Weak solutions of conservation laws and energy/entropy conservation

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Introduction: the principle of conservation of energy for classical solutions

Let us first focus our attention on the incompressible Euler system

$$\partial_t u + \operatorname{div}(u \otimes u) + \nabla p = 0,$$
$$\operatorname{div} u = 0,$$

If u is a classical solution, then multiplying the balance equation by u we obtain

$$\frac{1}{2}\partial_t |u|^2 + \frac{1}{2} u \cdot \nabla |u|^2 + u \cdot \nabla p = 0.$$

Integrating the last equality over the space domain Ω yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{1}{2} |u(x,t)|^2 \, \mathrm{d}x = 0.$$

Consequently, integrating over time in $(0,t)$, gives

$$\int_\Omega \frac{1}{2} |u(x,t)|^2 \, \mathrm{d}x = \int_\Omega \frac{1}{2} |u(x,0)| \, \mathrm{d}x.$$

Weak solutions

However, if u is a weak solution, then

$$\int_\Omega \frac{1}{2}|u(x,t)|^2 \,\mathrm{d}x = \int_\Omega \frac{1}{2}|u(x,0)| \,\mathrm{d}x.$$

might not hold. Technically, the problem is that u might not be regular enough to justify integration by parts in the above derivation.

Motivated by the laws of turbulence Onsager postulated that there is a critical regularity for a weak solution to be a conservative one:

Conjecture, 1949

Let u be a weak solution of incompressible Euler system

- If $u \in \mathrm{C}^\alpha$ with $\alpha > \frac{1}{3}$, then the energy is conserved.
- For any $\alpha < \frac{1}{3}$ there exists a weak solution $u \in C^\alpha$ which does not conserve the energy.

Onsager conjecture for incompressible Euler system

Weak solutions of the incompressible Euler equations which do not conserve energy were constructed:

- Scheffer '93, Shnirelman '97 constructed examples of weak solutions in $L^2(\mathbb{R}^2 \times \mathbb{R})$ compactly supported in space and time
- De Lellis and Székelyhidi 2010 showed how to construct weak solutions for given energy profile

Still incompressible case

Onsager conjecture:

If weak solution v has $C^{0,\alpha}$ (for $\alpha > \frac{1}{3}$) regularity then it conserves energy. In the opposite case it may not conserve energy.

- The first part of this assertion was proved in
  - P. Constantin, W. E, and E. S. Titi. Onsager's conjecture on the energy conservation for solutions of Euler's equation. Comm. Math. Phys., 1994
  - G. L. Eyink. Energy dissipation without viscosity in ideal hydrodynamics. I. Fourier analysis and local energy transfer. Phys. D, 1994
  - A. Cheskidov, P. Constantin, S. Friedlander, and R. Shvydkoy. Energy conservation and Onsager's conjecture for the Euler equations. Nonlinearity, 2008

Besov spaces

The elements of Besov space $B_p^{\alpha,\infty}(\Omega)$, where $\Omega = (0, T) \times \mathbb{T}^d$ or $\Omega = \mathbb{T}^d$ are functions w for which the norm

$$\|w\|_{B_p^{\alpha,\infty}(\Omega)} := \|w\|_{L^p(\Omega)} + \sup_{\xi \in \Omega} \frac{\|w(\cdot + \xi) - w\|_{L^p(\Omega \cap (\Omega - \xi))}}{|\xi|^\alpha}$$

is finite (here $\Omega - \xi = \{x - \xi : x \in \Omega\}$).
It is then easy to check that the definition of the Besov spaces implies

$$\|w^\epsilon - w\|_{L^p(\Omega)} \leq C\epsilon^\alpha \|w\|_{B_p^{\alpha,\infty}(\Omega)}$$

and

$$\|\nabla w^\epsilon\|_{L^p(\Omega)} \leq C\epsilon^{\alpha-1} \|w\|_{B_p^{\alpha,\infty}(\Omega)}.$$

Idea of the proof: P. Constantin, W. E, and E. S. Titi. Onsager's conjecture on the energy conservation for solutions of Euler's equation. Comm. Math. Phys., 1994

- take as the test function doubly mollified solution $(v^\epsilon)^\epsilon$
- problem: estimate term $\int_{\mathbb{T}^d} \mathrm{Tr}(v \otimes v)^\epsilon \cdot \nabla v^\epsilon dx$
- use the identity:
  $(v \otimes v)^\epsilon = v^\epsilon \otimes v^\epsilon + r_\epsilon(v, v) - (v - v^\epsilon) \otimes (v - v^\epsilon)$ where
  $\|r_\epsilon(v, v)\|_{L^{3/2}} \leq C\epsilon^{2\alpha} \|v\|^2_{B_p^{\alpha,\infty}}$

Onsager's conjecture for compressible Euler system

Compressible Euler system

We consider now the isentropic Euler equations,

$$\begin{aligned} \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) = 0, \\ \partial_t \rho + \operatorname{div}(\rho u) = 0. \end{aligned} \tag{1}$$

We will use the notation for the so-called pressure potential defined as

$$P(\rho) = \rho \int_1^\rho \frac{p(r)}{r^2} dr.$$

Theorem (Feireisl, Gwiazda, Ś.-G., Wiedemann, ARMA 2017)

Let ϱ, u *be a solution of* (1) *in the sense of distributions. Assume*

$$u \in B_3^{\alpha,\infty}((0,T)\times\mathbb{T}^d),\ \varrho, \varrho u \in B_3^{\beta,\infty}((0,T)\times\mathbb{T}^d), 0 \le \underline{\varrho} \le \varrho \le \overline{\varrho}$$

for some constants $\underline{\varrho}$, $\overline{\varrho}$, *and* $0 \le \alpha, \beta \le 1$ *such that*

$$\beta > \max\left\{1 - 2\alpha; \frac{1-\alpha}{2}\right\}. \tag{2}$$

Assume further that $p \in C^2[\underline{\varrho}, \overline{\varrho}]$, *and, in addition*

$$p'(0) = 0 \text{ as soon as } \underline{\varrho} = 0.$$

Then the energy is locally conserved in the sense of distributions on $(0,T)\times\Omega$, *i.e.*

$$\partial_t\left(\frac{1}{2}\varrho|u|^2 + P(\varrho)\right) + \operatorname{div}\left[\left(\frac{1}{2}\varrho|u|^2 + p(\varrho) + P(\varrho)\right)u\right] = 0.$$

Sharpness of assumptions

Shocks provide examples that show that our assumptions are sharp:

- A shock solution dissipates energy, but ρ and u are in $BV \cap L^\infty$, which embeds into $B_3^{1/3,\infty}$.
- Hence such a solution satisfies (2) with equality but fails to satisfy the conclusion.

Time regularity

The hypothesis on temporal regularity can be relaxed provided

$$\underline{\varrho} > 0$$

Indeed, in this case $\frac{(\varrho u)^\epsilon}{\varrho^\epsilon}$ can be used as a test function in the momentum equation, cf.

T. M. Leslie and R. Shvydkoy. The energy balance relation for weak solutions of the density-dependent Navier- Stokes equations. JDE, 2016.

Some references

P. Constantin, W. E, and E. S. Titi.
Onsager's conjecture on the energy conservation for solutions of Euler's equation. *Comm. Math. Phys.*, 1994.

G. L. Eyink.
Energy dissipation without viscosity in ideal hydrodynamics. I. Fourier analysis and local energy transfer. *Phys. D*, 1994.

J. Duchon and R. Robert.
Inertial energy dissipation for weak solutions of incompressible Euler and Navier-Stokes equations. *Nonlinearity*, 2000.

A. Cheskidov, P. Constantin, S. Friedlander, and R. Shvydkoy.
Energy conservation and Onsager's conjecture for the Euler equations. *Nonlinearity*, 2008.

R. E. Caflisch, I. Klapper, and G. Steele.
Remarks on singularities, dimension and energy dissipation for ideal hydrodynamics and MHD. *Comm. Math. Phys.*, 1997.

E. Kang and J. Lee.
Remarks on the magnetic helicity and energy conservation for ideal magneto-hydrodynamics. *Nonlinearity*, 2007.

R. Shvydkoy.
On the energy of inviscid singular flows. *J. Math. Anal. Appl.*, 2009.

R. Shvydkoy.
Lectures on the Onsager conjecture. *Discrete Contin. Dyn. Syst. Ser. S*, 2010.

E. Feireisl, P. Gwiazda, A. Świerczewska-Gwiazda, and E. Wiedemann.
Regularity and Energy Conservation for the Compressible Euler Equations. *Arch. Rational Mech. Anal.*, 2017.

J. Robinson, J. L. Rodrigo, J. W. D. Skipper.
Energy conservation in the 3D Euler equations on $T^2 \times \mathbb{R}_+$ *arXiv:*, (1611.00181), 2017

C. Yu.
Energy conservation for the weak solutions of the compressible Navier–Stokes equations. *Arch. Rational Mech. Anal.*, 2017.

T. M. Leslie and R. Shvydkoy.
The energy balance relation for weak solutions of the density-dependent Navier-Stokes equations. *J. Differential Equations*, 2016.

T. D. Drivas and G. L. Eyink.
An Onsager singularity theorem for turbulent solutions of compressible Euler equations. *to appear in Comm. in Math. Physics*, 2017.

C. Bardos, E. Titi.
Onsager's Conjecture for the Incompressible Euler Equations in Bounded Domains. *arXiv*, (1707.03115), 2017.

General conservation laws

- It is easy to notice similarities in the statements regarding sufficient regularity conditions guaranteeing energy/entropy conservation for various systems of equations of fluid dynamics.
- Especially the differentiability exponent of $\frac{1}{3}$ is a recurring condition.
- One might therefore anticipate that a general statement could be made, which would cover all the above examples and more. Indeed, consider a general conservation law of the form

$$\mathrm{div}_X(G(U(X))) = 0.$$

We consider the conservation law of the form

$$\mathrm{div}_X(G(U(X))) = 0. \tag{3}$$

Here $U : \mathcal{X} \to \mathcal{O}$ is an unknown and $G : \mathcal{O} \to \mathbb{M}^{n\times(d+1)}$ is a given, where $\mathcal{X}$ is an open subset of $\mathbb{R}^{d+1}$ or $\mathbb{T}^3 \times \mathbb{R}$ and the set $\mathcal{O}$ is open in $\mathbb{R}^n$. It is easy to see that any classical solution to (3) satisfies also

$$\mathrm{div}_X(Q(U(X))) = 0, \tag{4}$$

where $Q : \mathcal{O} \to \mathbb{R}^{s\times(d+1)}$ is a smooth function such that

$$D_U Q_j(U) = \mathfrak{B}(U) D_U G_j(U), \quad \text{for all } U \in \mathcal{O},\ j \in 0, \cdots, k, \tag{5}$$

for some smooth function $\mathfrak{B} : \mathcal{O} \to \mathbb{M}^{s\times n}$. The function Q is called a *companion* of G and equation (4) is called a *companion law* of the conservation law (3).

How much regularity of a weak solution is required so that it also satisfies the companion law?

Theorem

Let $U \in B_3^{\alpha,\infty}(\mathcal{X};\mathcal{O})$ be a weak solution of (3) *with $\alpha > \frac{1}{3}$. Assume that $G \in \mathrm{C}^2(\mathcal{O};\mathbb{M}^{n\times(d+1)})$ is endowed with a companion law with flux $Q \in \mathrm{C}(\mathcal{O};\mathbb{M}^{s\times(d+1)})$ for which there exists $\mathfrak{B} \in \mathrm{C}^1(\mathcal{O};\mathbb{M}^{s\times n})$ related through identity* (5) *and the essential image of U is compact in $\mathcal{O}$.*
Then U is a weak solution of the companion law (4) *with the flux Q.*

P. Gwiazda, M. Michálek and A. Świerczewska-Gwiazda,
A note on weak solutions of conservation laws and energy/entropy conservation. arxiv.1706.10154, 2017.

Remarks

- the generality of the above theorem is achieved at the expense of optimality of the assumptions.
- However given additional information on the structure of the problem at hand one might be able to relax some of these assumptions.
- the theorem provides for instance a conservation of energy result for the system of polyconvex elastodynamics, compressible hydrodynamics et al.
- T. Debiec, P. Gwiazda, and A. Świerczewska-Gwiazda,
A tribute to conservation of energy for weak solutions
arXiv:1707.09794, 2017.

Thank you for your attention