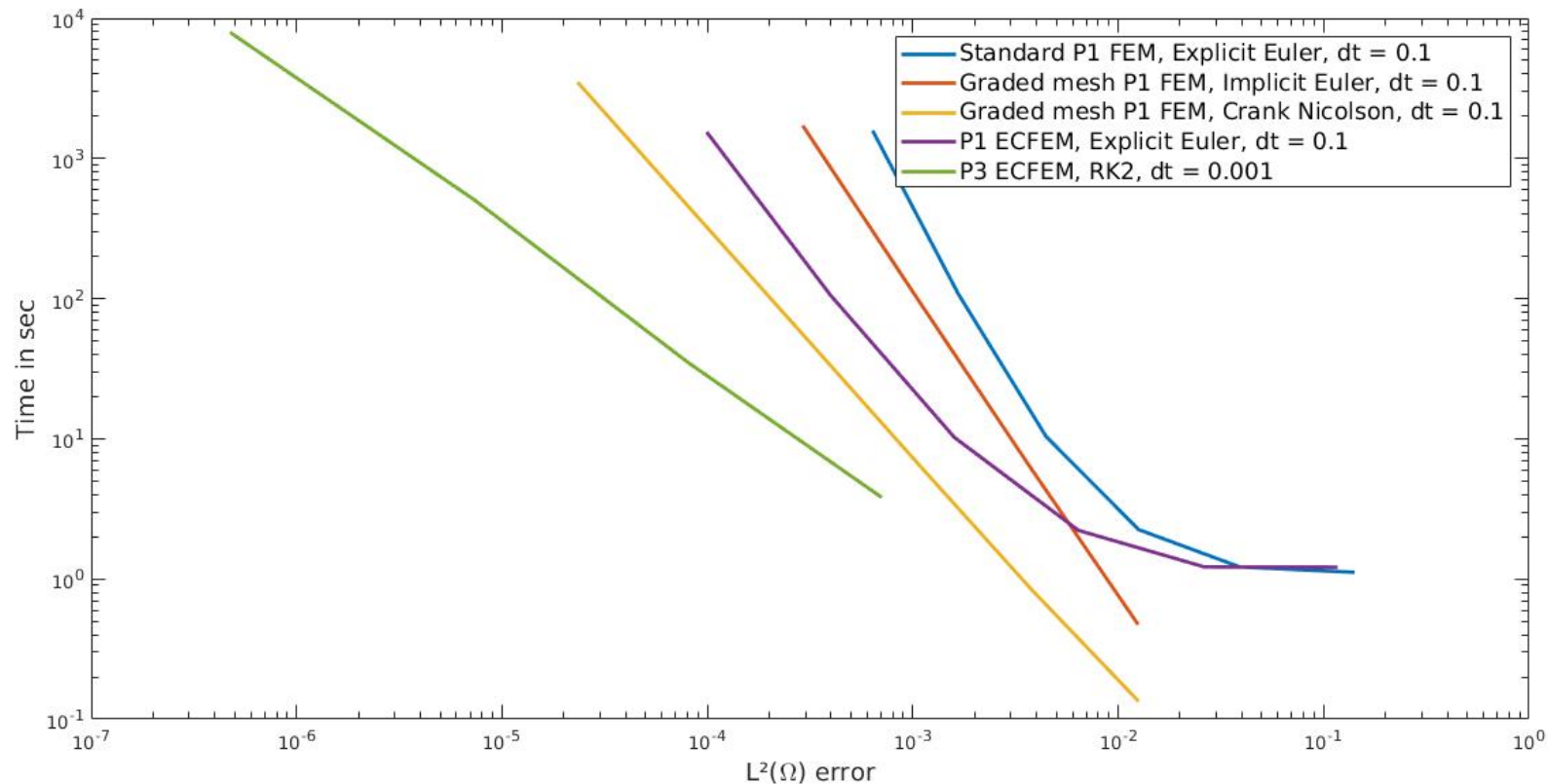
Recap: minimally enriched conforming FE spaces [Cohen et.al. 2001]



- The set of quadrature points needs to be unisolvent
- Quadrature needs to be exact for P_{2k-2}
- Quadrature nodes are nodal DOF
- Quadrature weights need to be **strictly positive**

Then the mass matrix is **diagonal, positive definite** and no order loss occurs

Comparison of different pairings



For Explicit schemes we set $dt = ch^2$, and we find numerically **second and fourth order accuracy** at endpoint T after post-processing in the standard L^2 -norm.

Summary and conclusion

- **Local modifications** can have a huge impact in efficiency
 - improved convergence order in weighted norms
 - post-process recovers standard norms
- **Higher order accuracy** in the energy is sufficient for getting rid of pollution

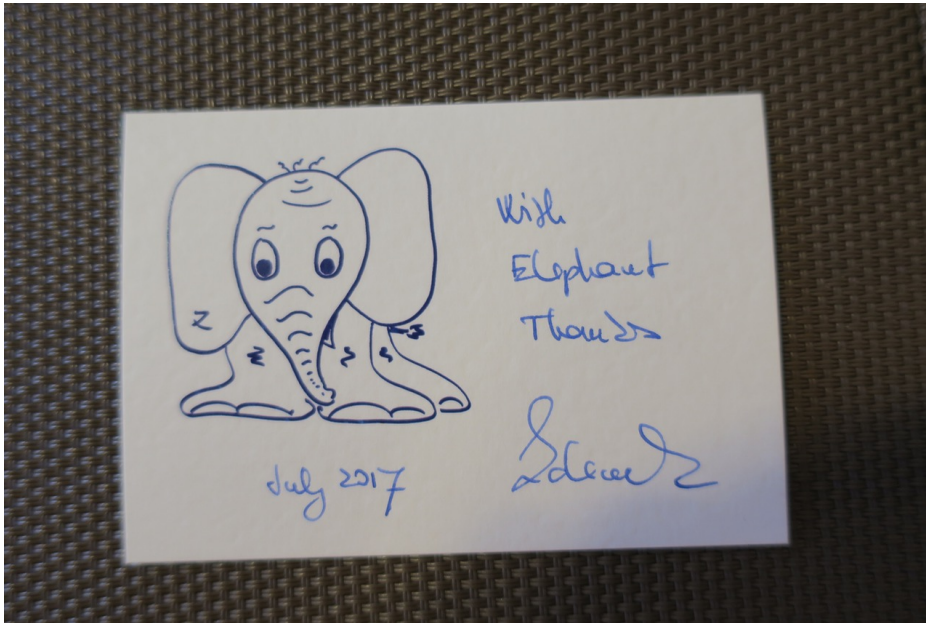
Summary and conclusion

- **Local modifications** can have a huge impact in efficiency
 - improved convergence order in weighted norms
 - post-process recovers standard norms
- **Higher order accuracy** in the energy is sufficient for getting rid of pollution

Happy birthday to
Zdeněk Strakoš
and
Eduard Feireisl



What can happen if you do only forward stability analysis?



A phase transition of one of Zdeněk's elephants to a nutria and you end up with Isogeometric Analysis

